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A fourth order method for finding a simple root of univariate function *

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Abstract: In this paper, we describe an iterative method for approximating a simple zero z of a real defined function. This method is a essentially based on the idea to extend Newton's method to be the inverse quadratic interpolation. We prove that for a sufficiently smooth function f in a neighborhood of z the order of the convergence is quartic. Using Mathematica with its high precision compatibility, we present some numerical examples to confirm the theoretical results and to compare our method with the others given in the literature.

Key Words: Nonlinear equation, Newton's method, Iterative methods, Order of convergence.

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1. Introduction

Newton's method is one of the fundamental tools in numerical analysis, operations research, optimization and control. It has numerous applications in management science, industrial and financial research, data mining. Newton's method (also known as the Newton-Raphson method), is a method for finding successively better approximations to the roots (or zeroes) of a real-valued function. The Newton-Raphson method in one variable has a second order of convergence and is implemented as follows: Given a function f defined over the reals x, and its derivative f', we begin with a first guess x_0 for a root of the function f. Provided the function is reasonably well-behaved a better approximation x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

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Geometrically, $(x_1, 0)$ is the intersection with the x-axis of the line tangent to f at $(x_0, f(x_0))$. The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

until a sufficiently accurate value is reached.

An other manner to obtain the Newton method is that x_{n+1} can be computed by a simple osculatory linear interpolation of the inverse function of f at the point $y_n = f(x_n)$ using the known data: the value and the first derivative at y_n (for more details on osculatory interpolation of the inverse function of f, one can see [21]).

There exist many iterative methods improving Newton's method for solving nonlinear equations. However, many of those iterative methods depend on the second or higher derivatives in computing process which make their practical application restricted strictly. As a result, Newton's method is frequently and alternatively used to solve nonlinear equations because of higher computational efficiency. There has been some progress on iterative methods improving Newton's method with cubic convergence that require the function and its first derivative evaluations for solving nonlinear equations, see [4,7,9,10,11,16,18,22] and the reference therein.

To further improve the order of convergence, some fourth-order iterative methods have been proposed and analyzed (see [1,5,13,14,21], for instance). These methods are free from second derivatives and require only three evaluations of both the function and its first derivative.

In this paper, we propose an efficient and rapid fourth order iterative method for approximating a simple zero of f, and use exactly the same type and number of data of the third order method given by Kasturiarachi [11] see also [8]. Our method is essentially based on the idea to extend the Newton's method, since it can be obtained by the inverse linear interpolation, to be an inverse quadratic interpolation. To do this, firstly we compute an intermediate approximation t_n of x_n using the Newton's approximation, and secondly we compute x_{n+1} using the quadratic polynomial in $y = f(x)$ which interpolates the inverse function of f at x_n and t_n and its first derivative at x_n. The expression of our iterative method can be viewed as a simple modification of the Leap-frogging method's expression [11].

The paper is organized as follows: In Section 2, we give some tools needed to prove the convergence of the proposed iterative method. In Section 3, we propose the method for finding a simple zero of a real-valued function, starting with a reasonable initial guess and we state the main result concerning the fourth order of convergence and asymptotic error constant. In Section 4 we give some numerical example to illustrate the theoretical results and to compare our method to Leap-frogging method and some existing fourth order methods in the literature.

2. Preliminary

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be function having a simple zero z and sufficiently smooth in a neighborhood of this zero. An alternative sufficient method for computing a solution of the equation $f(x) = 0$ is given by rewriting it in the equivalent form $g(x)-x=0$ where g is a certain real-valued function, defined and sufficiently smooth in a neighborhood of z. Upon such a transformation the problem of solving the equation $f(x)=0$ is converted into one of finding the fixed point of g. In order to approximate this fixed point, we use the following theorem.

Theorem 2.1. *There exists an interval* $\mathbf{J}$ *centered at* z*, such that for any* $x_0 \in \mathbf{J}$*, the the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated according to the following recurrence relation*

$$x_{n+1} = g(x_n) \quad n \geq 0, \tag{2.1}$$

converges to z *as* $n \to \infty$*.*

Proof:

Let $r \in \mathbb{N}$ with $\mathbb{N}$ be the set of natural numbers. Assume that g is of class C^r in a neighborhood of z and

$$\begin{cases} |g^{(r)}(z)| < 1, & \text{if } r = 1; \\ g^{(i)}(z) = 0 \text{ for } i \in \{1 \dots r-1\} \text{ and } g^{(r)}(z) \neq 0, & \text{if } r \geq 2. \end{cases} \tag{2.2}$$

Suppose also that x_n belong to a sufficiently small neighborhood of z for $n \geq 0$. By applying Taylor's theorem we obtain

$$x_{n+1} = g(x_n) = g(z) + \frac{g^{(r)}(\xi)(x_n - z)^r}{r!}, \tag{2.3}$$

where $\xi \in (a,b)$ with $a = \min(z, x_n)$ and $b = \max(z, x_n)$.
Since $g(x)$ is continuous at z, there exists, for all given $\varepsilon > 0$, a number δ such that

$$|x_{n+1} - z| = |g(x_n) - g(z)| = |g^{(r)}(\xi)||\frac{(x_n - z)^{r-1}}{r!}|x_n - z| < \varepsilon, \tag{2.4}$$

for every x_n with $|x_n - z| < \delta$.
Let $\mathbf{J} = \{x \in \mathbb{R} / |x - z| < \delta\}$. Since $g^{(r)}$ is continuous on $\mathbf{J}$, there exists a strictly positif real M such that $|g^{(r)}(x)| \leq M, \quad \forall x \in \mathbf{J}$. In the case $r = 1$, using Relation (2.2) we can take $M < 1$.
If we choose

$$\delta < \begin{cases} \varepsilon, & \text{if } r = 1; \\ \min\left(\varepsilon, (r!/M)^{1/(r-1)}\right), & \text{if } r \geq 2, \end{cases}$$

then $|x_{n+1} - z| = |g(x_n) - g(z)| < |x_n - z|$,.Therefore, $g(\mathbf{J}) \subset \mathbf{J}$.
From Relation (2.4), we obtain

$$|x_{n+1} - z| = |g(x_n) - g(z)| \leq K|x_n - z|, \tag{2.5}$$

where

$$K = \begin{cases} M, & \text{if } r = 1; \\ M\delta^{(r-1)}/r!, & \text{if } r \geq 2, \end{cases}$$

It is easy to see that the function g is a contraction on $\mathbf{J}$ for any r. Consequently g has a unique fixed point in $\mathbf{J}$. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by Relation

(2.1) converges to z as $n \to \infty$ for any starting value x_0 in $\mathbf{J}$ (see [2,3,6,12,19,20], for instance). □

From [6,19,21], we have the following definition

Definition 2.2. *Put $e_n = |x_n - z|$, then for the method (2.1) the order of the convergence is equal to r and the asymptotic error constant η is given by*

$$\eta = \lim_{n\to\infty} |\frac{e_{n+1}}{e_n^r}| = \frac{|g^{(r)}(z)|}{r!}. \tag{2.6}$$

3. Fourth order modified Newton's method

In this Section, we propose a method for finding a simple zero of a real-valued function, starting with a reasonable initial guess. For this, let f be a real function having a simple zero z and six times differentiable in a neighborhood of z and x_0 be an initial guess. We compute t_0 using the Newton's approximation, i.e.,

$$t_0 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

t_0 can be obtained as the intersection of the line $P_1(y)$ interpolating the inverse function and its first derivative at $f(x_0)$ with the x-axis. This interpolant is written in recursive Newton's interpolation form. Then if we denote by F the inverse function of f, it is easy to see that

$$P_1(y) = P_0(y) + (y - f(x_0)[f(x_0), f(x_0)]F,$$

where $P_0(y)$ is the interpolant of degree 0 at $f(x_0)$, i.e, $P_0(y) = x_0$, and $[f(x_0), f(x_0)]F$ is the divided difference of F given by

$$[f(x_0), f(x_0)]F = F'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Now if we denote by $P_2(y)$ the quadratic polynomial which interpolates the inverse function of f at the points $f(x_0)$ and $f(t_0)$ and its first derivative at $f(x_0)$, then $P_2(y)$ can be written as follows:

$$\begin{aligned} P_2(y) &= P_1(y) + (y - f(x_0))^2[f(x_0), f(x_0), f(t_0)]F. \\ &= x_0 + \tfrac{y-f(x_0)}{f'(x_0)} - \tfrac{(y-f(x_0))^2 f(t_0)}{f'(x_0)(f(t_0)-f(x_0))^2}. \end{aligned}$$

Since we are looking for a zero of f, so we replace $y = f(x)$ by 0 in this equation, then we can compute x_1 by the following relation

$$x_1 = P_2(0) = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f(x_0)^2 f(t_0)}{f'(x_0)(f(t_0) - f(x_0))^2}. \tag{3.1}$$

Repeating this process, we obtain a sequence of numbers $x_1, x_2, \dots, x_n, \dots$ the following iterative formula:

$$\begin{aligned} x_{n+1} &= t_n - \frac{f(x_n)^2 f(t_n)}{f'(x_n)(f(t_n) - f(x_n))^2}, \\ t_n &= x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{3.2}$$

To prove the convergence of the iterative method defined by Relation (3.2) we need some notations and lemmas. Put

$$g(x) = t(x) - \frac{f(x)^2 f(t(x))}{f'(x)(f(t(x)) - f(x))^2}. \tag{3.3}$$

where $t(x) = x - \frac{f(x)}{f'(x)}$.
Define the following functions:

$$L(x) = (f(t(x)) - f(x))^2, \tag{3.4}$$

$$R(x) = f(t(x)) f^2(x), \tag{3.5}$$

$$S(x) = f'(x) L(x). \tag{3.6}$$

Using the Relations (3.5), (3.4) and (3.6), we rewrite the function g and we obtain

$$(g(x) - t(x)) S(x) = -R(x). \tag{3.7}$$

The iterative method (3.2) can be written in a new form:

$$x_{n+1} = g(x_n). \tag{3.8}$$

Therefore, to approximate the zero z of f it suffice to approximate the fixed point of the function g.

Lemma 3.1. *The value and the derivatives of t of order $1, \dots, 3$ at $x = z$ are given by*

$$t(z) = z, \quad t'(z) = 0, \quad t''(z) = \frac{f''(z)}{f'(z)} \text{ and } t'''(z) = -3\frac{f''^2(z)}{f'^2(z)} + 2\frac{f'''(z)}{f'(z)}.$$

Proof: Since $t(x) = x - \dfrac{f(x)}{f'(x)}$, then

$$t'(x) = \frac{f(x) f''(x)}{f'^2(x)}$$

and

$$t''(x) = \frac{f''(x)}{f'(x)} + \frac{f(x) f'''(x)}{f'(x)^2} - \frac{2 f(x) f''^2(x)}{f'(x)^3}.$$

Using the fact that $f(z)=0$, it is easy to see that $t(z)=z$, $t'(z)=0$ and $t''(z)=\dfrac{f''(z)}{f'(z)}$. By differentiating the function t'' and using the fact that $f(z)=0$, we obtain easily that

$$t'''(z)=-3\frac{f''^2(z)}{f'^2(z)}+2\frac{f'''(z)}{f'(z)}.$$

□

Lemma 3.2. *The value and derivatives of L of order $1,\ldots,4$ at $x=z$ are given by*

$$L(z)=L'(z)=L'''(z)=0,\quad L''(z)=2f'^2(z)$$

and

$$L^{(4)}(z)=8f'(z)(f'''(z)-t'''(z)f'(z)).$$

Proof: Since $L(x)=(f(t(x))-f(x))^2$, then

$$\begin{aligned}
L'(x) &= 2[t'(x)f'(t(x))-f'(x)][f(t(x))-f(x)],\\
L''(x) &= 2[t''(x)f'(t(x))+t'^2(x)f''(t(x))-f''(x)][f(t(x))-f(x)]\\
&\quad +2[t'(x)f'(t(x))-f'(x)]^2,\\
L'''(x) &= 2[t'''(x)f'(t(x))+3t''(x)t'(x)f''(t(x))\\
&\quad +t'^3f'''(t(x))-f'''(x)][f(t(x))-f(x)]\\
&\quad +6[t''(x)f'(t(x))+t'^2(x)f''(t(x))-f''(x)][t'(x)f'(t(x))-f'(x)].
\end{aligned}$$

Using Lemma 3.1, we get

$$L(z)=L'(z)=L'''(z)=0,\ \text{and}\ L''(z)=2f'^2(z).$$

By differentiating the function L''' and using the fact that $f(t(z))-f(z)=0$, $t'(z)=0$ and $t''(z)f'(t(z))+t'^2(z)f''(t(z))-f''(z)=0$, we obtain

$$L^{(4)}(z)=8f'(z)\left(f'''(z)-t'''(z)f'(z)\right).$$

□

Lemma 3.3. *The value and derivatives of S of order $1,\ldots,4$ at $x=z$ are given by*

$$S(z)=S'(z)=0,\quad S''(z)=2f'^3(z),\quad S'''(z)=6f''(z)f'^2(z)$$

and

$$S^{(4)}(z)=-8t'''(z)f'^3(z)+20f'''(z)f'^2(z).$$

Proof: It suffices to use Leibniz formula and Lemma 3.2. □

Lemma 3.4. *The value and derivatives of R of order $1,\dots,4$ at $x=z$ are given by*

$$R(z)=R'(z)=R''(z)=R'''(z)=0,$$
$$R^{(4)}(z)=12t''(z)f'^3(z),$$
$$R^{(5)}(z)=20t'''(z)f'^3(z)+60t''(z)f'^2(z)f''(z),$$

and

$$\begin{aligned}R^{(6)}(z) &= 30t^{(4)}(z)f'^3(z)+150t'''(x)f'^2(z)f''(z)+120t''(z)f'''(z)f'^2(z)\\ &\quad +90t''(z)f''^2(z)f'(z)+90t''^2(z)f''(z)f'^2(z).\end{aligned}$$

Proof: Put $U(x)=f^2(x)$ and $V(x)=f(t(x))$. Then $R(x)=U(x)V(x)$. To compute the derivatives of R at $x=z$, we need the derivatives of order $1,\dots,4$ of U and V.
For the function U, we have

$$U'(x)=2f(x)f'(x),$$
$$U''(x)=2f'^2(x)+2f(x)f''(x),$$
$$U'''(x)=6f'(x)f''(x)+2f(x)f'''(x)$$

and

$$U^{(4)}(x)=8f'(x)f'''(x)+6f''^2(x)+2f(x)f^{(4)}(x).$$

Then

$$U(z)=U'(z)=0, U''(z)=2f'^2(z), U'''(z)=6f'(z)f''(z) \tag{3.9}$$

and

$$U^{(4)}(z)=8f'(z)f'''(z)+6f''^2(z). \tag{3.10}$$

For the function V, we have

$$V'(x)=t'(x)f'(t(x)),$$
$$V''(x)=t''(x)f'(t(x))+t'^2(x)f''(t(x)),$$

and

$$V'''(x)=t'''(x)f'(t(x))+3t''(x)t'(x)f''(x)+t'^3(x)f'''(t(x)),$$

Then

$$V(z)=V'(z)=0, V''(z)=t''(z)f'(z) \text{ and } V'''(z)=t'''(z)f'(z). \tag{3.11}$$

By differentiating the function V''' and using the fact that $t'(z)=0$, it is easy to see that

$$V^{(4)}(z)=t^{(4)}(z)f'(z)+t'''(z)f''(z)+3t''^2(z)f''(z). \tag{3.12}$$

Applying the Leibniz formula to the function R, we get

$$R^{(n)}(z)=\sum_{k=0}^{n}\binom{n}{k}U^{(k)}(z)V^{(n-k)}(z),$$

then, using relations (3.9), (3.10), (3.11) and (3.12) we obtain

$$R(z) = R'(z) = R''(z) = R'''(z) = 0, \tag{3.13}$$

$$R^{(4)}(z) = 6U''(z)V''(z) = 12t''(z)f'^3(z), \tag{3.14}$$

$$\begin{aligned} R^{(5)}(z) &= 10U''(z)V'''(z) + 10U'''(z)V''(z) \\ &= 20t'''(z)f'^3(z) + 60t''(z)f'^2f''(z), \end{aligned} \tag{3.15}$$

we have also

$$\begin{aligned} R^{(6)}(z) &= 15U^{(4)}(z)V''(z) + 20U'''(z)V'''(z) + 15U''(z)V^{(4)}(z) \\ &= 15t''(z)f'(z)\left(8f'''(z)f'(z) + 6f''^2(z)\right) + 120t'''(z)f'^2(z)f''(z) \\ &\quad +30\left(t^{(4)}(z)f'^3(z) + t'''(z)f'^2(z)f''(z) + 3t''^2(z)f''(z)f'^2(z)\right) \\ &= 120t''(z)f'''(z)f'^2(z) + 90t''(z)f''^2(z)f'(z) + 120t'''(z)f'^2(z)f''(z) \\ &\quad +30t^{(4)}(z)f'^3(z) + 30t'''(z)f'^2(z)f''(z) + 90t''^2(z)f''(z)f'^2(z). \end{aligned}$$

Then

$$\begin{aligned} R^{(6)}(z) &= 30t^{(4)}(z)f'^3(z) + 150t'''(x)f'^2(z)f''(z) + 120t''(z)f'''(z)f'^2(z) \\ &\quad +90t''(z)f''^2(z)f'(z) + 90t''^2(z)f''(z)f'^2(z). \end{aligned} \tag{3.16}$$

□

Now we state the main result.

Theorem 3.5. *Let f be real valued function which has a simple real zero z and which is six times differentiable in a small neighborhood of z. Then the Modified Newton's method given by Relation 3.2 converges with order 4 and its asymptotic error constant is given by*

$$\eta = \left|\frac{3f''^3(z) - f'(z)f''(z)f'''(z)}{12f'^3(z)}\right| \tag{3.17}$$

Proof: To prove this theorem, it suffice to prove that

$$g'(z) = g''(z) = g'''(z) = 0 \text{ and } g^{(4)}(z) = \frac{6f''^3(z) - 2f'(z)f''(z)f'''(z)}{f'^3(z)}.$$

To do this, we begin by computing the derivatives of order 3 of both sides of Equation (3.7). Then, by the Leibnitz formula we obtain

$$\sum_{k=0}^{3}\binom{3}{k}\left(g^{(k)}(z) - t^{(k)}(z)\right)S^{(3-k)}(z) = -R'''(z).$$

Using the fact that $g(z) = z$ and Lemmas 3.1, 3.3 and 3.4, we get $6g'(z)f'^3(z) = 0$. Therefore, $g'(z) = 0$. By the same technique we compute the derivatives of order 4 of both sides of Equation (3.7). Then we get

$$12\left(g''(z) - t''(z)\right) f'^3(z) = -12t''(z)f'^3(z).$$

Consequently $g''(z) = 0$. We continue with the Leibnitz formula and we differentiate Equation (3.7) to the order 5 and 6 at $x = z$. Then, for the order 5, we obtain

$$20(g'''(z)-t'''(z))\, f'^3(z)-60t''(z)f'^2(z)f''(z)=-\left(20t'''(z)f'^3(z)+60t''(z)f'^2f''(z)\right).$$

Therefore

$$g'''(z) = 0.$$

For the order 6, we have

$$15\left(g^{(4)}(z) - t^{(4)}(z)\right) S''(z) - 20t'''(z)S'''(z) - 15t''(z)S^{(4)}(z) = -R^{(6)}(z). \quad (3.18)$$

Using Lemmas 3.1, 3.3 and 3.4 and by eliminating the terms $-30t^{(4)}(z)f'^3(z)$ and $-120t'''(z)f'^2(z)f''(z)$, Equality (3.18) becomes

$$15g^{(4)}(z)S''(z)-270f''^3(z)-120f'(z)f''(z)f'''(z)=-90f''^3(z)-180f'(z)f''(z)f'''(z)$$

and therefore

$$g^{(4)}(z) = \frac{6f''^3(z) - 2f'(z)f''(z)f'''(z)}{f'^3(z)}.$$

By the results given in Section 2, it is easy to see that the iterative method (3.2) converge to the unique fixed point of g which is the simple zero of f. Furthermore, the order of convergence is 4 and the asymptotic error constant is given by Relation (3.17). □

Remark 3.6. *The method proposed in this paper can be viewed as a simple modification of the Leap-frogging Newton's method by taking the same type and number of data. We mention that our method is the fourth order of convergence. In the other side the Leap-frogging Newton's method is the third one. Furthermore, the efficiency index for our method is* $4^{1/3} \approx 1.5874$ *which is better than* $3^{1/3} \approx 1.4422$ *the one for the Leap-frogging Newton's method.*

4. Numerical examples

In this section we give some numerical examples for testing the performance of the proposed method. To programming this method we used Mathematica which allows us to find the zero with higher precision. In the first part of this Section, we compare our results with the results given by the Leap-frogging Newton's method. For this, we consider two test functions and we compute $|x_n - z|$ the error between the zero obtained by our method and the exact zero for $n = 0, \dots, 6$.

Exemple1:

In this example we consider the function f defined by

$$f(x) = x^3 - 3x^2 - 5,$$

which has as a simple zero

$$z = 1 + \left(\frac{7 - 3\sqrt{5}}{2}\right)^{1/3} + \left(\frac{7 + 3\sqrt{5}}{2}\right)^{1/3}.$$

We choose as initial guess $x_0 = 5$.

Exemple2:

In this example we consider the function f defined by

$$f(x) = (x^6 - x + 27)\sin(\pi x),$$

which has as a simple zero $z = 2$. We choose as initial guess $x_0 = 2.5$.

In Table 1 and Table 2, we give, respectively in each column, the errors obtained by the Leap-frogging Newton's method, the errors obtained by our method, the quantity e_{n+1}/e_n^4 which gives the practical order of convergence and the value of the theoretical quantity asymptotic error constant η. These examples present the higher performance of our method comparing with the Leap-frogging Newton's method and illustrate the fourth order of convergence and confirm the value of the asymptotic error constant shown in the previous Section.

Table 1: Comparative rates of convergence between Leap-frogging Newton's method (noted LFN) and our method for $f(x) = x^3 - 3x^2 - 5$.

n	LFN error	$\lvert e_n\rvert$ for our method	e_{n+1}/e_n^4	η
0	1.57401	1.57401	0.0237551419	0.2110192770
1	2.50482×10^{-1}	1.14581×10^{-1}	0.1630703290	–
2	2.98355×10^{-3}	7.37107×10^{-5}	0.2109911146	–
3	7.59060×10^{-9}	6.22855×10^{-18}	0.2110192770	–
4	6.85976×10^{-26}	3.17592×10^{-70}	0.2110192770	–
5	7.95976×10^{-77}	2.14686×10^{-279}	0.2110192770	–
6	1.24358×10^{-229}	$4.48272 \times 10^{-1116}$	–	–

In the second part of this Section we present some numerical experiments using our iterative method and we compare these results to well known fourth-order schemes. All computations were done using Mathematica programming using 128 digit floating point arithmetics. We accept an approximate solution rather than the exact root, depending on the precision ϵ of the computer. We use the following stopping criteria for computer programs:

Table 2: Comparative rates of convergence between Leap-frogging Newton's method (noted LFN) and our method for $f(x) = (x^6 - x + 27)\sin(\pi x)$.

n	LFN error	$\lvert e_n\rvert$ for our method	e_{n+1}/e_n^4	η
0	0.50000	0.50000	0.2788407969	17.5108704694
1	1.85038×10^{-2}	1.74275×10^{-2}	15.2601020680	–
2	2.66167×10^{-5}	1.40767×10^{-6}	17.5106746899	–
3	8.68347×10^{-14}	6.87565×10^{-23}	17.5108704694	–
4	3.01555×10^{-39}	3.91348×10^{-88}	17.5108704694	–
5	1.26295×10^{-115}	4.10735×10^{-349}	17.5108704694	–
6	1.03088×10^{-265}	$4.98734 \times 10^{-1393}$	–	–

i) $|x_{n+1} - x_n| < \epsilon$;

ii) $|f(x_{n+1})| < \epsilon$,

and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact zero z.

For numerical illustrations in this section we used the fixed stopping criterion $\epsilon < 10^{-25}$. We used the test functions given in [5,15,22].

We present some numerical test results for various fourth-order iterative schemes in Table 4. We compare our method with King's method with $\beta = 3$ [13] (KM), Kou's method [14] (KouM) and fourth order Chun's method [5](CM).

The comparison is about the number of iterations to approximate the zero (IT), the number of functional evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the value $f(z)$ and the distance δ of two consecutive approximations for the zero. The test results in Table 4 show that for most of the functions we tested, the iterative method introduced in this work have equal or better performance as compared to the other methods of the same order and use the same number and type of data.

Table 3: Test functions, zeros and initial guess.

Test functions	Initial guess	Zeros
$f_1(x) = x^3 + 4x^2 - 10$	1.6	1.3652300134140968457608068290
$f_2(x) = sin^2(x) - x^2 + 1$	1.0	1.4044916482153412260350868178
$f_3(x) = (x-1)^3 - 1$	3.5	2.0
$f_4(x) = x^3 - 10$	4.0	2.1544346900318837217592935665
$f_5(x) = xe^{x^2} - \sin^2 x$ $+3\cos x + 5$	-1.0	$-1.2076478271309189270094167584$
$f_6(x) = \sin x - \frac{x}{2}$	2.0	1.8954942670339809471440357381
$f_7(x) = x^5 + x - 10000$	4.0	6.3087771299726890947675717718
$f_8(x) = \sqrt{x} - \frac{1}{x} - 3$	9.0	9.6335955628326951924063127092
$f_9(x) = e^x + x - 20$	0.0	2.8424389537844470678165859402
$f_{10}(x) = ln(x) + \sqrt{x} - 5$	10.0	8.3094326942315717953469556827

5. Conclusion

In this paper, we have described an efficient iterative method for approximating a simple zero z of a real defined function. This method is a essentially based on the idea to extend Newton's method to be the inverse quadratic interpolation. We have proved that for a sufficiently smooth function f in a neighborhood of z, this method is a fourth order of convergence. We have used Mathematica programming with its high precision compatibility and we have presented some example tests to confirm the theoretical results. We also have showed the performance of our method comparing with the third order and fourth order existing methods in the literature.

Table 4: A Comparison of various existing fourth order methods and our method.

f		KM	KouM	CM	Our Method
f_1	IT	4	4	4	4
	NFE	12	12	12	12
	$f(z)$	-6.0×10^{-127}	-6.0×10^{-127}	-6.0×10^{-127}	0.0
	δ	4.94×10^{-48}	7.83×10^{-55}	1.64×10^{-45}	2.59×10^{-58}
f_2	IT	9	5	6	5
	NFE	27	15	18	15
	$f(z)$	-1.0×10^{-127}	-2.0×10^{-127}	-1.0×10^{-127}	0.0
	δ	5.27×10^{-76}	1.71×10^{-42}	1.15×10^{-94}	6.35×10^{-73}
f_3	IT	6	5	6	5
	NFE	18	15	18	15
	$f(z)$	0.0	1.11×10^{-120}	0.0	0.0
	δ	4.28×10^{-85}	6.10×10^{-31}	1.10×10^{-88}	5.23×10^{-36}
f_4	IT	5	5	5	4
	NFE	15	15	15	12
	$f(z)$	0.0	0.0	0.0	0.0
	δ	3.78×10^{-42}	7.40×10^{-56}	1.23×10^{-32}	2.62×10^{-67}
f_5	IT	5	5	4	4
	NFE	15	15	12	12
	$f(z)$	-1.94×10^{-101}	-1.20×10^{-126}	-1.10×10^{-126}	-4.0×10^{-123}
	δ	1.46×10^{-26}	9.01×10^{-90}	1.04×10^{-55}	8.67×10^{-32}
f_6	IT	4	4	4	4
	NFE	12	12	12	12
	$f(z)$	-2.0×10^{-128}	-6.0×10^{-128}	-2.0×10^{-128}	0.0
	δ	4.59×10^{-64}	1.40×10^{-70}	3.84×10^{-62}	1.36×10^{-73}
f_7	IT	48	12	14	12
	NFE	144	36	42	36
	$f(z)$	0.0	-5.93×10^{-102}	0.0	0.0
	δ	1.12×10^{-63}	9.85×10^{-27}	2.12×10^{-40}	1.63×10^{-39}
f_8	IT	4	3	4	3
	NFE	12	9	12	9
	$f(z)$	0.0	-3.98×10^{-109}	-3.10×10^{-126}	0.0
	δ	1.28×10^{-93}	1.69×10^{-26}	1.55×10^{-31}	1.68×10^{-33}
f_9	IT			14	10
	NFE			42	30
	$f(z)$	div	div	-3.98×10^{-109}	0.0
	δ			2.72×10^{-57}	3.44×10^{-88}
f_{10}	IT	4	4	4	4
	NFE	12	12	12	12
	$f(z)$	-1.0×10^{-127}	-1.0×10^{-127}	-7.17×10^{-116}	0.0
	δ	1.23×10^{-57}	2.62×10^{-71}	4.92×10^{-29}	3.76×10^{-85}

References

1. I.K. Argyros, D. Chen, Q. Qian, *The Jarratt method in Banach space setting*, J. Comput. Appl. Math. 51, 103–106, (1994).

2. R.G. Bartle, *The Elements of Real Analysis*, second ed., John Wiley and Sons, New York, 1976.

3. W. Cheney, D. Kincaid, *Numerical Mathematics and Computing*, Brooks/Cole Publishing Company, Monterey, California, 1980.

4. C. Chun, *Iterative methods improving Newton's method by the decomposition method*, Comput. Math. Appl. 50, 1559–1568, (2005).

5. C. Chun, B. Neta, *Certain improvements of Newton's method with fourth-order convergence*, Applied Mathematics and Computation 215, 821–828, (2009).

6. S. D. Conte, C. de Boor,*Elementary Numerical Analysis*, McGraw-Hill Inc., 1980.

7. M. Frontini, E. Sormani, *Some variants of Newton's method with third-order convergence*, J. Comput. Appl. Math. 140, 419–426, (2004).

8. Y. H. Geum, *The asymptotic error constant of leap-frogging Newton's method locating a simple real zero*, Applied Mathematics and Computation 189, 963–969, (2007).

9. H.H.H. Homeier, *A modified Newton method with cubic convergence*, J. Comput. Appl. Math. 157, 227–230, (2003).

10. H.H.H. Homeier, *On Newton-type methods with cubic convergence*, J. Comput. Appl. Math. 176, 425–432, (2005).

11. A. B. Kasturiarachi, *Leap-frogging Newton's Method*, Int. J. Math. Education. Sci. Technol. 33 (4), 521–527, (2002).

12. A. R. Kenneth, *Elementary Analysis*, Springer-Verlag Inc., New York, 1980.

13. R. King, *A family of fourth-order methods for nonlinear equations*, SIAM J. Numer. Anal. 10 (5), 876–879, (1973).

14. J. Kou, L. Yitian, X. Wang, *A composite fourth-order iterative method for solving non-linear equations*, Appl. Math. Comput. 184, 471–475, (2007).

15. B. Neta, *Several new schemes for solving equations*, Int. J. Comput. Math. 23, 265–282, (1987).

16. F.A. Potra, V. Pta'k, *Nondiscrete induction and iterative processes*, Research Notes in Mathematics, vol. 103, Pitman, Boston, 1984.

17. A. Ralston, P. Rabinowitz, *A First Course in Numerical Analysis*, McGraw-Hill, 1978.

18. J.R. Sharma, *A composite third order NewtonŨSteffensen method for solving nonlinear equations*, Appl. Math. Comput. 169, 242–246, (2005).

19. J. Stoer, R. Bulirsh, *Introduction to Numerical Analysis*, Springer-Verlag Inc., New York, 1980.

20. E. Süli, D. Mayers,*An Introduction to Numerical Analysis*, Cambridge University Press, 2003.

21. J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1964.

22. S. Weerakoon, G.I. Fernando, *A variant of Newton's method with accelerated third-order convergence*, Appl. Math. Lett. 17, 87–93, (2000).

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