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Modified Finite Difference Method for Solving Fractional Delay Differential Equations

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abstract: In this paper, a new numerical scheme for solving fractional delay differential equations is presented. Finite difference method is extended to study this problem, where the derivatives are defined in the Caputo fractional sense. The proposed method is also employed for solving some scientific models. The obtained results show that the propose method is very effective and convenient.

Key Words: Finite difference method; Caputo derivative; Fractional delay differential equations; Boundary values.

Contents

1. Introduction

Delay differential equation (DDE) is a generalization of the ordinary differential equation, which is suitable for physical system that also depends on the past data [1]. The solution of delay differential equations not only requires information of current state, but also requires some information about the previous state. Delay differential equations have numerous applications in mathematical modeling [2,3]: for example, physiological and pharmaceutical kinetics, chemical kinetics, the navigational control of ships and spacecraft [4], population dynamics and infectious diseases. During the last decade,several papers have been devoted to the study of the numerical solution of delay differential equations. Therefore different numerical methods [5,6,7,8,9] have been developed and applied for providing approximate solutions.
The fractional delay differential equation (FDDE) is a generalization of the delay differential equation to arbitrary non-integer order. During the last decade, several papers have been devoted to the study of the numerical solution of fractional delay differential equations. For most of fractional order delay differential equations, exact solutions are not known. Therefore different numerical methods [10,11,12,13,14] have been developed and applied for providing approximate solutions. Yang and

Cao [15] studied the existence and uniqueness of initial value problems for nonlinear higher fractional equations with delay by fixed point theory. Wang [16] combined Adams Bash forth Moulton method with the linear interpolation method to approximate FDDEs. Moghaddam and Mostaghim [17] developed a numerical method based on finite difference for solving fractional delay differential equations. Also, Moghaddam and Mostaghim [18] discussed and introduced a novel matrix approach to fractional finite difference for solving models based on nonlinear fractional delay differential equations. Recently, there has been increasing interest in the investigation of fractional delay differential equations with boundary conditions [19,20,21]. Saeed and et. al. [2] developed the Chebyshev wavelet method for solving the fractional delay differential equations and integro-differential equations. In [22], the fractional comparison result of order $\beta \in (1,2)$ established and investigated the existence of extremal solutions for a nonlinear fractional differential equation with three-point nonlinear boundary conditions. Pimenov and Hendy [23] presented a new method of backward differentiation formula type for solving FDDEs. Further, Moghaddam and Mostaghim [24] adapted A Matrix scheme based on fractional finite difference method for solving FDDEs with boundary conditions.
In this paper, we focus on FDDE which has the following form

$$D_*^{\beta} y(t) = f(t, y(t), y(t-\tau), D_*^{\alpha} y(t), D_*^{\alpha} y(t-\tau)) \tag{1.1}$$

on $a \leq t \leq b$, $0 < \alpha \leq 1$, $1 < \beta \leq 2$ and under the following interval and boundary conditions:

$$\begin{aligned} y(t) &= \varphi(t) \qquad -\tau \leq t \leq a, \\ y(b) &= \gamma, \end{aligned} \tag{1.2}$$

where $D_*^{\beta} y(t)$, $D_*^{\alpha} y(t)$ and $D_*^{\alpha} y(t-\tau)$ are the standard Caputo fractional derivatives, φ represent smooth function and $\tau \in R^+$ denotes the delay.
This paper is organized as follows, we recall some necessary definitions of the fractional calculus in section 2. In section 3, we generalize the finite difference method and use it for solving FDDEs with boundary conditions. The adaptation of a variety of differential equations in the mathematical modeling process of difference applications will be considered; for example, the problem of the convection [25], the problem of the muscle reflex mechanism in during snoring [26], the problem of the electromechanical systems [27]; the interesting point is that these fractional version of models are similar to real phenomena in section 4. Finally, we give some brief conclusions in section 5.

2. Definitions of fractional derivatives and integrals

In this section, we present notation, definitions, and recall well-known results about fractional differential equations. For more details the interested reader is referred to the book by Podlubny [28].

Definition 2.1. *The Riemann-Lowville fractional integral operator* J^α *of order* α *is given by*

$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} y(x)dx, \qquad \alpha > 0, \qquad t > 0,$

$J^0 y(t) = y(t).$
Its properties as following

(i) $J^\alpha J^\beta = J^{\alpha+\beta}$, *(ii)* $J^\alpha J^\beta = J^\beta J^\alpha$.

Definition 2.2. *The Caputo definition of fractional differential operator is given by*

$$D_*^\gamma y(t) = \begin{cases} \frac{d^r y(t)}{dt^r}, & \gamma = r\epsilon Z^+; \\ \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} d\tau, & 0 \le r-1 < \gamma < r. \end{cases}$$

The Caputo fractional derivatives of order γ is also defined as $D_*^\gamma y(t) = J^{r-\gamma} D^r y(t)$, where D^γ is the usual integer differential operator of order . Note that for $r-1 < \gamma < r$ and $k \in N$,

$D_*^\gamma J^\gamma y(t) = y(t),$

$J^\gamma D_*^\gamma y(t) = y(t) - \sum_{k=0}^{r-1} y^{(k)}(0^+) \frac{t^k}{k!}, \qquad t > 0.$

3. Numerical Method

In this section we will present a numerical method for solving the boundary value problems in delay differential equations (1-1)-(1-2). If $y(t)$ is a smooth answer in problem (1.1), it should be verified in the boundary problems of (1.1)-(1.2); they also should be continuous in the interval $[a, b]$ and be continuously differentiable in the interval (a, b). This numerical method includes the finite difference operator on a specific consistent mesh. Consider a uniform grid $\{t_n = nh : n = -m, -m+1, \ldots, -1, 0, 1, \ldots, N\}$ where m and N are integers such that $h = \frac{b-a}{N}$ and $h = \frac{\tau}{m}$ in which $m = pq$ and p is an affirmative integer and q is the τ Mantissa. This difference method for boundary problem (1.1)-(1.2) is based on the following relations:

$$for \quad i = 1, \ldots, N-1 \qquad y(t) \to y_i \tag{3.1}$$

$$for \quad i = 1, \ldots, N-1 \qquad y(t-\tau) \to y_{i-m} \tag{3.2}$$

$$for \quad i = -m, -m+1, \ldots, 0 \qquad y_i = \varphi_i \tag{3.3}$$

$$y_N = \gamma \tag{3.4}$$

From definition 2.2, for $r = 1$, we get

$$D_*^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} y(\tau) d\tau, \qquad 0 < \alpha \le 1.$$

Therefore, we take the following finite difference approximation for time fractional derivative:

$$\begin{aligned} D_*^{\alpha} y(t) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i} \frac{y(t_{j+1}) - y(t_j)}{h} \int_{jh}^{(j+1)h} \frac{d\tau}{(t-\tau)^{\alpha}} + O(h) \\ &= \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i} \frac{y(t_{i-j+1}) - y(t_{i-j})}{h} [(j+1)^{1-\alpha} - j^{1-\alpha}] + O(h), \end{aligned}$$

then

$$D_*^{\alpha} y(t) \approx \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}] D_+ y_{i-j}. \tag{3.5}$$

Also, from definition 2.2, we have

$$D_*^{\beta} y(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-\tau)^{1-\beta} \frac{d^2}{d\tau^2} y(\tau) d\tau, \qquad 1 < \beta \leq 2.$$

Therefore

$$\begin{aligned} D_*^{\beta} y(t) &= \frac{1}{\Gamma(2-\beta)} \sum_{j=0}^{i} \frac{y(t_{j+2}) - 2y(t_{j+1}) + y(t_j)}{h^2} \int_{jh}^{(j+2)h} \frac{d\tau}{(t-\tau)^{\beta}} + O(h^2) \\ &= \frac{h^{2-\beta}}{2\Gamma(3-\beta)} \sum_{j=0}^{i} \frac{y(t_{i-j+2}) - 2y(t_{i-j+1}) + y(t_{i-j})}{h^2} [(j+2)^{2-\beta} - j^{2-\beta}] \\ &\quad + O(h^2), \end{aligned}$$

then

$$D_*^{\beta} y(t) = \frac{h^{2-\beta}}{2\Gamma(3-\beta)} \sum_{j=0}^{i} [(j+2)^{2-\beta} - j^{2-\beta}] D_+ D_- y_{i-j}, \tag{3.6}$$

$$D_*^{\alpha} y(t-\tau) \approx \frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}] D_+ y_{i-m-j}. \tag{3.7}$$

Table 1: Approximate solutions with different values α, β and $\tau = 1$ for Model 1.

t	$\alpha = 0.25, \beta = 1.5$	$\alpha = 0.5, \beta = 1.75$	$\alpha = 0.95, \beta = 1.85$
0	1	1	1
0.2	-0.06935648085	-0.4676735321	-0.6663489350
0.4	-0.01396181245	0.1560783015	0.3740576937
0.6	-0.005461778382	-0.06378143359	-0.2232814045
0.8	-0.003863134010	0.01793320380	0.1250802883
1	-0.002767405544	-0.008975557245	-0.07399244528

In which

$$D_+D_-y_{i-j} = \frac{y_{i-j+2} - 2y_{i-j+1} + y_{i-j}}{h^2},$$

$$D_+y_{i-j} = \frac{y_{i-j+1} - y_{i-j}}{h},$$

and

$$D_+y_{i-m-j} = \frac{y_{i-j-m+1} - y_{i-j-m}}{h}.$$

By substituting relations (3.1)-(3.7) in the equation (1.1), we obtain generalized-form fractional difference quotient formula. It should be noted that when using the generalized-form fractional difference quotient formula, special attention should be paid to the applicability of different approximation schemes and sometimes modifications needs to be made.

4. Numerical results

In this section, three models are considered and solved by using the proposed method. Then, the graphs and tables will be shown according to different amounts of α and β with step size $h = 0.01$.

Model 1. This model which is related to equation of the convection with delay in the convection term and its equation is as [25]:

$$\varepsilon D_*^{\beta} y(t) = D_*^{\alpha} y(t-\tau) - y(t), \qquad 0 < t < 1, \tag{4.1}$$

under the boundary conditions:

$$y(t) = 1 \qquad -\tau \le t \le 0 \qquad y(1) = 0,$$

where $\varepsilon = 0.01$.

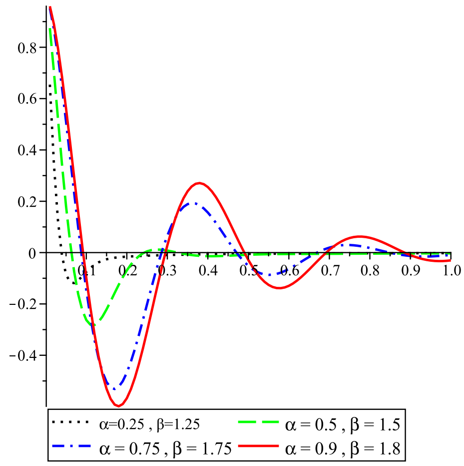

Figure 1: The numerical solution of Model 1 with different values α, β and $\tau = 1$.

Model 2. Huang and Williams [26] coupled snoring mechanics with neurological responses. If the elastic forces are insufficient to maintain the stability of the airway, neuromuscular functions become crucial. However, these functions are very much reduced during sleep, and the muscle reflex mechanism may have a time delay of several cycles of oscillations experienced during snoring. Huang and Williams assumed that following a delayed signal from the neural receptors, the muscle opposes the collapsing tendency by increasing the walls stiffness by an amount proportional to the negative pressure at time $(t-\tau)$. Its equation is as:

$$D_*^{\beta} y(t) = -\delta D_*^{\alpha} y(t) - y(t) + \frac{\mu q^2}{y^3(t)}(y(t) - \gamma y(t-\tau)), \qquad 0 < t < 1, \qquad (4.2)$$

under the boundary conditions:

$$y(t) = 1 \qquad -\tau \le t \le 0 \qquad y(1) = 3,$$

where $\delta = 0.3$, $\mu = 1$, $q = 0.4$ and $\gamma = 0.2$.

Table 2: Approximate solutions with different values α, β and $\tau = 5$ for Model 2.

t	$\alpha = 0.5, \beta = 1.5$	$\alpha = 0.75, \beta = 1.75$	$\alpha = 0.95, \beta = 1.95$
0	1	1	1
0.2	1.230053053	1.212958460	1.229225020
0.4	1.521339669	1.497033332	1.519859723
0.6	1.882538740	1.847562711	1.878952412
0.8	2.330459527	2.280021417	2.322731560
1	2.917052561	3.018221882	2.974566072

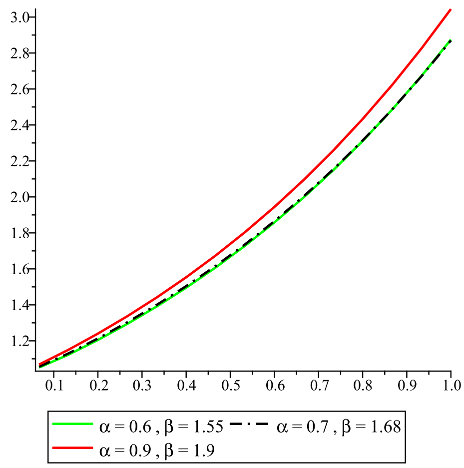

Figure 2: The numerical solution of Model 2 with different values α, β and $\tau = 5$.

Table 3: Approximate solutions with different values α, β and $\tau = 3$ for Model 3.

t	$\alpha = 0.5, \beta = 1.25$	$\alpha = 0.75, \beta = 1.5$	$\alpha = 0.8, \beta = 1.9$
0	0.5	0.5	0.5
2	0.1804719309	0.0625380460	0.05463796349
4	0.4279463895	0.1931450691	0.2213159498
6	0.7138737783	0.4480035882	0.6688060172
8	1.269600334	1.056835698	1.134553314
10	2.477536643	2.515526144	2.507524849

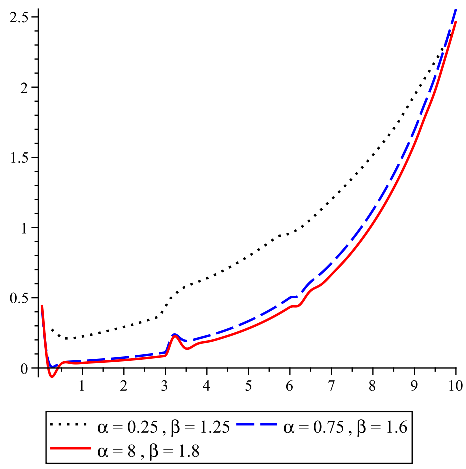

Figure 3: The numerical solution of Model 3 with different values α, β and $\tau = 3$.

Model 3. Johnson and Moon [27] investigated experimentally an electromechanical system. Their experiments were compared to the numerical solutions of the following equation:

$$D_*^{\beta} y(t) = -aD_*^{\alpha} y(t) - bD_*^{\alpha} y(t-\tau) + by(t), \qquad 0 < t < 10, \tag{4.3}$$

where $a = 2.623$, $b = 2.6$, under the boundary conditions:

$$y(t) = 0.5 \qquad -\tau \leq t \leq 0 \qquad y(10) = 2.5,$$

5. Conclusions

The fundamental goal of this work was to construct a numerical method to find the solution of linear and nonlinear delay differential equations of fractional order with boundary values. According to this scheme, which is based on the finite difference method, an approximate solution for solving different kinds of boundary problems with fractional order was obtained. The figures of the solutions of the considered examples for different values of delay α, β and τ were plotted and the effect of delay in producing fluctuations has been examined.

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