Bol. Soc. Paran. Mat. (3s.) v. 34 1 (2016): 203–212.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v34i1.25102

$g_{{}^*\lambda_\mu}$−closed sets in Generalized Topological Spaces

P. Jeyanthi, P. Nalayini, M. Mocana

abstract: In this paper we introduce some new classes of generalized closed sets called ${}^*\lambda_\mu - g-$closed, ${}^*\lambda_\mu - g_\mu-$closed and $g_{{}^*\lambda_\mu}-$closed sets in generalized topological spaces, which are related to the classes of $g_\mu-$closed sets, $g - \lambda_\mu-$closed sets and $\lambda_\mu - g-$closed sets. We investigate the properties of the newly introduced classes, as well as the connections among the above mentioned classes of generalized closed sets. Also, we give a unified framework for the study of several types of generalized closed sets in a space endowed with two generalized topologies.

Key Words: Generalized topology, $\lambda_\mu-$closed, ${}^*\lambda_\mu-$closed, ${}^*\lambda_\mu - g-$closed, ${}^*\lambda_\mu - g_\mu-$closed and $g_{{}^*\lambda_\mu}-$closed sets.

Contents

1. Introduction

In 1997, *Á.Császár* [2] introduced the concept of generalization of topological space, which is a generalized topological space. A subset μ of $exp(X)$ is called a generalized topology [4] on X if $\emptyset \in \mu$ and μ is closed under arbitrary union. Elements of μ are called $\mu-$open sets. A set X with a GT μ on it is said to be a generalized topological space (briefly GTS) and is denoted by (X, μ).For a subset A of X, we denote by $c_\mu(A)$ the intersection of all closed sets containing A and by $i_\mu(A)$ the union of all $\mu-$open sets contained in A. Then $c_\mu(A)$ is the smallest closed set containing A and $i_\mu(A)$ is the largest $\mu-$open set contained in A. A point $x \in X$ is called a $\mu-$cluster point of A if for every $U \in \mu$ with $x \in U$ we have $A \cap U \neq \emptyset$. It is known from [4] that $c_\mu(A)$ is the set of all $\mu-$cluster points of A. A GTS (X, μ) is called a quasi-topological space [3] if μ is closed under finite intersections. A subset A of X is said to be $\pi-$regular [5] (resp. $\sigma-$regular) if $A = i_\mu c_\mu(A)$ (resp. $A = c_\mu i_\mu(A)$).

2000 *Mathematics Subject Classification:* 54A05

Definition 1.1. [6] *Let (X,μ) be a GTS and $A \subset X$. Then the subsets $\wedge_\mu(A)$ and $\vee_\mu(A)$ are defined as follows:*

$$\wedge_\mu(A) = \begin{cases} \cap\{G : A \subseteq G, G \in \mu\} & \text{if there exists } G \in \mu \text{ such that } A \subseteq G; \\ X & \text{otherwise.} \end{cases}$$

and

$$\vee_\mu(A) = \begin{cases} \cup\{H : H \subseteq A, H^c \in \mu\} & \text{if there exists } H \text{ such that } H^c \in \mu \text{ and } H \subseteq A; \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition 1.2. [1] *A subset A of a GTS (X,μ) is said to be a $(\wedge,\mu)-$closed (briefly $\lambda_\mu-$closed) set if $A = T \cap C$, where T is a $\wedge_\mu-$set and C is a $\mu-$closed set. The complement of a $\lambda_\mu-$closed set is called a $\lambda_\mu-$open set.*

Given a GTS (X,μ), we denote the class of $\mu-$closed sets by $C(X,\mu)$ and the class of $\wedge_\mu-$sets by $\wedge_\mu(X,\mu)$. Moreover, following [1] we denote the class of $\lambda_\mu-$closed sets (resp. $\lambda_\mu-$open sets) by $\wedge_\mu - C(X,\mu)$ (resp. $\wedge_\mu - O(X,\mu)$).

It is known that $\mu \subset \wedge_\mu(X,\mu)$ and $\wedge_\mu(X,\mu)$ is a GT closed to arbitrary intersections, by Proposition 2.3 of [6] and Theorem 2.7, Remark 2.8 [6]. Note that $X \in \wedge_\mu(X,\mu) \cap C(X,\mu)$ by Theorem2.7 [6]. By Theorem 2.6 and 2.7 [1], we have

$$\wedge_\mu - C(X,\mu) \supset \wedge_\mu(X,\mu) \cup C(X,\mu)$$

and $\wedge_\mu - C(X,\mu)$ is closed to arbitrary intersections, hence $\wedge_\mu - O(X,\mu)$ is a GT. From $\mu \subset \wedge_\mu - C(X,\mu)$ and $C(X,\mu) \subset \wedge_\mu - C(X,\mu)$ we conclude that $\mu-$closed sets and $\mu-$open sets are simultaneously $\lambda_\mu-$closed and $\wedge_\mu-$open, i.e.

$$\mu \cup C(X,\mu) \subset \wedge_\mu - C(X,\mu) \cap \wedge_\mu - O(X,\mu).$$

Definition 1.3. [1] *Let (X,μ) be a GTS. A point $x \in X$ is called a $(\wedge,\mu)-$cluster point of A if for every $(\wedge,\mu)-$ open set U of X containing x, we have $A \cap U \neq \emptyset$. The set of all $(\wedge,\mu)-$ cluster points of A is called the $(\wedge,\mu)-$closed of A and is denoted by $A^{(\wedge,\mu)}$.*

Lemma 1.4. [1] *Let (X,μ) be a GTS. Then the following properties hold:*
(a). $A \subseteq A^{(\wedge,\mu)}$;
(b). $A^{(\wedge,\mu)} = \cap\{F : A \subseteq F and F is (\wedge,\mu) - closed\}$;
(c). $A \subseteq B \Rightarrow A^{(\wedge,\mu)} \subseteq B^{(\wedge,\mu)}$;
(d). A is $(\wedge,\mu)$- closed iff $A = A^{(\wedge,\mu)}$;
(e). $A^{(\wedge,\mu)}$ is $(\wedge,\mu)-$ closed.

Also, throughout the paper , it should be "$A^{(\wedge,\mu)}$" instead "$c_{\lambda_\mu}(A)$"

Definition 1.5. [10] *A subset A of a GTS (X,μ) is said to be $^*\wedge_\mu-$set if $A = {}^*\wedge_\mu(A)$ where $^*\wedge_\mu(A) = \cap\{U \subset X | A \subset U$ and U is a $\lambda_\mu-$open$\}$. Denote the class of $^*\wedge_\mu-$sets by $^*\wedge_\mu(X,\mu)$.*

We have $\wedge_\mu - O(X,\mu) \subset^* \wedge_\mu(X,\mu)$ and $X \in^* \wedge_\mu(X,\mu)$. Since $^*\wedge_\mu$ $(\cup_{i\in I} A_i) = \cup^*_{i\in I} \wedge_\mu(A_i)$ whenever A_i, $i \in I$, are subsets of X, the class $^*\wedge_\mu(X,\mu)$ is a generalized topology. Since $^*\wedge_\mu(\cap_{i\in I} A_i) \subset \cap^*_{i\in I} \wedge_\mu(A_i)$ whenever A_i, $i \in I$, are subsets of X, the class $^*\wedge_\mu(X,\mu)$ is closed to arbitrary intersections.

Definition 1.6. [10] *A subset A of a GTS* (X,μ) *is said to be a* ${}^*\lambda_\mu-$*closed set if* $A = T\cap C$, *where T is a* ${}^*\wedge_\mu-$*set and C is a* $\lambda_\mu-$*closed set. The complement of a* ${}^*\lambda_\mu-$*closed set is called a* ${}^*\lambda_\mu-$*open set.*

Denote the class of ${}^*\lambda_\mu-$closed sets (resp. ${}^*\lambda_\mu-$open sets) by ${}^*\wedge_\mu - C(X,\mu)$ (resp. ${}^*\wedge_\mu - O(X,\mu)$).

The class of ${}^*\lambda_\mu-$closed sets is closed to arbitrary intersections, since ${}^*\wedge_\mu$ (X,μ) and $\wedge_\mu - C(X,\mu)$ have this property. In addition, X is a ${}^*\lambda_\mu-$closed set. Therefore, the class of ${}^*\lambda_\mu-$open sets is a generalized topology.

Lemma 1.7. *For a GTS* (X,μ), $\wedge_\mu(X,\mu) \subset^* \wedge_\mu(X,\mu) \subset^* \wedge_\mu - C(X,\mu)$.

Proof: a) Since every $\mu-$open set is $\lambda_\mu-$open, we have ${}^*\wedge_\mu(A) \subset \wedge_\mu(A)$ for every $A \subset X$. Now assume that A is a $\wedge_\mu-$set, i.e. $\wedge_\mu(A) \subset A$. Then ${}^*\wedge_\mu(A) \subset A$, hence A is a ${}^*\wedge_\mu$-set.
b) Let $B \in^* \wedge_\mu(X,\mu)$. Writing $B = T\cap C$, where $T = B$ is a ${}^*\wedge_\mu-$set and $C = X$ is a $\lambda_\mu-$closed set, we see that $B \in^* \wedge_\mu - C(X,\mu)$. □

Theorem 1.8. [10] *For a GTS* (X,μ), $\lambda_\mu-$*closed sets and* $\lambda_\mu-$*open sets are* ${}^*\lambda_\mu-$*closed and* ${}^*\lambda_\mu-$*open, i.e.*

$$\wedge_\mu - C(X,\mu) \cup \wedge_\mu - O(X,\mu) \subset^* \wedge_\mu - C(X,\mu) \cap^* \wedge_\mu - O(X,\mu).$$

Proof. a) Let $A \in \wedge_\mu - C(X,\mu)$.Writing $A = T\cap C$, where $T = X$ is a ${}^*\wedge_\mu-$set and $C = A$ is a $\lambda_\mu-$closed set, we see that $A \in^* \wedge_\mu - C(X,\mu)$.
b) Since $\wedge_\mu - O(X,\mu) \subset^* \wedge_\mu(X,\mu) \subset^* \wedge_\mu - C(X,\mu)$, all the $\lambda_\mu-$open sets are ${}^*\lambda_\mu-$closed.
c) By a) and b), $\lambda_\mu-$closed sets and $\lambda_\mu-$open sets are ${}^*\lambda_\mu-$closed, hence they are ${}^*\lambda_\mu-$open.

The above disscusion can be summarized in the following diagram:

μ-open ⇒ λ_μ-open
⇓ ⇓
$\wedge_\mu$-set ⇒ ${}^*\wedge_\mu$-set
⇓ ⇓
μ-closed ⇒ λ_μ-closed ⇒ ${}^*\lambda_\mu$-closed

From the above diagram, we have the following observation, which are used in the subsequent chapters.

Observation 1.9. *For a GTS* (X,μ), *the following hold:*
i). Since X is $\mu-$*closed and* Φ *is a* $\wedge_\mu-$*set, X and* Φ *are* $\lambda_\mu-$*closed.*
ii). $\mu-$*closed sets and* $\mu-$*open sets are* $\lambda_\mu-$*closed and* $\lambda_\mu-$*open.*
iii). Every $\lambda_\mu-$*open set is a* ${}^*\wedge_\mu-$ *set.*
iv). $\lambda_\mu-$*closed sets and* $\lambda_\mu-$*open sets are* ${}^*\lambda_\mu-$*closed and* ${}^*\lambda_\mu-$*open.*

Definition 1.10. *A subset A of a GTS* (X,μ) *is called* $g_\mu-$*closed* [11] *(resp.* $g-\lambda_\mu-$*closed* [9], $\lambda_\mu - g-$*closed* [9]*) if* $c_\mu(A) \subset U$ *(resp.*$c_{\lambda_\mu}(A) \subset U$,$c_\mu(A) \subset U$*) whenever* $A \subset U$ *and U is* μ*-open (resp. U is* μ*-open, U is* λ_μ*-open) in* (X,μ).

For $A \subset X$ we denote by $c_{^*\lambda_\mu}(A)$ the intersection of all $^*\lambda_\mu-$closed subsets of X containing A,we have

$$c_{^*\lambda_\mu}(A) \subset c_{\lambda_\mu}(A) \subset c_\mu(A)$$

for every $A \subset X$.

The purpose of this present paper is to define some new class of generalized closed sets called $^*\lambda_\mu - g-$closed, $^*\lambda_\mu - g_\mu-$closed and $g_{^*\lambda_\mu}-$closed sets and also we obtain some basic properties of these closed sets in generalized topological spaces. Moreover the relations between these classes of sets are established.

2. $^*\lambda_\mu - g-$closed sets

Definition 2.1. *Let (X, μ) be a GTS. A subset A of X is called $^*\lambda_\mu - g-$closed set if $c_\mu(A) \subset U$ whenever $A \subset U$ and U is $^*\lambda_\mu-$open in X. The complements of a $^*\lambda_\mu - g-$closed sets are called $^*\lambda_\mu - g-$open sets.*

Theorem 2.2. *Every $\mu-$closed set is a $^*\lambda_\mu - g-$closed set.*

Proof: Let A be μ-closed set and U be any $^*\lambda_\mu-$open set containing A. Since A is μ-closed, $c_\mu(A) = A$. Therefore $c_\mu(A) \subset U$ and hence A is $^*\lambda_\mu - g-$closed.

Example 2.3 shows that the converse of the above theorem is not true.

Example 2.3. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, X\}$. Then $\{a, c\}$ is $^*\lambda_\mu - g-$closed but not$\mu-$closed.*

Theorem 2.4 shows that every $\lambda^*_\mu - g-$closed set is a $g_\mu-$closed set (a $g - \lambda_\mu-$closed set, a $\lambda_\mu - g-$closed set) and Example 2.5 shows that no converse is **true.**

Theorem 2.4. *Let(X, μ) be a GTS. Then the following hold:*
a). Every $^*\lambda_\mu - g-$closed set is a $g_\mu-$closed set.
b). Every $^*\lambda_\mu - g-$closed set is a $g - \lambda_\mu-$closed set.
c). Every $^*\lambda_\mu - g-$closed set is a $\lambda_\mu - g-$closed set.

Proof: a). Let A be a $^*\lambda_\mu - g-$closed set and U be an $\mu-$open set containing A in(X, μ). Since every μ- open set is $^*\lambda_\mu-$open and A is $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset U$. Therefore A is $g_\mu-$closed.
b). Let A be a $^*\lambda_\mu - g-$closed set and U be an $\mu-$open set containing A in(X, μ). From the above part, $c_\mu(A) \subset U$. Since $c_{\lambda_\mu}(A) \subset c_\mu(A)$,$c_{\lambda_\mu}(A) \subset U$ and hence A is $g - \lambda_\mu-$closed.
c). Let A be a $^*\lambda_\mu - g-$closed set and U be an $\lambda_\mu-$open set containing A in(X, μ). Since every $\lambda_\mu-$open set is $^*\lambda_\mu-$open and A is $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset U$. Therefore A is $\lambda_\mu - g-$closed.

Example 2.5. *Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b, c\}\{b, c, d\}, X\}$. If $A = \{a, b, d\}$ then A is both g_μ- closed and $g - \lambda_\mu-$closed but not $^*\lambda_\mu - g-$closed. Further if $B = \{a, c, d\}$ is $\lambda_\mu - g-$closed but not $^*\lambda_\mu - g-$closed.*

Theorem 2.6 gives a characterization of $^*\lambda_\mu - g-$closed sets.

Theorem 2.6. *Let (X,μ) be a GTS. Then a subset A of X is $^*\lambda_\mu - g-$closed set if and only if $F \subset c_\mu(A)\backslash A$ and F is $^*\lambda_\mu-$closed imply that F is empty.*

Proof: Let F be a $^*\lambda_\mu-$closed subset of $c_\mu(A)\backslash A$. Then $A \subset X\backslash F$ and $X\backslash F$ is $^*\lambda_\mu-$open. Since A is $^*\lambda_\mu - g-$closed, we have $c_\mu(A) \subset X\backslash F$. Consequently $F \subset X\backslash c_\mu(A)$. Hence F is empty.
Conversely, Suppose the implication holds and $A \subset U$,where U is $^*\lambda_\mu-$open. If $c_\mu(A) \not\subset U$, then $c_\mu(A) \cap (X - U)$ is a non-empty $^*\lambda_\mu-$closed subset of $c_\mu(A)\backslash A$. Therefore A is $^*\lambda_\mu - g-$closed.

Theorem 2.7. *If A is $^*\lambda_\mu - g-$closed set in a GTS (X,μ), then $c_\mu(A)\backslash A$ contains no non-empty $\lambda_\mu-$closed ($\lambda_\mu-$open / $\mu-$open / $\mu-$closed) subset of X.*

Proof: Suppose $c_\mu(A)\backslash A$ contains on non empty $\lambda_\mu-$closed ($\lambda_\mu-$open /$\mu-$open /$\mu-$closed) subset of X. Since every $\lambda_\mu-$closed ($\lambda_\mu-$open /$\mu-$open /$\mu-$closed) is $^*\lambda_\mu-$closed , we have F is a non-empty $^*\lambda_\mu-$closed set contained in $c_\mu(A)\backslash A$, which is a contradiction to Theorem 2.6. Hence the proof.

Example 2.8 shows that the converse of the above theorem is not true.

Example 2.8. *Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b,c\}\{b,c,d\}, X\}$. If $A = \{a,b,d\}$ then $c_\mu(A)\backslash A = \{c\}$, which is neither contain a nonempty $\lambda_\mu-$closed set nor contain a nonempty $\lambda_\mu-$open set, but A is not a $^*\lambda_\mu - g-$closed set.*
If $B = \{a,c,d\}$ then $c_\mu(B)\backslash B = \{b\}$, which is neither contain a nonempty $\mu-$closed set nor contain a nonempty $\mu-$open set, but B is not a $^*\lambda_\mu - g-$closed set.

Lemma 2.9. [7] *For a GTS (X,μ) and $S,T \subset X$, the following properties hold:*
(a).$i_\mu(s \cap T) \subset i_\mu(S) \cap i_\mu(T)$.
(b).$c_\mu(s) \cup c_\mu(T) \subset c_\mu(S) \cup T)$.

Remark 2.10. [7] *$i_\mu(s \cap T) \supset i_\mu(S) \cap i_\mu(T)$ is not true in general for subset S and T of a GTS (X,μ).*

Theorem 2.11. [5] *Let (X,μ) be a quasi-topological space. Then $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$ for every A and B of X.*

Theorem 2.12. *Let (X,μ) be a quasi-topological space. Then $A \cup B$ is a $^*\lambda_\mu - g-$closed set whenever A and B are $^*\lambda_\mu - g-$closed sets.*

Proof: Let U be a $^*\lambda_\mu-$closed set such that $A \cup B \subset U$. Then $A \subset U$ and $B \subset U$. Since A and B are $^*\lambda_\mu - g-$closed, $c_\mu(A) \subset U$ and $c_\mu(B) \subset U$. Hence $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B) \subset U$ and so that proof follows.

Example 2.13. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a,b\}\{b,c\}, X\}$. Then μ is a GT but not a quasi-topology. If $A = \{a\}$ and $B = \{c\}$, then A and B are $^*\lambda_\mu - g-$closed sets but their union is not a $^*\lambda_\mu - g-$closed set.*

Example 2.14. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a\}\}$. If $A = \{a,b\}$ and $B = \{a,c\}$, then A and B are $^*\lambda_\mu - g-$closed sets but $A \cap B = \{a\}$ is not a $^*\lambda_\mu - g-$closed set.*

Theorem 2.15. *Let (X,μ) be a GTS. If A is ${}^*\lambda_\mu - g-$closed and B is $\mu-$closed, then $A\cap B$ is a ${}^*\lambda_\mu - g-$closed set.*

Proof: Suppose $A\cap B \subset U$ where U is ${}^*\lambda_\mu-$open. Then $A \subset U \cup (X\backslash B)$. Since A is ${}^*\lambda_\mu - g-$ closed, $c_\mu(A) \subset U \cup (X\backslash B)$ and so $c_\mu(A) \cap B \subset U$. Hence $c_\mu(A\cap B) \subset U$, which implies that $A\cap B$ is a ${}^*\lambda_\mu - g-$closed set.

Theorem 2.16. *Let (X,μ) be a GTS. If A is ${}^*\lambda_\mu-$open and ${}^*\lambda_\mu - g-$closed, then A is $\mu-$closed.*

Proof: Since A is ${}^*\lambda_\mu-$open and ${}^*\lambda_\mu - g-$closed, $c_\mu(A) \subset A$ and hence A is $\mu-$closed.

3. ${}^*\lambda_\mu - g_\mu-$ Closed sets

Definition 3.1. *Let (X,μ) be a GTS. A subset A of X is called ${}^*\lambda_\mu - g_\mu-$closed set if $c_{\lambda_\mu}(A) \subset U$ whenever $A \subset U$ and U is ${}^*\lambda_\mu-$open set in X. The complements of a ${}^*\lambda_\mu - g_\mu-$closed sets are called ${}^*\lambda_\mu - g_\mu-$open sets.*

Theorem 3.2. *For a GTS (X,μ), Every $\lambda_\mu-$closed set is ${}^*\lambda_\mu - g_\mu-$closed.*

Proof: Let A be $\lambda_\mu-$closed set and U be any ${}^*\lambda_\mu - g-$open set containing A. Since A is $\lambda_\mu-$closed, $c_{\lambda_\mu}(A) = A$. Therefore $c_{\lambda_\mu}(A) \subset U$and hence A is ${}^*\lambda_\mu - g_\mu-$closed.

Corollary 3.3. *For a GTS (X,μ), the following hold:*
a).Every $\mu-$closed set is ${}^*\lambda_\mu - g_\mu-$closed.
b).Every $\mu-$open set is ${}^*\lambda_\mu - g_\mu-$closed.

Example 3.4 shows that the converse of the above theorem is not true.

Example 3.4. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a\}\}$. If $A = \{c\}$ then A is neither $\mu-$ closed nor $\mu-$open but it is ${}^*\lambda_\mu - g_\mu-$closed. If $B = \{a,b\}$, then B is ${}^*\lambda_\mu - g_\mu-$closed but not ${}^*\lambda_\mu-$closed.*

Theorem 3.5. *Let(X,μ) be a GTS. If $A \subset X$ and A is a ${}^*\lambda_\mu - g_\mu-$closed set, then A is a $g - \lambda_\mu-$closed set.*

Proof: Let U be a $\mu-$open set containing A in(X,μ). Since every μ- open set is ${}^*\lambda_\mu-$open and A is ${}^*\lambda_\mu - g_\mu-$closed, $c_{\lambda_\mu}(A) \subset U$. Therefore A is $g - \lambda_\mu-$closed.
Example3.6 shows that the converse of Theorem 3.5 is not true.

Example 3.6. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a,b\}\{b,c\}, X\}$. If $A = \{a,c\}$ then A is $g - \lambda_\mu-$closed but not ${}^*\lambda_\mu - g_\mu-$closed.*

Theorem 3.7 shows that the relation between ${}^*\lambda_\mu - g-$closed set and ${}^*\lambda_\mu - g_\mu-$closed set.

Theorem 3.7. *In a GTS (X,μ), every ${}^*\lambda_\mu - g-$closed set is ${}^*\lambda_\mu - g_\mu-$closed.*

Proof: Let A be a $^*\lambda_\mu - g-$closed set and U be an $\lambda_\mu-$open set containing A in (X,μ). Then $c_\mu(A) \subset U$. Since $c_{\lambda_\mu}(A) \subset c_\mu(A)$, we have $c_{\lambda_\mu}(A) \subset U$. Therefore A is $^*\lambda_\mu - g_\mu-$closed.

Example 3.8 shows that the converse of the above theorem is not true.

Example 3.8. *Let $X = \{a,b,c\}$ and $\mu = \{\emptyset, \{a,b\}, X\}$. If $A = \{a\}$, then A is $^*\lambda_\mu - g_\mu-$closed set but it is not $^*\lambda_\mu - g-$ closed.*

Remark 3.9. *The concepts of $g_\mu-$closed set and $^*\lambda_\mu - g_\mu-$closed set are independent.*

Example 3.10. *Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. Then $\{b,d\}$ is $g_\mu-$closed but not $^*\lambda_\mu - g_\mu-$closed and $\{c\}$ is $^*\lambda_\mu - g_\mu-$closed but not $g_\mu-$closed.*

Theorem 3.11 gives a characterization of $^*\lambda_\mu - g_\mu-$closed sets.

Theorem 3.11. *Let (X,μ) be a GTS. Then a subset A of X is $^*\lambda_\mu - g_\mu-$ closed set if and only if $F \subset c_{\lambda_\mu}(A)\backslash A$ and F is $^*\lambda_\mu-$closed imply that F is empty.*

Proof: The proof is similar to that of Theorem 2.6 so that it is omitted.

Theorem 3.12. *If A is $^*\lambda_\mu - g_\mu-$ closed set in a GTS (X,μ), then $c_{\lambda_\mu}(A)\backslash A$ contains no non empty $\lambda_\mu-$ closed ($\lambda_\mu-$open / $\mu-$open / $\mu-$closed) subset of X.*

Proof: The proof is similar to that of Theorem 2.7 so that it is omitted.

Example 3.13 shows that the converse of above Theorem is not true.

Example 3.13. *Let $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b,c\}\{b,c,d\}, X\}$. If $A = \{a,d\}$, $c_{\lambda_\mu}(A)\backslash A = \{b,c\}$,which does not contain both a nonempty $\mu-$open set and $\lambda_\mu-$open set but A is not $^*\lambda_\mu - g_\mu-$closed set. If $B = \{c,d\}$,$c_{\lambda_\mu}(B)\backslash B = \{b\}$,which does not contain both a nonempty $\mu-$closed set and $\lambda_\mu-$closed set but B is not a $^*\lambda_\mu - g_\mu-$closed set.*

Theorem 3.14. *Let (X,μ) be a GTS and A and B be subsets of X. If $A \subset B \subset c_{\lambda_\mu}(A)$ and A is $^*\lambda_\mu - g_\mu-$closed set then B is $^*\lambda_\mu - g_\mu-$closed.*

Proof: If F is $^*\lambda_\mu-$closed such that $F \subset c_{\lambda_\mu}(B)\backslash B$, then $F \subset c_{\lambda_\mu}(A)\backslash A$. Since A is $^*\lambda_\mu - g_\mu-$closed, by Theorem 3.10, $F = \emptyset$ and so B is $^*\lambda_\mu - g_\mu-$closed.

Theorem 3.15. *Let A be a $^*\lambda_\mu - g_\mu-$closed set in a quasi- topological space (X,μ). Then the following hold:*
a).If A is a $\pi-$regular set, then $i_\pi(A)$ and $c_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.
b).If A is a $\sigma-$regular set, then $c_\pi(A)$ and $i_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.

Proof: a). Since A is $\pi-$regular set, $c_\sigma(A) = A \cup i_\mu c_\mu(A) = A$ and $i_\pi(A) = A \cap i_\mu c_\mu(A) = A$. Thus $i_\pi(A)$ and $c_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.
b). Since A is a $\sigma-$regular set, $c_\pi(A) = A$ and $i_\sigma(A) = A$. Thus $c_\pi(A)$ and $i_\sigma(A)$ are $^*\lambda_\mu - g_\mu-$closed sets.

Remark 3.16. *The union (resp. intersection) of two* $^*\lambda_\mu - g_\mu-$*closed sets need not be* $^*\lambda_\mu - g_\mu-$*closed set.*

Example 3.17. *Let* $X = \{a, b, c, d, e\}$ *and* $\mu = \{\emptyset, \{a\}, \{a, b, c\}, X\}$. *Then* $\{a\}$ *and* $\{d, e\}$ *are* $^*\lambda_\mu - g_\mu-$*closed sets but their union is not a* $^*\lambda_\mu - g_\mu-$*closed set. Further* $\{a, b, d\}$ *and* $\{a, c, d, e\}$ *are* $^*\lambda_\mu - g_\mu-$*closed sets but their intersection is not a* $^*\lambda_\mu - g_\mu-$*closed set.*

4. $g_{^*\lambda_\mu}-$ Closed sets

Definition 4.1. *Let* (X, μ) *be a GTS. A subset A of X is called* $g_{^*\lambda_\mu}-$*closed set if* $c_{^*\lambda_\mu}(A) \subset U$ *whenever* $A \subset U$ *and U is* $^*\lambda_\mu-$*open set in X. The complements of a* $g_{^*\lambda_\mu}-$*closed sets are called* $g_{^*\lambda_\mu}-$*open sets.*

Theorem 4.2. *For a GTS* (X, μ)*, Every* $^*\lambda_\mu-$*closed set is* $g_{^*\lambda_\mu}-$*closed.*

Proof: Let A be $^*\lambda_\mu-$closed set and U be any $^*\lambda_\mu-$open set containing A.Since A is $^*\lambda_\mu-$closed, $c_{^*\lambda_\mu}(A) = A$. Therefore $c_{^*\lambda_\mu}(A) \subset U$ and hence A is $g_{^*\lambda_\mu}-$closed.

Corollary 4.3. *For a GTS* (X, μ)*, the following hold:*
a).Every $\lambda_\mu-$*closed set is* $g_{^*\lambda_\mu}-$*closed.*
b).Every $\lambda_\mu-$*open set is* $g_{^*\lambda_\mu}-$*closed.*
c).Every $\mu-$*closed set is* $g_{^*\lambda_\mu}-$*closed.*
d).Every $\mu-$ *open set is* $g_{^*\lambda_\mu}-$*closed.*

Example 4.4 shows that the converse of Theorem 4.2 and the converses of a)-d) from corollary 4.3 are not true.

Example 4.4. *Let* $X = \{a, b, c\}$ *and* $\mu = \{\emptyset, \{a\}\}$. *If* $A = \{c\}$ *then A is neither* $\mu-$*closed nor* $\mu-$*open but it is* $g_{^*\lambda_\mu}-$*closed. If* $B = \{a, b\}$*, then B is neither* $\lambda_\mu-$*closed but nor* $\lambda_\mu-$*open but it is* $g_{^*\lambda_\mu}-$ *closed. If* $C = \{a, c\}$ *then C is* $^*\lambda_\mu-$*colsed but is not* $g_{^*\lambda_\mu}-$*closed.*

Remark 4.5. *The concepts of* $g - \lambda_\mu-$*closed set and* $g_{^*\lambda_\mu}-$*closed set are independent.*

Example 4.6. *Let* $X = \{a, b, c, d\}$ *and* $\mu = \{\emptyset, \{a, b, c\}\{b, c, d\}, X\}$. *Then* $\{a, d\}$ *is* $g_{^*\lambda_\mu}-$*closed but* $g - \lambda_\mu-$*closed and* $\{c\}$ *is* $g - \lambda_\mu-$*closed but not* $g_{^*\lambda_\mu}-$*closed.*

Theorem 4.7 shows that the relation between $g_{^*\lambda_\mu}-$closed set and generalized closed sets defined in section 3.

Theorem 4.7. *For a GTS* (X, μ)*, Every* $^*\lambda_\mu - g_\mu-$*closed set is* $g_{^*\lambda_\mu}-$*closed.*

Proof: Let A be a $^*\lambda_\mu - g_\mu-$closed sets and U be an $^*\lambda_\mu-$open set containing A in(X, μ). Then $c_{\lambda_\mu}(A) \subset U$. Since $c_{^*\lambda_\mu}(A) \subset c_{\lambda_\mu}(A)$,we have $c_{^*\lambda_\mu}(A) \subset U$. Therefore A is $g_{^*\lambda_\mu}-$closed.
Example 4.8 shows that the converse of Theorem 4.7 is not true.

Example 4.8. *Let* $X = \{a, b, c, d\}$ *and* $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. *If* $A = \{a, c\}$ *then A is* $g_{^*\lambda_\mu}-$*closed but not* $^*\lambda_\mu - g_\mu-$*closed.*

5. A Unified framework of generalized closed sets in GTS'

Given an ordered pair of topologies (τ_1, τ_2) on X, a subset $A \subset X$ is said to be (i,j)-g-closed if $\tau_j - cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau_i$, where $i, j \in \{1, 2\}$ with $i \neq j$ [6]. This definition could be easily generalized: Given a pair of generalized topologies (μ_1, μ_2) on X, a subset $A \subset X$ is said to be $g(\mu_i, \mu_j)-$closed if $c_{\mu_j}(A) \subset U$ whenever $A \subset U$ and $U \in \mu_i$, where $i, j \in \{1, 2\}$.
If $i = j$, the class of g(μ_i, μ_i)-closed sets is the class of $g_{\mu_i}-$closed sets.

Observation 5.1. *For a space (X, μ_1, μ_2),the following are hold:*
a). If $\mu_1 = \mu$ and μ_2 is the class of $\lambda_\mu-$open sets, then A is $g(\mu_1, \mu_2)-$closed if and only if A is $g - \lambda_\mu-$closed.
b). If μ_1 is the class of $\lambda_\mu-$open sets and $\mu_2 = \mu$, then A is $g(\mu_1, \mu_2)-$closed if and only if A is $\lambda_\mu - g-$closed.
c). If μ_1 is the class of $^*\lambda_\mu-$open sets and $\mu_2 = \mu$, then A is $g(\mu_1, \mu_2)-$closed if and only if A is $^*\lambda_\mu - g-$closed.
d). If μ_1 is the class of $^*\lambda_\mu-$open sets and μ_2 is the class of $\lambda_\mu-$open sets, then A is $g(\mu_1, \mu_2)-$closed if and only if A is $^*\lambda_\mu - g_\mu-$closed.
e). If $\mu_1 = \mu_2$ represents the class of $^*\lambda_\mu-$open sets, then A is $g(\mu_1, \mu_2)-$closed if and only if A is $g_{^*\lambda_\mu}-$closed.

Theorem 5.2. *If $\mu_1 \subset \mu_2$, then:*
a). A is $g(\mu_2, \mu_1)-$closed $\Rightarrow$ A is $g_{\mu_1}-$closed and $g_{\mu_2}-$closed.
b). A is $g_{\mu_1}-$closed $\Rightarrow$ A is $g(\mu_1, \mu_2)-$closed.
c). A is μ_1-closed $\Rightarrow$ A is $g(\mu_2, \mu_1)-$closed.

Proof: Since $\mu_1 \subset \mu_2$, we have $c_{\mu_2}(A) \subset c_{\mu_1}(A)$ for every $A \subset X$.
a) Let A be $g(\mu_2, \mu_1)-$closed. Assume that $A \subset U$ and $U \in \mu_1$. We prove that $c_{\mu_1}(A) \subset U$. Since $\mu_1 \subset \mu_2$, we have $U \in \mu_2$. But A is $g(\mu_2, \mu_1)-$closed, therefore $c_{\mu_1}(A) \subset U$.
Assume that $A \subset V$ and $V \in \mu_2$. We prove that $c_{\mu_2}(A) \subset V$. Since A is $g(\mu_2, \mu_1)-$closed, we have $c_{\mu_1}(A) \subset V$. But $c_{\mu_2}(A) \subset c_{\mu_1}(A)$, hence $c_{\mu_2}(A) \subset V$.
b) Let A be $g_{\mu_1}-$closed. Assume that $A \subset U$ and $U \in \mu_1$. We prove that $c_{\mu_2}(A) \subset U$. By our assumption, we have $c_{\mu_1}(A) \subset U$, but $c_{\mu_2}(A) \subset c_{\mu_1}(A)$, hence the claim follows.
c) Let A be μ_1-closed. If $A \subset W$ and $W \in \mu_2$, then $c_{\mu_1}(A) = A \subset W$.

Theorem 5.3. *Assume that $\mu_1' \subset \mu_1''$ and $\mu_2'' \subset \mu_2'$. If A is $g(\mu_1'', \mu_2'')-$closed, then A is $g(\mu_1', \mu_2')-$closed.*

Proof : Let A be $g(\mu_1'', \mu_2'')-$closed. Consider $U \in \mu_1'$ such that $A \subset U$. We prove that $c_{\mu_2'}(A) \subset U$. Since $U \in \mu_1''$ and A is $g(\mu_1'', \mu_2'')-$closed, we have $c_{\mu_2''}(A) \subset U$. But $\mu_2'' \subset \mu_2'$ implies $c_{\mu_2'}(A) \subset c_{\mu_2''}(A)$, therefore the claim holds true.

Remark 5.4. *Assume that μ_1 and μ_2 are generalized topologiesbon X.*
(i). Let $\mu_1 = \mu$ and μ_2 be the class of $^*\lambda_\mu-$open sets. From remark 5.2 a), we obtain Theorem 2.4 (a) and Theorem 4.7.

(ii). Let $\mu_1 = \mu$ *and* μ_2 *be the class of* $\lambda_\mu-$*open sets. From remark 5.2 b), we recover the fact every* $g_\mu-$*closed set* $g - \lambda_\mu-$*closed. So Theorem 2.4 b), follows from this and Theorem* 2.4.
(iii). Let μ_1' *be the class of* $\lambda_\mu-$*open sets and* μ_1'' *be the class of* ${}^*\lambda_\mu-$*open sets and take* $\mu_2' = \mu_2'' = \mu$. *From Theorem 5.3, we obtain Theorem* 2.4 *c.*
(iv) Let μ_1 *be the class of* ${}^*\lambda_\mu-$*open sets and* μ_2 *be the class of* $\lambda_\mu-$*open sets. Since* $\mu_2 \subset \mu_1$, *by Theorem 5.2 we obtain that every* ${}^*\lambda_\mu - g-$*closed set is* $g_{\lambda_\mu}-$*closed (and* $g_{*\lambda_\mu}-$*closed). By Theorem* 5.2 *c) we get Theorem 3.2 a).*
(v) Let $\mu_1' = \mu$ *and* μ_1'' *be the class of* λ_μ^*-*open sets, and let* $\mu_2' = \mu_2''$ *be the class of* $\lambda_\mu-$*open sets.From Theorem 5.3 we obtain Theorem 3.4.*
(vi) Let $\mu_1' = \mu_1''$ *be the class of* λ_μ^*-*open sets and let* μ_2' *be the class of* $\lambda_\mu-$*open sets, while* $\mu_2'' = \mu$. *From Theorem 5.3 we get Theorem 3.6.*

References

1. . Bishwarbhar Roy and Erdal Ekici, On $(\wedge, \mu)$- closed sets in Generalized topological spaces, Methods of Functional Analysis and Topology.,17(2)(2011), 174-179.
2. . *Á.Császár*, Generalized open sets, Acta Math.Hungar., 75(1-2) (1997), 65-87.
3. . *Á.Császár*, Further remark on the formula for γ-interior, Acta Math.Hungar., 113(4):(2006), 325-332.
4. . *Á.Császár*, Generalized open sets in generalized topologies, Acta Math.Hungar., 2005, 106(53-66).
5. . *Á.Császár*, $\delta -$ *and* $\theta-$ modifications of generalized topologies, Acta Math.Hungar., 120 (2008), 275-279.
6. . E. Ekici and B. Roy , New generalized topologies on generalized topological spaces due to *Császár*, Acta Math.Hungar., 132(1-2) (2011),117-124.
7. . E. Ekici, Generalized submaximal spaces, Acta Math.Hungar., 134(1-2) (2012),132-138.
8. . A. Ghareeb and T. Noiri, $\wedge-$generalized closed sets in bitopological spaces, Journal of Egyptian Mathematical Society., 19 (2011) 142-145.
9. . R. Jamunarani, P. Jeyanthi and M. Velrajan, Generalized $\lambda_\mu-$closed sets in Generalized Topological Spaces, Journal of Advance Studies in Topology, Vol 4, No 3, 2013, 39-46.
10. . P.Jeyanthi and P.Nalayini, ${}^*\wedge_\mu$- sets and ${}^*\vee_\mu$- sets in Generalized Topological Spaces., (Communicated).
11. . S.Maragathavalli,M.Sheiljohn and D.Sivaraj, On g-closed sets in generalized topological spaces., Journal of Advanced Research in Pure Mathematics, Vol 2, 2010, (57-64).

P. Jeyanthi, P. Nalayini
Research Centre
Department of Mathematics
Govindammal Aditanar College for Women
Tiruchendur-628 215, Tamil Nadu, India
E-mail address: jeyajeyanthi@rediffmail.com
E-mail address: nalayini4@gmail.com

and

Marcelina Mocana
Department of Mathematics, informatics and Education Science
V. Alecsadri University of Bacau, Romania
E-mail address: mmocanu@ub.ro