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Travelling wave solutions of nonlinear systems of PDEs by using the functional variable method

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Abstract: In this paper, we will use the functional variable method to construct exact solutions of some nonlinear systems of partial differential equations, including, the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation, the Whitham-Broer-Kaup-Like systems and the Kaup-Boussinesq system. This approach can also be applied to other nonlinear systems of partial differential equations which can be converted to a second-order ordinary differential equation through the travelling wave transformation.

Key Words: Functional variable method; Bogoyavlenskii's breaking soliton equation; Whitham-Broer-Kaup-Like systems; Kaup-Boussinesq system.

Contents

1. Introduction

Finding the exact solutions nonlinear systems of partial differential equations plays an important role in the study of many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics and optical. Thus, it is important to investigate the exact explicit solutions of nonlinear systems of partial differential equations. In recent years, various powerful methods have been presented for finding exact solutions of the nonlinear systems of partial differential equations in mathematical physics, such as modified simple equation method [1], Algebraic method [2], sine-cosine method [3], F-expansion method [4], generalized hyperbolic function [5] and functional variable method [6]. Among these methods, the functional variable method is a powerful mathematical tool to solve nonlinear systems of partial differential equations. This method were first proposed by Zerarka et al [7] to find the exact solutions for a wide class of linear and nonlinear wave equations. The functional variable method was further developed

2000 *Mathematics Subject Classification:* 83C15, 74J35, 93C10 by many authors [8,9]. The advantage of this method is that one treats nonlinear problems by essentially linear methods, based on which it is easy to construct in full the exact solutions such as soliton-like waves, compacton and noncompacton solutions, trigonometric function solutions, pattern soliton solutions, black solitons or kink solutions, and so on. The aim of this paper is to construct exact solutions of the the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation, the Whitham-Broer-Kaup-Like systems and the Kaup-Boussinesq system by using the functional variable method. Also, we presented two useful theorems of the functional variable method for finding traveling wave solutions of nonlinear partial differential equations.

The rest of this paper is organized as follows. In Section 2, brief description of the functional variable method for finding traveling wave solutions of nonlinear system of partial differential equations is given. In Section 3, the method is employed for obtaining the exact solutions of the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation, the Whitham-Broer-Kaup-Like systems and the Kaup-Boussinesq system. Finally, some conclusions are given in Section 4.

2. The functional variable method

Now, we describe the main steps of the functional variable method for finding exact solutions of nonlinear system of partial differential equations.

Consider the following nonlinear system of partial differential equations with independent variables x and t and dependent variables u and v

$$\begin{aligned} &P_1(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, u_{xt}, \ldots) = 0, \\ &P_2(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, u_{xt}, \ldots) = 0. \end{aligned} \tag{2.1}$$

Applying the travelling wave transformations $u(x,t) = U(\xi)$ and $v(x,t) = V(\xi)$ where $\xi = x - wt$, converts Eq.(2.1) into a system of ordinary differential like

$$\begin{aligned} &G_1(U, V, U_\xi, V_\xi, U_{\xi\xi}, V_{\xi\xi}, \ldots) = 0, \\ &G_2(U, V, U_\xi, V_\xi, U_{\xi\xi}, V_{\xi\xi}, \ldots) = 0. \end{aligned} \tag{2.2}$$

Using some mathematical operations, the system (2.2) is converted into a second-order ordinary differential equation as

$$H(U, U_{\xi\xi}) = 0. \tag{2.3}$$

Then we make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U), \tag{2.4}$$

and

$$U_{\xi\xi} = \frac{1}{2}(F^2)', \tag{2.5}$$

where "$'$" stands for $\frac{d}{dU}$.

Substituting (2.5) into Eq.(2.3) and after the mathematical manipulations, we reduce the ordinary differential equation (2.3) in terms of U, F as

$$K(U, F) = 0. \tag{2.6}$$

The key idea of this particular form Eq.(2.6) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq.(2.6) provides the expression of F, and this, together with Eq.(2.4), give appropriate solutions to the original problem.

Theorem 2.1. *Consider the following second-order ordinary differential equation*

$$U_{\xi\xi} = k_1 U - k_2 U^2, \tag{2.7}$$

where k_1 and $k_2 \neq 0$ are constants and U is a functional variable in the form (2.4). Then using (2.5) transformation, the exact solutions of the Eq.(2.7) are obtained as

Type I. *When $k_1 > 0$, the soliton solutions of Eq.(2.7) are*

$$U_1(\xi) = \frac{3k_1}{2k_2}\mathrm{sech}^2(\frac{\sqrt{k_1}}{2}\xi), \tag{2.8}$$

$$U_2(\xi) = -\frac{3k_1}{2k_2}\mathrm{csch}^2(\frac{\sqrt{k_1}}{2}\xi), \tag{2.9}$$

Type II. *When $k_1 < 0$, the periodic wave solutions of Eq.(2.7) are*

$$U_3(\xi) = \frac{3k_1}{2k_2}\sec^2(\frac{\sqrt{-k_1}}{2}\xi), \tag{2.10}$$

$$U_4(\xi) = \frac{3k_1}{2k_2}\csc^2(\frac{\sqrt{-k_1}}{2}\xi). \tag{2.11}$$

Proof: According to Eq.(2.5), we get from (2.7) an expression for the function $F(U)$

$$\frac{1}{2}\left(F^2(U)\right)' = k_1 U - k_2 U^2, \tag{2.12}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(2.12) with respect to U and after the mathematical manipulations, we have

$$F(U) = \pm U\sqrt{k_1 - \frac{2k_2}{3}U}, \tag{2.13}$$

or

$$F(U) = \pm\sqrt{k_1}U\sqrt{1 - \frac{2k_2}{3k_1}U}. \tag{2.14}$$

After changing the variables

$$Z = \frac{2k_2}{3k_1}U, \tag{2.15}$$

or

$$\frac{3k_1}{2k_2}Z = U, \tag{2.16}$$

with differentiation from Eq.(2.16)

$$\frac{3k_1}{2k_2}dZ = dU(\xi). \tag{2.17}$$

Now, using (2.17), (2.4) and (2.14), we have

$$\frac{dZ}{Z\sqrt{1-Z}} = \pm\sqrt{k_1}d\xi, \tag{2.18}$$

with integrating from Eq.(2.18) and with setting the constant of integration as zero

$$\ln\left|\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}}\right| = \pm\sqrt{k_1}\xi. \tag{2.19}$$

In this case we have

$$\left|\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}}\right| = e^{\pm\sqrt{k_1}\xi}. \tag{2.20}$$

If $\theta = \pm\sqrt{k_1}\xi$, two cases will be considered separately.
Case 1. suppose that $k_1 > 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = e^{\theta}, \tag{2.21}$$

thus, according to (2.21), we have

$$Z = \frac{4}{e^{-\theta}+e^{\theta}+2} = \frac{2}{\cosh\theta+1} = \frac{1}{\cosh^2\left(\frac{\theta}{2}\right)+1} = \sec h^2\left(\frac{\theta}{2}\right),$$

so

$$Z = \operatorname{sech}^2(\frac{\sqrt{k_1}}{2}\xi). \tag{2.22}$$

Now, suppose that $k_1 < 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = e^{i\theta}, \tag{2.23}$$

thus, according to (2.23), we have

$$Z = \frac{4}{e^{-i\theta}+e^{i\theta}+2} = \frac{2}{\cos\theta+1} = \frac{1}{\cos^2\left(\frac{\theta}{2}\right)+1} = \sec^2\left(\frac{\theta}{2}\right),$$

hence

$$Z = \sec^2(\frac{\sqrt{-k_1}}{2}\xi). \tag{2.24}$$

Case 2. suppose that $k_1 > 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = -e^{\theta}, \tag{2.25}$$

therefore, according to (2.25), we have

$$Z = -\frac{4}{e^{-\theta}+e^{\theta}+2} = \frac{2}{\cosh\theta - 1} = \frac{1}{\sinh^2\left(\frac{\theta}{2}\right)+1} = -\csc h^2\left(\frac{\theta}{2}\right),$$

so

$$Z = -\mathrm{csch}^2(\frac{\sqrt{k_1}}{2}\xi). \tag{2.26}$$

Now, assume that $k_1 < 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = -e^{i\theta}, \tag{2.27}$$

thus, according to (2.27), we have

$$Z = -\frac{4}{e^{-i\theta}+e^{i\theta}-2} = \frac{2}{1-\cos\theta} = \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} = \csc^2\left(\frac{\theta}{2}\right),$$

so

$$Z = \csc^2(\frac{\sqrt{-k_1}}{2}\xi). \tag{2.28}$$

Here, using the relations (2.16), (2.22), (2.24), (2.26) and (2.28), the solutions of Eq.(2.7) are in the following forms

When $k_1 > 0$, the soliton solutions of Eq.(2.7) are

$$U_1(\xi) = \tfrac{3k_1}{2k_2}\mathrm{sech}^2(\tfrac{\sqrt{k_1}}{2}\xi),$$

$$U_2(\xi) = -\tfrac{3k_1}{2k_2}\mathrm{csch}^2(\tfrac{\sqrt{k_1}}{2}\xi).$$

When $k_1 < 0$, the periodic wave solutions of Eq.(2.7) are

$$U_3(\xi) = \tfrac{3k_1}{2k_2}\sec^2(\tfrac{\sqrt{-k_1}}{2}\xi),$$

$$U_4(\xi) = \tfrac{3k_1}{2k_2}\csc^2(\tfrac{\sqrt{-k_1}}{2}\xi).$$

$\square$

Theorem 2.2. *Consider the following second-order ordinary differential equation*

$$U_{\xi\xi} = k_1 \left[k_2^2 U + 3k_2 U^2 + 2U^3\right], \tag{2.29}$$

where $k_1 \neq 0$, and k_2 are constants and U is a functional variable in the form (2.4). Then using (2.5) transformation, the exact solutions of the Eq.(2.29) are obtained as

Type I. *When $k_1 > 0$, the soliton solutions of Eq.(2.29) are*

$$U_1(\xi) = -\frac{k_2}{2}\left[1 + \coth(\frac{k_2\sqrt{k_1}}{2}\xi)\right], \tag{2.30}$$

$$U_2(\xi) = -\frac{k_2}{2}\left[1 + \tanh(\frac{k_2\sqrt{k_1}}{2}\xi)\right]. \tag{2.31}$$

Type II. *When $k_1 < 0$, the periodic wave solutions of Eq.(2.29) are*

$$U_3(\xi) = -\frac{k_2}{2}\left[1 - i\cot(\frac{k_2\sqrt{-k_1}}{2}\xi)\right], \tag{2.32}$$

$$U_4(\xi) = -\frac{k_2}{2}\left[1 + i\tan(\frac{k_2\sqrt{-k_1}}{2}\xi)\right]. \tag{2.33}$$

Proof: According to Eq.(2.5), we get from (2.29) an expression for the function $F(U)$

$$\frac{1}{2}\left(F^2(U)\right)' = k_1\left[k_2^2 U + 3k_2 U^2 + 2U^3\right], \tag{2.34}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(2.34) with respect to U and after the mathematical manipulations, we have

$$F(U) = \pm\sqrt{k_1}U\sqrt{k_2^2 + 2k_2 U + U^2}, \tag{2.35}$$

or

$$F(U) = \pm\sqrt{k_1}U(U + k_2). \tag{2.36}$$

Now, using (2.4) and (2.36), we have

$$\frac{dU}{U(U + k_2)} = \pm\sqrt{k_1}d\xi, \tag{2.37}$$

with integrating from Eq.(2.37) and with setting the constant of integration as zero

$$\frac{1}{k_2}\ln\left|\frac{U}{U + k_2}\right| = \pm\sqrt{k_1}\xi. \tag{2.38}$$

In this case we have

$$\left|\frac{U}{U + k_2}\right| = e^{\pm k_2\sqrt{k_1}\xi}. \tag{2.39}$$

If $\theta = \pm k_2\sqrt{k_1}\xi$, two cases will be considered separately.
Case 1. suppose that $k_1 > 0$, then

$$\frac{U}{U + k_2} = e^{\theta}, \tag{2.40}$$

thus, according to (2.40), we have

$$U = -\frac{k_2 e^{\theta}}{e^{\theta} - 1} = -\frac{k_2}{2}(\frac{2e^{\theta}}{e^{\theta} - 1}) = -\frac{k_2}{2}(1 + \frac{e^{\theta} + 1}{e^{\theta} - 1}) = -\frac{k_2}{2}\left[1 + \coth(\frac{\theta}{2})\right],$$

so

$$U = -\frac{k_2}{2}\left[1 + \coth(\frac{k_2\sqrt{k_1}}{2}\xi)\right]. \tag{2.41}$$

Now, suppose that $k_1 < 0$, then

$$\frac{U}{U + k_2} = e^{i\theta}, \tag{2.42}$$

thus, according to (2.42), we have

$$U = -\frac{k_2 e^{\theta}}{e^{\theta} - 1} = -\frac{k_2}{2}(\frac{2e^{i\theta}}{e^{i\theta} - 1}) = -\frac{k_2}{2}(1 + \frac{e^{i\theta} + 1}{e^{i\theta} - 1}) = -\frac{k_2}{2}\left[1 - i\cot(\frac{\theta}{2})\right],$$

hence

$$U = -\frac{k_2}{2}\left[1 - i\cot(\frac{k_2\sqrt{-k_1}}{2}\xi)\right]. \tag{2.43}$$

Case 2. suppose that, $k_1 > 0$ then

$$\frac{U}{U + k_2} = -e^{\theta}, \tag{2.44}$$

therefore, according to (2.44), we have

$$U = -\frac{k_2 e^{\theta}}{e^{\theta} + 1} = -\frac{k_2}{2}(\frac{2e^{\theta}}{e^{\theta} + 1}) = -\frac{k_2}{2}(1 + \frac{e^{\theta} - 1}{e^{\theta} + 1}) = -\frac{k_2}{2}\left[1 + \tanh(\frac{\theta}{2})\right],$$

so

$$U = -\frac{k_2}{2}\left[1 + \tanh(\frac{k_2\sqrt{k_1}}{2}\xi)\right]. \tag{2.45}$$

Now, assume that $k_1 < 0$, then

$$\frac{U}{U + k_2} = -e^{i\theta}, \tag{2.46}$$

thus, according to (2.46), we have

$$U = -\frac{k_2 e^{\theta}}{e^{\theta} + 1} = -\frac{k_2}{2}(\frac{2e^{i\theta}}{e^{i\theta} + 1}) = -\frac{k_2}{2}(1 + \frac{e^{i\theta} - 1}{e^{i\theta} + 1}) = -\frac{k_2}{2}\left[1 + i\tan(\frac{\theta}{2})\right],$$

so

$$U = -\frac{k_2}{2}\left[1 + i\tan(\frac{k_2\sqrt{-k_1}}{2}\xi)\right]. \tag{2.47}$$

Here, using the relations (2.41), (2.43), (2.45) and (2.47), the solutions of Eq.(2.29) are in the following forms

When $k_1 > 0$, the soliton solutions of Eq.(2.29) are

$$U_1(\xi) = -\tfrac{k_2}{2}\left[1 + \coth(\tfrac{k_2\sqrt{k_1}}{2}\xi)\right],$$

$$U_2(\xi) = -\tfrac{k_2}{2}\left[1 + \tanh(\tfrac{k_2\sqrt{k_1}}{2}\xi)\right].$$

When $k_1 < 0$, the periodic wave solutions of Eq.(2.29) are

$$U_3(\xi) = -\tfrac{k_2}{2}\left[1 - i\cot(\tfrac{k_2\sqrt{-k_1}}{2}\xi)\right],$$

$$U_4(\xi) = -\tfrac{k_2}{2}\left[1 + i\tan(\tfrac{k_2\sqrt{-k_1}}{2}\xi)\right].$$

□

3. Applications

Here, we will apply the functional variable method to obtain the exact solutions for the following three nonlinear systems.

3.1. The (2+1)-dimensional Bogoyavlenskii's breaking soliton equation

Let us consider the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation which reads

$$u_t + u_{xxy} + 4uu_y + 4u_x\partial^{-1}u_y = 0, \tag{3.1}$$

its equivalent form [10]

$$\begin{cases} u_t + u_{xxy} + 4uv_x + 4u_xv = 0, \\ v_x = u_y, \end{cases} \tag{3.2}$$

which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave long the x axis. The $u = u(x, y, t)$ represents the physical field and $v = v(x, y, t)$ some potential. This equation is typical of the so-called "breaking soliton" equation and was studied by Bogoyavenskii, where overlapping solutions were generated [11]. Now, to look for travelling wave solutions of eq. (3.2), we first make the transformations

$$u(x, y, t) = U(\xi), \quad v(x, y, t) = V(\xi), \quad \xi = x + y - wt. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain ordinary differential equations

$$-wU_\xi + U_{\xi\xi\xi} + 4UU_\xi + 4U_\xi V = 0, \tag{3.4}$$

$$U_\xi = V_\xi. \tag{3.5}$$

By integrating the Eq.(3.5) with respect to ξ, and neglecting the constant of integration, we have

$$V = U. \tag{3.6}$$

Substituting Eq.(3.6) into Eq.(3.4), after integrating with respect to ξ choosing constant of integration to zero, we obtain

$$U_{\xi\xi} + 4U^2 - \omega U = 0, \tag{3.7}$$

or

$$U_{\xi\xi} = -4U^2 + \omega U. \tag{3.8}$$

Then we use the transformation

$$U_\xi = F(U), \tag{3.9}$$

substituting Eq.(2.5) into Eq.(3.8) we obtain

$$\frac{1}{2}\left(F^2(U)\right)' = -4U^2 + \omega U, \tag{3.10}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(3.10) with respect to U and after the mathematical manipulations, we have

$$F(U) = \pm\sqrt{-\frac{8}{3}U^4 + \omega U^2} = \pm\sqrt{\omega}U\sqrt{1 - \frac{8w}{3}U^2}. \tag{3.11}$$

Using the relations (3.9), (2.8), (2.9), (2.10) and (2.11), when $w > 0$, the solutions of Eq.(3.7) are in the following forms

$$U_1(\xi) = \frac{3w}{8}\operatorname{sech}^2(\frac{\sqrt{w}}{2}\xi), \tag{3.12}$$

$$U_2(\xi) = -\frac{3w}{8}\operatorname{csch}^2(\frac{\sqrt{w}}{2}\xi), \tag{3.13}$$

and, when $w < 0$, the solutions of Eq.(3.7) are in the following forms

$$U_3(\xi) = \frac{3w}{8}\sec^2(\frac{\sqrt{-w}}{2}\xi), \tag{3.14}$$

$$U_4(\xi) = \frac{3w}{8}\csc^2(\frac{\sqrt{-w}}{2}\xi). \tag{3.15}$$

For $w > 0$, using the travelling wave transformations (3.3), we obtain the following soliton solutions of the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation

$$u_1(x, y, t) = v_1(x, y, t) = \frac{3w}{8}\operatorname{sech}^2(\frac{\sqrt{w}}{2}(x + y - wt)), \tag{3.16}$$

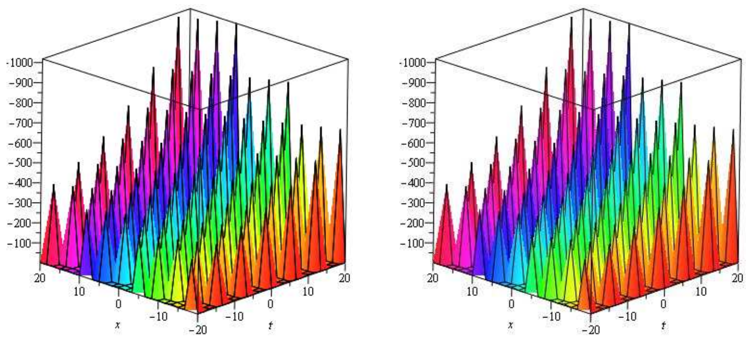

Figure 1: Plot of Eq.(3.2): $u_4(x, y, t)$ on the left and $v_4(x, y, t)$ on the right, where $y = 0.5, w = -2$.

$$u_2(x, y, t) = v_2(x, y, t) = -\frac{3w}{8}\operatorname{csch}^2(\frac{\sqrt{w}}{2}(x + y - wt)). \tag{3.17}$$

For $w < 0$, we obtain the periodic wave solutions

$$u_3(x, y, t) = v_3(x, y, t) = \frac{3w}{8}\sec^2(\frac{\sqrt{-w}}{2}(x + y - wt)), \tag{3.18}$$

$$u_4(x, y, t) = v_4(x, y, t) = \frac{3w}{8}\csc^2(\frac{\sqrt{-w}}{2}(x + y - wt)). \tag{3.19}$$

These solutions are all new exact solutions.

Figure 1 shown that the periodic wave solution $u_4(x, y, t)$ and $v_4(x, y, t)$ of the Eq.(3.2) with $y = 0.5, w = -2$ and x in the interval $[-20, 20]$ and time in the interval $[-20, 20]$.

3.2. The Whitham-Broer-Kaup-Like systems

Let us consider the the Whitham-Broer-Kaup-Like systems [12,13], in the form

$$\begin{cases} u_t + uu_x + \gamma v_x + \beta u_{xx} = 0, \\ v_t + (vu)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{cases} \tag{3.20}$$

where $u = u(x, t)$ is the field of horizontal velocity, $v = v(x, t)$ is the height that deviates from the equilibrium position of the liquid and α, β, γ are constants. It is necessary to point out that when the parameters are taken as different values, the following celebrated nonlinear systems can be derived from Eq.(3.20).

(i) When $\gamma = 1$, we have the Whitham-Broer-Kaup equations [14,15]

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + (vu)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{cases} \tag{3.21}$$

(ii) When $\alpha = 0, \gamma = 1$, we get the approximate equations for long wave equations

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + (vu)_x - \beta v_{xx} = 0, \end{cases} \tag{3.22}$$

(iii) When $\alpha = \gamma = 1, \beta = 0$, we obtain the variant Boussinesq equations [16]

$$\begin{cases} u_t + uu_x + u_{xx} = 0, \\ v_t + (vu)_x + \alpha u_{xxx} = 0, \end{cases} \tag{3.23}$$

(iv) When $\alpha = \frac{1}{3}, \gamma = 1, \beta = 0$, we get the dispersive long wave equations [17]

$$\begin{cases} u_t + uu_x + v_x = 0, \\ v_t + (vu)_x + \frac{1}{3}u_{xxx} = 0. \end{cases} \tag{3.24}$$

It is clear to see that Eq.(3.20) is very important in the field of mathematical physics. Therefore, it is a significant task to search for explicit solutions of the Eq.(3.20). Now, we apply the functional variable method to find the solitary wave solutions for Whitham-Broer-Kaup-Like systems. Firstly, we let

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = x - wt. \tag{3.25}$$

Then (3.20) is converted to ordinary differential equations

$$-wU_\xi + UU_\xi + \gamma V_\xi + \beta U_{\xi\xi} = 0, \tag{3.26}$$

$$-wV_\xi + (VU)_\xi + \alpha U_{\xi\xi\xi} - \beta V_{\xi\xi} = 0. \tag{3.27}$$

By integrating the Eq.(3.26) with respect to ξ, and neglecting the constant of integration, we have

$$-wU + \frac{1}{2}U^2 + \gamma V + \beta U_\xi = 0, \tag{3.28}$$

or

$$V = \frac{1}{\gamma}\left[wU - \frac{1}{2}U^2 - \beta U_\xi\right]. \tag{3.29}$$

Integrating Eq.(3.27) with respect to ξ choosing constant of integration to zero, we obtain

$$-wV + VU + \alpha U_{\xi\xi} - \beta V_\xi = 0. \tag{3.30}$$

Substituting Eq.(3.29) into Eq.(3.30) yields

$$(\alpha\gamma + \beta^2)U_{\xi\xi} - \frac{1}{2}U^3 + \frac{3w}{2}U^2 - w^2U = 0, \tag{3.31}$$

or

$$U_{\xi\xi} = \frac{1}{(\alpha\gamma + \beta^2)}\left[\frac{1}{2}U^3 - \frac{3w}{2}U^2 + w^2U\right]. \tag{3.32}$$

Then we use the transformation

$$U_\xi = F(U), \tag{3.33}$$

and Eq.(2.5) to convert Eq.(3.32) to

$$\frac{1}{2}\left(F^2(U)\right)' = \frac{1}{(\alpha\gamma+\beta^2)}\left[\frac{1}{2}U^3 - \frac{3w}{2}U^2 + w^2U\right], \tag{3.34}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(3.34) with respect to U and after the mathematical manipulations, we have

$$F(U) = \sqrt{\frac{1}{\alpha\gamma+\beta^2}}U\sqrt{w^2 - wU + \frac{1}{4}U^2} = \frac{1}{2}\sqrt{\frac{1}{\alpha\gamma+\beta^2}}U(U-2w). \tag{3.35}$$

Using the relations (3.33), (2.30), (2.31), (2.32) and (2.33), when $\frac{1}{\alpha\gamma+\beta^2} > 0$, the solutions of Eq.(3.32) are in the following forms

$$U_1(\xi) = w\left(1 - \coth(\frac{1}{2}\frac{w}{\sqrt{\alpha\gamma+\beta^2}}\xi)\right), \tag{3.36}$$

$$U_2(\xi) = w\left(1 - \tanh(\frac{1}{2}\frac{w}{\sqrt{\alpha\gamma+\beta^2}}\xi)\right), \tag{3.37}$$

and, when $\frac{1}{\alpha\gamma+\beta^2} < 0$, the solutions of Eq.(3.32) are in the following forms

$$U_1(\xi) = w\left(1 + i\cot(\frac{1}{2}\frac{w}{\sqrt{-\alpha\gamma-\beta^2}}\xi)\right), \tag{3.38}$$

$$U_2(\xi) = w\left(1 - i\tan(\frac{1}{2}\frac{w}{\sqrt{-\alpha\gamma-\beta^2}}\xi)\right). \tag{3.39}$$

For $\frac{1}{\alpha\gamma+\beta^2} > 0$, we can easily obtain following solito solutions

$$u_1(x,t) = w\left(1 - \coth(\frac{1}{2}\frac{w}{\sqrt{\alpha\gamma+\beta^2}}(x-wt))\right), \tag{3.40}$$

$$v_1(x,t) = -\frac{1}{2}\frac{w^2(\sqrt{\alpha\gamma+\beta^2}+\beta)}{\gamma\sqrt{\alpha\gamma+\beta^2}\left(\cosh^2(\frac{1}{2}\frac{w}{\sqrt{\alpha\gamma+\beta^2}}(x-wt)) - 1\right)}, \tag{3.41}$$

$$u_2(x,t) = w\left(1 - \tanh(\frac{1}{2}\frac{w}{\sqrt{\alpha\gamma+\beta^2}}(x-wt))\right), \tag{3.42}$$

$$v_2(x,t) = \frac{1}{2}\frac{w^2(\sqrt{\alpha\gamma+\beta^2}+\beta)}{\gamma\sqrt{\alpha\gamma+\beta^2}\left(\cosh^2(\frac{1}{2}\frac{w}{\sqrt{\alpha\gamma+\beta^2}}(x-wt))\right)}. \tag{3.43}$$

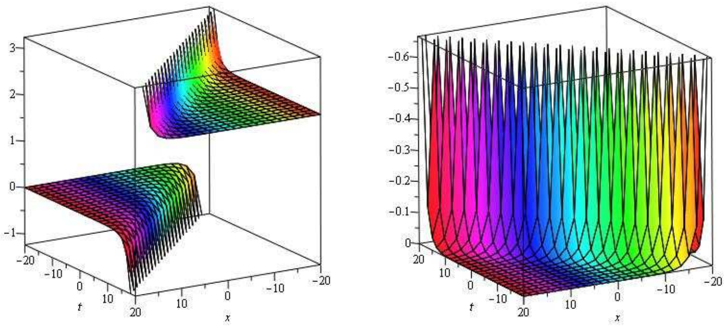

Figure 2: Plot of Eq.(3.2): $u_1(x,t)$ on the left and $v_1(x,t)$ on the right, where $\alpha = 1, \beta = 0, \gamma = 3$ and $w = 1$.

For $\frac{1}{\alpha\gamma+\beta^2} < 0$, we obtain the periodic wave solutions

$$u_3(x,t) = w\left(1 + i\cot(\frac{1}{2}\frac{w}{\sqrt{-\alpha\gamma-\beta^2}}(x-wt))\right), \tag{3.44}$$

$$v_3(x,t) = -\frac{1}{2}\frac{w^2(\beta\sqrt{\alpha\gamma+\beta^2}+\alpha\gamma+\beta^2)}{\gamma(\alpha\gamma+\beta^2)\left(\cos^2(\frac{1}{2}\frac{w}{\sqrt{-\alpha\gamma-\beta^2}}(x-wt))-1\right)}, \tag{3.45}$$

$$u_4(x,t) = w\left(1 - i\tan(\frac{1}{2}\frac{w}{\sqrt{-\alpha\gamma-\beta^2}}(x-wt))\right), \tag{3.46}$$

$$v_4(x,t) = \frac{1}{2}\frac{w^2(\beta\sqrt{\alpha\gamma+\beta^2}+\alpha\gamma+\beta^2)}{\gamma(\alpha\gamma+\beta^2)\left(\cos^2(\frac{1}{2}\frac{w}{\sqrt{-\alpha\gamma-\beta^2}}(x-wt))\right)}. \tag{3.47}$$

Figure 2 shown that the soliton solution $u_1(x,t)$ and $v_1(x,t)$ of the Eq.(3.2) with $\alpha = 1, \beta = 0, \gamma = 3, w = 1$ and x in the interval $[-20, 20]$ and time in the interval $[-20, 20]$.
Note that Eq.(3.40) is same as obtained in [13].

3.3. The Kaup-Boussinesq system

Consider the Kaup-Boussinesq system [18]

$$\begin{cases} u_t - u_{xxx} - 2vu_x - 2uv_x = 0, \\ v_t - u_x - 2vv_x = 0, \end{cases} \tag{3.48}$$

where $= u(x,t)$ denotes the height of the water surface above a horizontal bottom and $v = v(x,t)$ is related to the horizontal velocity field. At this time, by means of the functional variable method, we will find some solitary wave solutions of the Kaup-Boussinesq system. By considering the wave transformations

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = x - wt. \tag{3.49}$$

We change Eq.(3.49) into a system of ordinary differential equations given by

$$-wU_\xi - V_{\xi\xi\xi} - 2VU_\xi - 2UV_\xi = 0, \tag{3.50}$$

$$-wV_\xi - U_\xi - 2VV_\xi = 0. \tag{3.51}$$

Integrating Eq.(3.51) with respect to ξ once, considering the zero constant for the integration, yields

$$U = -wV - V^2, \tag{3.52}$$

substituting Eq.(3.52) into Eq.(3.50) yields

$$w^2V_\xi + 6wVV_\xi + 6V^2V_\xi - V_{\xi\xi\xi} = 0. \tag{3.53}$$

Now, integrating Eq.(3.53) with respect to ξ and choosing constant of integration to zero, we obtain

$$V_{\xi\xi} = w^2V + 3wV^2 + 2V^3. \tag{3.54}$$

Then we use the transformation

$$V_\xi = F(V), \tag{3.55}$$

and Eq.(2.5) to convert Eq.(3.54) to

$$\frac{1}{2}\left(F^2(V)\right)' = w^2V + 3wV^2 + 2V^3, \tag{3.56}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq.(3.56) with respect to V and after the mathematical manipulations, we have

$$F(V) = \pm V\sqrt{w^2 + 2wV + V^2} = \pm V(V + w). \tag{3.57}$$

Since $k_1 = 1$ and $k_2 = w$, then, using the relations (3.55), (2.30) and (2.31), the solutions of Eq.(3.54) are in the following forms

$$V_1(\xi) = \frac{-w}{2}\left(1 + \coth(\frac{w}{2}\,\xi)\right), \tag{3.58}$$

$$V_2(\xi) = \frac{-w}{2}\left(1 + \tanh(\frac{w}{2}\,\xi)\right). \tag{3.59}$$

We can easily obtain following soliton solutions

$$v_1(x,t) = \frac{-w}{2}\left(1 + \coth(\frac{w}{2}\,(x - wt))\right), \tag{3.60}$$

$$u_1(x,t) = \frac{w^2}{2}\left(1 + \coth(\frac{w}{2}\,(x - wt))\right) - \frac{w^2}{4}\left(1 + \coth(\frac{w}{2}\,(x - wt))\right)^2. \tag{3.61}$$

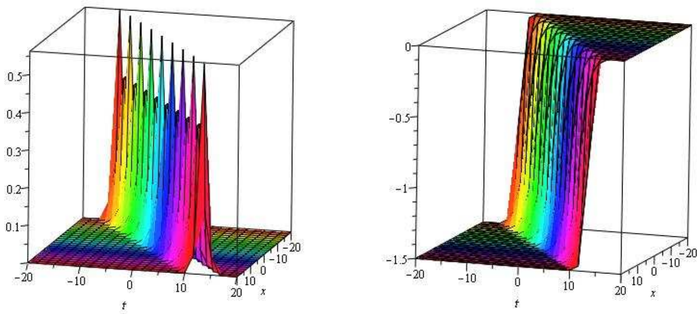

Figure 3: Plot of Eq.(3.48): $u_2(x,t)$ on the left and $v_2(x,t)$ on the right, where $w = 1.5$.

$$v_2(x,t) = \frac{-w}{2}\left(1 + \tanh(\frac{w}{2}\,(x - wt))\right), \tag{3.62}$$

$$u_2(x,t) = \frac{w^2}{2}\left(1 + \tanh(\frac{w}{2}\,(x - wt))\right) - \frac{w^2}{4}\left(1 + \tanh(\frac{w}{2}\,(x - wt))\right)^2. \tag{3.63}$$

Figure 3 shown that the soliton solution $u_2(x,t)$ and $v_2(x,t)$ of the Eq.(3.48) with $w = 1.5$ and x in the interval $[-20, 20]$ and time in the interval $[-20, 20]$. Solutions (3.60), (3.61) and (3.62), (3.63) are new types of exact traveling wave solutions to the Kaup-Boussinesq system.

Remark 3.1. *We have verified all the obtained solutions by putting them back into the original equations* (3.2)*,* (3.20) *and* (3.48) *with the aid of Maple 13.*

4. Conclusion

In this paper, the functional variable method was successfully applied to obtain travelling wave solutions of some important nonlinear systems, including, the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation, the Whitham-Broer-Kaup-Like systems and the Kaup-Boussinesq systems, which were not discussed elsewhere using that method. The travelling wave solutions were expressed by the hyperbolic functions and the trigonometric functions. Also, we conclude that the proposed method is powerful for nonlinear partial differential systems which can be converted to a second-order ordinary differential equations through the travelling wave transformation.

References

1. Taghizadeh, N., Mirzazadeh, M., Paghaleh, A. S. and Vahidi, J., *Exact solutions of nonlinear evolution equations by using the modified simple equation method*, ASEJ. 3 321-325, (2012).

2. Zhang, S. and Xia, T.C., *A further improved extended Fan sub-equation method and its application to the (3+1)-dimensional Kadomstev-Petviashvili equation*, Phys. Lett. A. 356 119-123, (2006).

3. Yusufoglu, E. and Bekir, A., *Solitons and periodic solutions of coupled nonlinear evolution equations by using the sineŰcosine method*, Int. J. Comput. Math. 83 915-924, (2006).

4. Wang, D. and Zhang, H. Q., *Further improved F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equation*, Chaos, Solitons and Fractals. 25 601-610, (2005).

5. Zhang, S. and Xia, T. C., *A generalized new auxiliary equation method and its applications to nonlinear partial differential equations*, Phys. Lett. A. 363 356-360, (2007).

6. Zerarka, A., Ouamane, S. and Attaf, A., *On the functional variable method for finding exact solutions to a class of wave equations*, Appl. Math. Comput. 217 2897-2904, (2010).

7. Zerarka, A. and Ouamane, S., *Application of the functional variable method to a class of nonlinear wave equations*, World J. Model. Simul. 6 150-160, (2010).

8. Zayed,E. M., Hoda Ibrahim, S. A., Simos, T. E., Psihoyios, G., Tsitouras, C. and Anastassi, Z, *The functional variable method and its applications for finding the exact solutions of nonlinear PDEs in mathematical physics*, AIP Conference Proceedings-American Institute of Physics. 1479, 2049-2053, (2012).

9. Aminikhah, H., Refahi Sheikhani, A. and Rezazadeh, H., *Functional Variable Method for Solving the Generalized Reaction Duffing Model and the Perturbed Boussinesq Equation*, Adv. Model. Optim. 17 55-65, (2015).

10. Fan, E.G., Hon, Y.C., *Quasiperiodic waves and asymptotic behavior for Bogoyavlenskii's breaking soliton equation in (2+1)-dimensions*, Phys. Rev. E. 78 036607, (2008).

11. Bogoyavlenskii, O.I., *Breaking solitons in (2+1)-dimensional integrable equations*, Russian Math. Surveys. 45 1-86, (1990).

12. ZHOU, Y.B., Li, C., *Application of modified* (G'/G)*-expansion method to traveling wave solutions for Whitham-Broer-Kaup-Like equations*, Commun. Theoret. Phys. 51 664-670, (2009).

13. Song, M., Gao, J. and Guan, X., *Application of the bifurcation method to the WhithamŰBroerŰKaup-Like equations*, Math. Comput. Model. 55 688-696, (2012).

14. Xie, F.D., Yan, Z.Y. and Zhang, H.Q., *Explicit and exact traveling wave solutions of Whitham-Broer-Kaup shallow water equations*, Phys. Lett. A 285 76-80, (2001).

15. El-Sayed, S.M., Kaya, D., *Exact and numerical traveling wave solutions of Whitham-Broer-Kaup equations*, Appl. Math. Comput. 167 1339-1349, (2005).

16. Yan, Z.Y., Zhang, H.Q., *New explicit and exact travelling wave solutions for a system of variant Boussinesq equations in mathematical physics*, Phys. Lett. A. 252 291-296, (1999).

17. Zheng, X., Chen, Y. and Zhang, H., *Generalized extended tanh-function method and its application to (1+1) dimensional dispersive long wave equation*, Phys. Lett. A. 311 145-157, (2003).

18. Zhou, J., Tian, L., Fan, X., *Solitary-wave solutions to a dual equation of the Kaup-Boussinesq system*, Nonlinear Anal. 11 3229-3235, (2010).

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