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Existence of solutions for a fourth order eigenvalue problem with variable exponent under Neumann boundary conditions

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abstract: In this work we will study the eigenvalues for a fourth order elliptic equation with $p(x)$-growth conditions $\Delta^2_{p(x)}u = \lambda|u|^{p(x)-2}u$, under Neumann boundary conditions, where $p(x)$ is a continuous function defined on the bounded domain with $p(x) > 1$. Through the Ljusternik-Schnireleman theory on C^1-manifold, we prove the existence of infinitely many eigenvalue sequences and $\sup\Lambda = +\infty$, where Λ is the set of all eigenvalues.

Key Words: Fourth order elliptic equation, p(x)-biharmonic operator, variable exponent, Neumann problem

Contents

1. Introduction

We are concerned here with the eigenvalue problem:

$$\begin{cases} \Delta^2_{p(x)}u = \lambda|u|^{p(x)-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial\nu} = \frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator, $\lambda \in \mathbb{R}$, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}}(p(x)) > 1$.

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions is an interesting topic, which arises from nonlinear electrorheological fluids and other phenomena related to image processing, elasticity and the flow in porous media. In this context we refer to ([8], [9], [10], [13], [11], [12]).

This work is motivated by recent results in mathematical modeling of non Newtonian fluids and elastic mechanics, in particular, the electrorheological fluids (Smart fluids). This important class of fluids is characterized by change of viscosity, which is not easy to manipulate and depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in electric mechanics,

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fluid dynamics etc...
In the case where $p(x) \equiv p$ (a constant), many authors have been interested in spectral problems including the p-Biharmonic operator (See [2], [3], [4], [5], [6], [7]), and in ([1]), the authors have studied the problem (1.1), they have showed the existence of solution for the equation $\Delta_p^2 u = \lambda m(x)|u|^{p-2}u$ under Neumann boundary conditions.
In the variable exponent case, the authors in ([13]) investigated the eigenvalues of the p(x)-biharmonic with Navier boundary conditions . In ([14]), they considered the problem

$$\begin{cases} \Delta^2_{p(x)} u = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where p, q are continuous functions on $\overline{\Omega}$. Using the mountain pass lemma and Ekeland variational principle, they prove the existence of a continuous family of eigenvalues.
The main goal of this paper is to show the existence of solutions for the problem (1.1). We first prove the existence of positive eigenvalues of the following perturbed problem

$$\begin{cases} \Delta^2_{p(x)} u + \epsilon |u|^{p(x)-2}u = \lambda |u|^{p(x)-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where ϵ is enough small $(0 < \epsilon < 1)$.
Through the Ljusternik-Schnireleman theory and by considering for each $t > 0$ the manifold

$$\mathcal{M}_t = \{u \in X : \ G(u) = t\},$$

where $G(u) = \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx$, we prove that for each $t > 0$ the problem (1.3) has a infinitely many eigenvalue sequences. And by tending $\epsilon \longrightarrow 0$, we deduce that the problem (1.1) has infinitely many eigenvalue sequences.
Our main results are stated in the following theorems:

Theorem 1.1. *For each $t > 0$ the problem (1.1) has infinitely many eigenpair sequences $\{(\mp u_{n,t}, \mp\lambda_{n,t})\}$ such that $\lambda_{n,t} \longrightarrow \infty$ as $n \longrightarrow \infty$.*

Define $\lambda_* = \inf(\Lambda)$, where $\Lambda = \{\lambda : \lambda \ \ \textit{is an eigenvalue of} \ \ (1.1)\}$.

Theorem 1.2. *If there exist an open subset $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) < (or \ \ >) p(x)$ for all $x \in \partial U$, then $\lambda_* = 0$*

2. Preliminaries

In order to deal with $p(x)-$biharmonic operator problems, we need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ and some properties of $p(x)-$biharmonic operator, which we will use later.
Define the generalized Lebesgue space by:

$$L^{p(x)}(\Omega) = \left\{u : \ \ \Omega \longrightarrow \mathbb{R}, \ \ \text{measurable and} \ \ \int_\Omega |u(x)|^{p(x)}dx < \infty\right\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}) : \quad h(x) > 1, \quad \forall x \in \overline{\Omega}\right\}.$$

Denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm

$$|u|_{p(x)} = \inf\left\{\mu > 0; \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

and the space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach.

Proposition 2.1. *[15] The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$ i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder's type inequality

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)}$$

holds true.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \quad D^\alpha u \in L^{p(x)}(\Omega), \quad |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{i=N} \alpha_i$.

The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer to ([16], [15], [11], [17]).
For all $\epsilon > 0$, we consider in $W^{2,p(x)}(\Omega)$ the norm

$$\|u\|_\epsilon = \inf\left\{\lambda > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\lambda}\right|^{p(x)} + \epsilon \left|\frac{u(x)}{\lambda}\right|^{p(x)}\right) dx \leq 1\right\}$$

Remark 2.2. *The norm $\|.\|_\epsilon$ is equivalent to the norm*

$$|\Delta u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\Omega)},$$

and $(W^{2,p(x)}(\Omega); \|.\|_\epsilon)$ is a Banach, separable and reflexive space.

Proposition 2.3. *[15]. For all $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping

$$\rho : L^{p(x)}(\Omega) \longrightarrow \mathbb{R},$$

defined by

$$\rho(x) = \int_\Omega |u|^{p(x)} dx$$

Proposition 2.4. *If $u_n, u \in L^{p(x)}(\Omega)$ then the following relations hold true*

(i) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$

(ii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$

(iii) $|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho(u_n - u) \to 0.$

Through this paper, we will consider the following space

$$X = \{u \in W^{2,p(x)}(\Omega) : \frac{\partial u}{\partial \nu} = 0\}.$$

which is considered by F.Moradi and all in ([20]). They have proved that X is a nonempty, well defined and closed subspace of $W^{2,p(x)}(\Omega)$. Firstly they have showed the following boundary trace embedding theorem for variable exponent Sobolev spaces.

Theorem 2.5. *[20] Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. If $2p(x) \geq N \geq 2$ for all $x \in \overline{\Omega}$, then for all $q \in C_+(\Omega)$ there is a continuous boundary trace embedding*

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega), \tag{2.1}$$

and

$$W^{2,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega), \tag{2.2}$$

Proof

(2.1) We choose $p, q \in C_+(\overline{\Omega})$ such that for all $x \in \overline{\Omega}$, $2p(x) \geq N$.
There exists the following continuous embedding

$$W^{2,p(x)}(\Omega) \hookrightarrow W^{2,p^-}(\Omega), \tag{2.3}$$

and

$$L^{q^+}(\partial\Omega) \hookrightarrow L^{q(x)}(\partial\Omega). \tag{2.4}$$

By using the classical boundary trace embedding theorem, since $2p^- \geq N$ and $q^+ \geq 1$, there exists the continuous embedding

$$W^{2,p^-}(\Omega) \hookrightarrow L^{q^+}(\partial\Omega). \tag{2.5}$$

And by combining (2.3), (2.4), (2.5) we deduce that $W^{2,p(x)}(\Omega)$ is continuously embedded into $L^{q(x)}(\partial\Omega)$.

(2.2) Since $2p^- > N$ and $p^+ > 1$, we have the continuous embedding (see [22])

$$W^{2,p^-}(\Omega) \hookrightarrow W^{1,p^+}(\partial\Omega). \tag{2.6}$$

Moreover

$$W^{1,p^+}(\partial\Omega) \hookrightarrow W^{1,p(x)}(\Omega). \tag{2.7}$$

Then from (2.3), (2.6) and (2.7) we deduct the result.

Proposition 2.6. *[20] If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set*

$$X = \{u \in W^{2,p(x)}(\Omega) \ \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

is a closed subspace of $W^{2,p(x)}(\Omega)$

Proof
Consider the operator

$$D: \quad W^{2,p(x)}(\Omega) \longrightarrow L^{p(x)}(\partial\Omega)$$

$$u \quad \longmapsto \quad \frac{\partial u}{\partial \nu}|_{\partial\Omega}.$$

We prove that D is continuous from $(W^{2,p(x)}(\Omega), \|.\|_\epsilon)$ to $(L^{p(x)}(\partial\Omega), |.|_{L^{p(x)}(\partial\Omega)})$.
For this, we prove the continuity of the operator

$$\nabla: \quad W^{2,p(x)}(\Omega) \longrightarrow (L^{p(x)}(\partial\Omega))^N$$

$$u \quad \longmapsto \quad (\nabla u)|_{\partial\Omega},$$

from $(W^{2,p(x)}(\Omega), \|.\|_\epsilon)$ to $((L^{p(x)}(\partial\Omega))^N, \|.\|_{p(x),N})$, with

$\|\overrightarrow{n}\|_{p(x),N} = \sum_{i=1}^{i=N} |n_i|_{p(x)}$.

Let $(u_n)_n \subset W^{2,p(x)}(\Omega)$ be a sequence such that $u_n \longrightarrow u$ in $W^{2,p(x)}(\Omega)$. Using the second assertion of theorem 2.5 , we have $u_n \longrightarrow u$ in $W^{1,p(x)}(\partial\Omega)$, which implies that $\nabla u_n \longrightarrow \nabla u$ in $(L^{p(x)}(\partial\Omega))^N$, and then ∇ is continuous. Moreover, $D = T \circ \nabla$ with T is the linear function defined as

$$T: \quad (L^{p(x)}(\partial\Omega))^N \longrightarrow L^{p(x)}(\partial\Omega)$$

$$\overrightarrow{n}(n_1, n_2, ..., n_N) \longmapsto \quad \overrightarrow{n}.\overrightarrow{v},$$

where $\overrightarrow{v}(x) = (\alpha_1(x), \alpha_2(x), ..., \alpha_N(x))$ is the outer unit normal vector and $\sum_{i=1}^{i=N} |\alpha_i(x)|^2 = 1$ for all $x \in \partial\Omega$.

The operator T is continuous, indeed, for $\overrightarrow{n} \in (L^{p(x)}(\partial\Omega))^N$, we have

$$|\overrightarrow{n}.\overrightarrow{v}|_{p(x)} = |\sum_{i=1}^{i=N} n_i\alpha_i| \leq \sum_{i=1}^{i=N} |n_i\alpha_i|_{p(x)}.$$

On the other hand, we have $\sum_{i=1}^{i=N} |\alpha_i(x)|^2 = 1$, then $|\alpha_i(x)| \leq 1$ for all $x \in \partial\Omega$, $i \in \{1, 2, ..., N\}$.

Consequently, we deduct that

$$|\overrightarrow{n}.\overrightarrow{v}|_{L^{p(x)}(\partial\Omega)} \leq \sum_{i=1}^{i=N} |n_i|_{p(x)} = \|\overrightarrow{n}\|_{p(x),N},$$

which assert that T is continuous and then D is also continuous. Finally, since $X = D^{-1}(\{0\})$, it result that X is closed in $W^{2,p(x)}(\Omega)$. Hence, the proof of the proposition is completed.

A pair $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of (1.1) provided that

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx = \lambda \int_\Omega |u|^{p(x)-2}uv dx, \quad \forall v \in X.$$

In the case where u is nontrivial, such a pair (u, λ) is called an eigenpair, λ is an eigenvalue and u is called an associated eigenfunction.

Proposition 2.7. *If $u \in X$ is a weak solution of (1.1) and $u \in C^4(\overline{\Omega})$ then u is a classical solution of (1.1).*

Proof

Let $u \in C^4(\overline{\Omega})$ be a weak solution of problem (1.1) then for every $\varphi \in X$, we have

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi dx = \lambda \int_\Omega |u|^{p(x)-2}u\varphi dx. \tag{2.8}$$

By applying Green formula, we obtain:

$$\int_\Omega \Delta(|\Delta u|^{p(x)-2}\Delta u)\varphi dx = -\int_\Omega \nabla(|\Delta u|^{p(x)-2}\Delta u).\nabla\varphi dx + \int_{\partial\Omega}\varphi\frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u)dx, \tag{2.9}$$

and

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi dx = -\int_\Omega \nabla(|\Delta u|^{p(x)-2}\Delta u).\nabla\varphi dx + \int_{\partial\Omega}|\Delta u|^{p(x)-2}\Delta u\frac{\partial}{\partial\nu}(\varphi)dx, \tag{2.10}$$

then we have

$$\int_\Omega \Delta(|\Delta u|^{p(x)-2}\Delta u)\varphi dx = \lambda\int_\Omega |u|^{p(x)-2}u\varphi dx + \int_{\partial\Omega}\varphi\frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u)dx - \int_{\partial\Omega}|\Delta u|^{p(x)-2}\Delta u\frac{\partial}{\partial\nu}(\varphi)dx.$$

As $\varphi \in X$, then $\frac{\partial}{\partial\nu}(\varphi) = 0$. And for all $\varphi \in \mathcal{D}(\overline{\Omega})$, we obtain

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u \quad a.e\ x \in \Omega.$$

We deduce that for each $\varphi \in X$

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u)\varphi dx = 0,$$

then for all $\varphi \in \mathcal{D}(\overline{\Omega})$, we have

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u)\varphi dx = 0,$$

which implies that

$$\frac{\partial}{\partial\nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 \quad a.e\ x \in \Omega$$

the result follows.

Definition 2.1. Let E be a real Banach space and A be a symmetric subset of $E \setminus \{0\}$ witch is closed in E. We define the genus of A the number:

$$\gamma(A) = inf\{m; \exists\ f \in C^0(A, \mathbb{R}^m \setminus \{0\}); f(-u) = f(u)\}$$

and $\gamma(A) = \infty$ if does not exist such a map f.
$\gamma(\emptyset) = 0$ by definition.

Lemma 2.8. *[18] Suppose that M is a closed symmetric C^1-manifold of a real Banach space E and $0 \notin M$. Suppose also that $f \in C^1(M, \mathbb{R})$ is even and bounded bellow.*
Define

$$c_j = \inf_{K \in \Gamma_j} \sup_{x \in K} f(x),$$

where $\Gamma_j = \{K \subset M : \ K \text{ is symmetric, compact and } \gamma(K) \geq j\}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies $(PS)_c$ for all $c = c_j$, $j = 1, ..., k$, then f has at least k distinct pairs of critical points.

3. Proof of main results

Let us consider a perturbation of problem (1.1) as follows

$$\begin{cases} \Delta^2_{p(x)} u + \epsilon |u|^{p(x)-2} u = \lambda |u|^{p(x)-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2} \Delta u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where ϵ is enough small $(0 < \epsilon < 1)$.
Consider the functional

$$J(u) = \int_\Omega \left(|\Delta u|^{p(x)} + \epsilon |u|^{p(x)} \right) dx, \quad \forall u \in X.$$

Then $\|u\|_\epsilon = \inf\{\mu > 0 : \ J\left(\frac{u}{\mu}\right) \leq 1\}$.
According to the proposition 2.4 we have

Proposition 3.1. *For all $u \in X$, we have*

(i) $\|u\|_\epsilon < 1 \ \ (= 1; > 1) \Longleftrightarrow J(u) < 1 \ \ (= 1; > 1)$,

(ii) $\|u\|_\epsilon \leq 1 \Longrightarrow \|u\|_\epsilon^{p^+} \leq J(u) \leq \|u\|_\epsilon^{p^-}$,

(iii) $\|u\|_\epsilon \geq 1 \Longrightarrow \|u\|_\epsilon^{p^-} \leq J(u) \leq \|u\|_\epsilon^{p^+}$,
for all $u_n \in X$, we have

(iv) $\|u_n\|_\epsilon \longrightarrow 0 \Longleftrightarrow J(u_n) \longrightarrow 0$,

(v) $\|u_n\|_\epsilon \longrightarrow \infty \Longleftrightarrow J(u_n) \longrightarrow \infty$

Proof
Similar to those of theorem 1.3 in ([15]).

Theorem 3.2. *For each $t > 0$, the problem (3.1) has infinitely many eigenpair sequences $\{(\mp u_{n,t,\epsilon}, \lambda_{n,t,\epsilon})\}$ such that $\lambda_{n,t,\epsilon} \longrightarrow \infty$ as $n \longrightarrow \infty$*

Proof
Let us consider the functionals $F_\epsilon, G : \ X \longrightarrow \mathbb{R}$ defined by

$$F_\epsilon(u) = \int_\Omega \frac{1}{p(x)} \left[|\Delta u|^{p(x)} + \epsilon |u|^{p(x)} \right] dx,$$

and

$$G(u)=\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx.$$

It is well known that $F_\epsilon, G\in C^1(X,\mathbb{R})$ and for all $u,v\in X$

$$\langle F'_\epsilon(u);v\rangle=\int_\Omega\left(|\Delta u|^{p(x)-2}\Delta u\Delta v+\epsilon|u|^{p(x)-2}uv\right)dx,$$

and

$$\langle G'(u);v\rangle=\int_\Omega |u|^{p(x)-2}uvdx$$

It is clear that (u,λ) is a weak solution of (3.1) if and only if

$$F'_\epsilon(u)=\lambda G'(u)\ \ in\ \ X'. \tag{3.2}$$

We need the following result

Proposition 3.3. *(1) $F'_\epsilon:\ \ X\longrightarrow X'$ is continuous, bounded and strictly monotone.*

(2) F'_ϵ is of type (S_+).

(3) F'_ϵ is homeomorphism.

Proof

(1) Since F'_ϵ is the Fréchet derivative of F_ϵ, it follows that F_ϵ is continuous and bounded.
Let's define the sets

$$U_p=\{x\in\Omega:\ \ p(x)\geq 2\}\ \text{and}\quad V_p=\{x\in\Omega:\ \ 1<p(x)<2\}.$$

Using the following inequalities

$$\begin{cases} |x-y|^\gamma\leq 2^\gamma(|x|^{\gamma-2}x-|y|^{\gamma-2}y)(x-y) & \text{if } \gamma\geq 2,\\ |x-y|^2\leq \frac{1}{(\gamma-1)}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x-|y|^{\gamma-2}y)(x-y) & \text{if } 1<\gamma<2,\end{cases} \tag{3.3}$$

for all $(x,y)\in(\mathbb{R}^N)^2$, where $x.y$ denotes the usual inner product in $\mathbb{R}^N$, we obtain for all $u,v\in X$ such that $u\neq v$

$$\langle F'_\epsilon(u)-F'_\epsilon(v),u-v\rangle>0,$$

which implies that F'_ϵ is strictly monotone.

(2) We consider $(u_n)_n$ a sequence of X such that

$$u_n\rightharpoonup u\ \ in\ X\ \text{and}\ \limsup_{n\to+\infty}\langle F'_\epsilon(u_n),u_n-u\rangle\leq 0.$$

From the proposition 3.1, it suffices to show that

$$\int_{\Omega}\left(|\Delta u_n-\Delta u|^{p(x)}+\epsilon|u_n-u|^{p(x)}\right)dx\longrightarrow 0. \tag{3.4}$$

by the monotonicity of F'_ϵ, we have

$$\langle F'_\epsilon(u_n)-F'_\epsilon(u),u_n-u\rangle\geq 0,$$

and since $u_n\rightharpoonup u$ in X, we deduce that

$$\limsup_{n\to+\infty}\langle F'_\epsilon(u_n)-F'_{\epsilon(u)},u_n-u\rangle=0.$$

We consider

$$\varphi_n(x)=\left(|\Delta u_n|^{p(x)-2}\Delta u_n-|\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n-\Delta u),$$

$$\xi_n(x)=\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u).$$

By the compact embedding of X into $L^{p(x)}(\Omega)$, it follows that

$$u_n\longrightarrow u\quad\text{in } L^{p(x)}(\Omega)$$

and

$$|u_n|^{p(x)-2}u_n\longrightarrow|u|^{p(x)-2}u\quad\text{in } L^{q(x)}(\Omega),$$

where $\dfrac{1}{q(x)}+\dfrac{1}{p(x)}=1$, for all $x\in\Omega$. It results that

$$\int_{\Omega}\xi_n(x)dx\longrightarrow 0$$

and

$$\limsup_{n\longrightarrow+\infty}\int_{\Omega}\varphi_n(x)dx=0. \tag{3.5}$$

Thus, from (3.3) we have

$$\int_{U_p}|\Delta u_n-\Delta u|^{p(x)}dx\leq 2^{p^+}\int_{U_p}\varphi_n(x)dx,$$

and

$$\int_{U_p}|u_n-u|^p(x)dx\leq 2^{p^+}\int_{U_p}\xi_n(x)dx.$$

Then

$$\int_{U_p}\left(|\Delta u_n-\Delta u|^{p(x)}+\epsilon|u_n-u|^{p(x)}\right)dx\longrightarrow 0\ \ as\ \ n\longrightarrow+\infty. \tag{3.6}$$

On the other hand, in V_p, setting $\delta_n = |\Delta u_n| + |\Delta u|$, we have

$$\int_{V_p} |\Delta u_n - \Delta u|^{p(x)} dx \leq \frac{1}{p^- - 1} \int_{V_p} (\varphi_n)^{\frac{p(x)}{2}} (\delta_n)^{\frac{p(x)}{2}(2-p(x))} dx,$$

and the Young's inequality yields that

$$\begin{aligned} d \int_{V_p} |\Delta u_n - \Delta u|^{p(x)} dx &\leq \frac{1}{p^- - 1} \int_{V_p} \left[d(\varphi_n)^{\frac{p(x)}{2}} \right] (\delta_n)^{\frac{p(x)}{2}(2-p(x))} dx \\ &\leq \frac{1}{p^- - 1} \left(\int_{V_p} \varphi_n (d)^{\frac{2}{p(x)}} dx + \int_{V_p} (\delta_n)^{p(x)} dx \right). \end{aligned} \tag{3.7}$$

From (3.5) and since $\varphi_n \geq 0$, we can consider that $0 \leq \int_{V_p} \varphi_n dx < 1$.

If $\int_{V_p} \varphi_n dx = 0$ then $\int_{V_p} |\Delta u_n - \Delta u|^{p(x)} = 0$.

If not, we take $d = \left(\int_{V_p} \varphi_n(x) dx \right)^{\frac{-1}{2}} > 1$, and the fact that $\frac{2}{p(x)} < 2$, the inequality (3.7) becomes

$$\begin{aligned} (p^- - 1) \int_{V_p} |\Delta u_n - \Delta u|^{p(x)} &\leq \frac{1}{d} \left(\int_{V_p} \varphi_n d^2 dx + \int_{\Omega} \delta_n^{p(x)} dx \right) \\ &\leq \left(\int_{V_p} \varphi_n dx \right)^{\frac{1}{2}} \left(1 + \int_{\Omega} \delta_n^{p(x)} dx \right). \end{aligned}$$

Note that, $\int_{\Omega} \delta_n^{p(x)} dx$ is bounded, which implies that

$$\int_{V_p} |\Delta u_n - \Delta u|^{p(x)} dx \longrightarrow 0 \;\; as \;\; n \longrightarrow +\infty.$$

Similarly, we have

$$\int_{V_p} |u_n - u|^{p(x)} dx \longrightarrow 0 \;\; as \;\; n \longrightarrow +\infty.$$

We conclude that

$$\int_{V_p} \left(|\Delta u_n - \Delta u|^{p(x)} + \epsilon |u_n - u|^{p(x)} \right) dx \longrightarrow 0 \;\; as \;\; n \longrightarrow +\infty. \tag{3.8}$$

Finally, (3.4) is given by combining (3.6) and (3.8).

(3) We prove now that F'_ϵ is an homeomorphism.
First, by the strict monotonicity, F'_ϵ is an injection.
Furthermore, for any $u \in X$ with $\|u\| > 1$, we have

$$\frac{\langle F'_\epsilon(u), u\rangle}{\|u\|} = \frac{J(u)}{\|u\|} \geq \|u\|^{p^- -1} \longrightarrow \infty \ \ as \ \ \|u\| \longrightarrow \infty,$$

i.e. F'_ϵ is coercive. Thus, F'_ϵ is a surjection in view of Minty-Browder theorem (see theorem 26.A(d) in [19]).
Hence, F'_ϵ has an inverse mapping $(F'_\epsilon)^{-1}: \ X' \longrightarrow X$.
Therefore, the continuity of $(F'_\epsilon)^{-1}$ is sufficient to ensure F'_ϵ to be an homeomorphism.
Let $(f_n)_n$ be a sequence of X' such that $f_n \longrightarrow f \ \ in \ \ X'$.
Let u_n and u in X such that

$$(F'_\epsilon)^{-1}(f_n) = u_n \quad and \quad (F'_\epsilon)^{-1}(f) = u,$$

by coercivity of F'_ϵ, one deducts that the sequence (u_n) is bounded in the reflexive space X. For a subsequence (u_n), we have $u_n \rightharpoonup \widehat{u}$ in X, which implies that

$$\lim_{n\longrightarrow+\infty} \langle F'_\epsilon(u_n) - F'_\epsilon(u), u_n - \widehat{u}\rangle = \lim_{n\longrightarrow+\infty} \langle f_n - f, u_n - \widehat{u}\rangle = 0.$$

It follows by the second assertion and the continuity of F'_ϵ that $u_n \longrightarrow \widehat{u} \ \ in \ \ X$ and $F'_\epsilon(u_n) \longrightarrow F'_\epsilon(\widehat{u}) = F'_\epsilon(u) \ \ in \ \ X'$.
Moreover since F'_ϵ is an injection, we conclude that $u = \widehat{u}$

Proposition 3.4. *$G' : X \longrightarrow X'$ is sequentially weakly-strongly continuous, namely,*

$$u_n \longrightarrow u \ \ in \ \ X \ \ implies \ \ G'(u_n) \longrightarrow G'(u) \ \ in \ \ X'.$$

Proof
Let $u_n \rightharpoonup u$ in X. For any $v \in X$, by Hölder's inequality in X and continuous embedding of X in to $L^{p(x)}(\Omega)$, it's follows that

$$\begin{aligned} |\langle G'(u_n) - G'(u), v\rangle| &= \left| \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)v dx \right| \\ &\leq C \left\| |u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u \right\|_{q(x)} \|v\|_{p(x)} \\ &\leq C' \left\| |u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u \right\|_{q(x)} \|v\|. \end{aligned}$$

By using the compact embedding of X in to $L^{p(x)}(\Omega)$, we have $u_n \rightharpoonup u$ in $L^{p(x)}(\Omega)$, thus

$$|u_n|^{p(x)-2}u_n \longrightarrow |u|^{p(x)-2}u \ \ in \ \ L^{q(x)}(\Omega) \ \blacksquare$$

To solve the eigenvalue problem (3.2), we will use the Ljusternik-schnirelmann theory on C^1-manifolds (see [18] corollary 4.1).
For any $t > 0$, denote by

$$\mathcal{M}_t = \{u \in X : \quad G(u) = t\}.$$

We know that for all $x \in \Omega$, we have $p^- \leq p(x) \leq p^+$, then for all $u \in \mathcal{M}_t$

$$\langle G'(u), u \rangle = \int_\Omega |u|^{p(x)} dx \geq tp^- > 0$$

Hence, $\mathcal{M}_t$ is a C^1-manifold of X with codimension one.
Denote by $\mathcal{T}_u(\mathcal{M}_t)$ the tangent space at $u \in \mathcal{M}_t$, i.e. $\mathcal{T}_u(\mathcal{M}_t) = KerG'(u)$, $\widetilde{F}_\epsilon : \mathcal{M}_t \longrightarrow \mathbb{R}$ the restriction of F_ϵ on $\mathcal{M}_t$ and $d\widetilde{F}_\epsilon(u)$ the derivative of $\widetilde{F}_\epsilon$ at $u \in \mathcal{M}_t$, i.e. the restriction of $F'_\epsilon(u)$ on $\mathcal{T}_u(\mathcal{M}_t)$.

Proposition 3.5. *$\widetilde{F}_\epsilon$ satisfies the (PS) condition, namely, any sequence $(u_n) \subset \mathcal{M}_t$, such that $\widetilde{F}_\epsilon(u_n) \longrightarrow c$ and $d\widetilde{F}_\epsilon(u_n) \longrightarrow 0$, contains a converging subsequence.*

Proof
Let $u \in \mathcal{M}_t$, then $G'(u) \neq 0$ and $v = (F'_\epsilon)^{-1}(G'(u)) \neq 0$, thus

$$\begin{aligned}\langle G'(u), F'^{-1}_\epsilon(G'(u))\rangle &= \langle F'_\epsilon(v), v\rangle \\ &= \int_\Omega \left(|\Delta v|^{p(x)} + \epsilon |v|^{p(x)}\right) dx > 0.\end{aligned}$$

Hence, $v \notin \mathcal{T}_u(\mathcal{M}_t)$; therefore

$$X = \mathcal{T}_u(\mathcal{M}_t) \oplus \{\beta v, \quad \beta \in \mathbb{R}\}.$$

We consider $P : \quad X \longrightarrow \mathcal{T}_u(\mathcal{M}_t)$ the natural projection. Then, for every $w \in X$, there exists a unique $\beta \in \mathbb{R}$ such that $w = Pw + \beta v$.
We have $\langle G'(u), Pw\rangle = 0$, then

$$\beta = \frac{\langle G'(u), w\rangle}{\langle G'(u), v\rangle}.$$

Consequently

$$\begin{aligned}\langle d\widetilde{F_\epsilon}(u), w\rangle &= \langle F'_\epsilon(u), Pw\rangle \\ &= \langle F'_\epsilon(u), w\rangle - \left\langle F'_\epsilon(u), \frac{\langle G'(u), w\rangle}{\langle G'(u), v\rangle} v \right\rangle \\ &= \left\langle F'_\epsilon(u) - \frac{\langle F'_\epsilon(u), v\rangle}{\langle G'(u), v\rangle} G'(u), w \right\rangle,\end{aligned}$$

then

$$d\widetilde{F}_\epsilon(u) = F'_\epsilon(u) - \frac{\langle F'_\epsilon(u), v\rangle}{\langle G'(u), v\rangle} G'(u) = F'_\epsilon(u) - \lambda(u) G'(u),$$

where

$$\lambda(u) = \frac{\langle F'_\epsilon(u), v\rangle}{\langle G'(u), v\rangle}.$$

Let $(u_n) \subset \mathcal{M}_t$ be such that $\widetilde{F}_\epsilon(u_n) \longrightarrow c$ and $d\widetilde{F}_\epsilon(u_n) \longrightarrow 0$. As F_ϵ is coercive, $(\|u_n\|)$ is bounded in X. By reflexivity of X, there exist $u_0 \in X$ and a subsequence of (u_n) such that $u_n \rightharpoonup u_0$ in X. Consequently, $u_n \longrightarrow u_0$ in $L^{p(x)}(\Omega)$, $G(u_n) \longrightarrow G(u_0)$ and $G'(u_n) \longrightarrow G'(u_0)$.
Hence $u_0 \in \mathcal{M}_t$.
Putting $v_n = (F'_\epsilon)^{-1})(G'(u_n))$, then $v_n \longrightarrow v_0 \neq 0$ in X, because $G'(u_n) \longrightarrow G'(u_0) \neq 0$ in X'.
Moreover,

$$\left\langle G'(u_n), (F'_\epsilon)^{-1}(G'(u_n))\right\rangle = \langle F'_\epsilon(v_n), v_n\rangle \longrightarrow \int_\Omega \left(|\Delta v_0|^{p(x)} + \frac{\epsilon}{k}|v_0|^{p(x)}\right) dx > 0,$$

and we have

$$\left|\left\langle F'_\epsilon(u_n), (F'_\epsilon)^{-1}(G'(u_n))\right\rangle\right| = |\langle G'(u_n), v_n\rangle| \leq k_1\|u_n\|\|v_n\| < k_2.$$

According to $\lambda(u) = \frac{\langle F'_\epsilon(u), v\rangle}{\langle G'(u), v\rangle}$, we deduce that $(\lambda(u_n))$ is bounded. Taking a subsequence, if necessary, we may assume that $\lambda(u_n) \longrightarrow \lambda_0$.
Then, $u_n \longrightarrow (F'_\epsilon)^{-1}(\lambda_0 G'(u_0))$, because $d\widetilde{F}_\epsilon \longrightarrow 0$. ■
Set $\Sigma_{n,t} = \{K \subset \mathcal{M}_t$: compact, symetric and $\gamma(K) \geq n\}$.
Define

$$c_{n,t,\epsilon} = \inf_{K\in\Sigma_{n,t}} \sup_{u\in K} \widetilde{F}_\epsilon(u), \qquad n = 1, 2, ...,$$

by the Ljusternik-Schnirlemann theory on $C^1 - manifolds$, we know that each $c_{n,t,\epsilon}$ is a critical value of $\widetilde{F}_\epsilon$ and $c_{n,t,\epsilon} \leq c_{n+1,t,\epsilon}$ $(n = 1, 2, ...)$.

Lemma 3.6.

$$\Sigma_{n,t} \neq \emptyset, \quad \forall n.$$

Proof
For given $n \in \mathbb{N}$. Let $x_1 \in \Omega$ and $r_1 > 0$ be small enough such that

$$\overline{B(x_1, r_1)} \subset \Omega \ \text{ and } \ \text{meas}(\overline{B(x_1, r_1)}) < \frac{\text{meas}(\Omega)}{2}.$$

First take $u_1 \in C_0^\infty(\Omega)$ with $\text{supp}(u_1) = \overline{B(x_1, r_1)}$.
Put $B_1 = \Omega \setminus \overline{B(x_1, r_1)}$, then $\text{meas}(B_1) > \frac{\text{meas}(B_1)}{2}$.
Next, let $u_2 \in C_0^\infty(\Omega)$, with $\text{supp}(u_2) = \overline{B(x_2, r_2)}$. After a finite number of steps, we get $u_1, u_2, ..., u_n \in C_0^\infty(\Omega)$ such that $\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset$ if $i \neq j$ and $\text{meas}(\text{supp}(u_i)) > 0$, for $i, j \in \{1, 2, ..., n\}$.
Let $E_n = \text{span}\{u_1, u_2, ..., u_n\}$ be the vector subspace of $C_0^\infty(\Omega)$ spanned by $\{u_1, u_2,$

..., $u_n\}$. Then, $dim(E_n) = n$.
Note that the map

$$w \longrightarrow |w| = \left\{\beta > 0 : \int_\Omega \left|\frac{w(x)}{\beta}\right|^{p(x)} dx \leq 1\right\}$$

define a norm in E_n.
Putting $S_n = \{v \in E_n : |v| = 1\}$ the unit sphere of E_n.
Let us introduce the functional $f : \mathbb{R}^+ \times E_n \longrightarrow \mathbb{R}$ by $f(s, u) = G(su)$.
It is clear that $f(0, u) = 0$ and $f(s, u)$ is nondecreasing with respect to s. More, for $s > 1$ we have

$$f(s, u) \geq s^{p^-} G(u),$$

and so $\lim_{s \longrightarrow +\infty} f(s, u) = +\infty$. Therefore, for every $u \in S_n$ fixed, there is a unique value $s = s(u) > 0$ such that $f(s(u), u) = 1$.
On the other hand, since

$$\frac{\partial f}{\partial s}(s(u), u) = \int_\Omega (s(u))^{p(x)-1} |u(x)|^{p(x)} dx \geq \frac{p^-}{s(u)} f(s(u), u) = \frac{p^-}{s(u)} > 0.$$

The implicit theorem implies that the map $u \longrightarrow s(u)$ is continuous and even by uniqueness.
Now we take the compact $K_n = \mathcal{M}_t \cap E_n$. Since the map $h : S_n \longrightarrow K_n$ defined by $h(u) = s(u).u$ is continuous and odd, it follows by the property of genus that $\gamma(K_n) \geq n$. This completes the proof. ■

Lemma 3.7.

$$\lim_{n \longrightarrow +\infty} c_{n,t,\epsilon} = +\infty.$$

Proof
X is a reflexive and separable space, there are $\{e_i\} \subset X$ and $\{f_i\} \subset X'$ such that $\langle f_i, e_i\rangle = \delta_{i,j}$ (Kronecker symbol).
We have $X = \overline{span}\{e_i : i \in \mathbb{N}^*\}$ and $X' = \overline{span}^{w^*}\{f_i : i \in \mathbb{N}^*\}$.
For $n = 1, 2, ...,$ denote by

$$X_n = span\{e_n\}, \quad Y_n = \bigoplus_{i=1}^{n} X_i \text{ and } Z_n = \overline{\bigoplus_{i=n}^{\infty} X_i},$$

Using the following

Proposition 3.8. *[21] Assume that $\varphi : X \longrightarrow \mathbb{R}$ is weakly-strongly continuous and $\varphi(0) = 0$, $r > 0$ is a given positive number. Then*

$$\lim_{n \longrightarrow +\infty} \sup_{u \in Z_n, \|u\| \leq r} |\varphi(u)| = 0,$$

we deduce that

$$\lim_{n \longrightarrow +\infty} \inf_{u \in \mathcal{M}_t \cap Z_n} \|u\| = +\infty. \tag{3.9}$$

Indeed, by contradiction, assume that there exist $c_0 > 0$ and $\{u_n\} \subset \mathcal{M}_t \cap Z_n$ such that $\|u_n\| \leq c_0$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\longrightarrow+\infty} \sup_{u\in Z_n, \|u\|\leq c_0} |G(u)| \geq \lim_{n\longrightarrow+\infty} |G(u_n)| = 1.$$

That is a contradiction with the proposition 3.8.
From (3.9), for each $c > 1$; there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $u \in \mathcal{M}_t \cap Z_n$, $\|u\| > c$.
On the other hand, for any $K \subset \mathcal{M}_t$, compact and symmetric, we have $\gamma(K \cap Y_{n-1}) \leq n-1$.
As $cod(Z_n) \leq n-1$ and by the property of genus, for $K \subset \Sigma_{n,t}$, we have $K \cap Z_n \neq \emptyset$.
Then,

$$c_{n,t,\epsilon} \geq \inf_{K\in\Sigma_{n,t}} \sup_{u\in K\cap Z_n} \widetilde{F}_\epsilon(u) \geq \inf_{K\in\Sigma_{n,t}} \sup_{u\in K\cap Z_n} \frac{\|u\|^{p^-}}{p^+} \geq \frac{c^{p^-}}{p^+}.$$

This achieves the proof. ■
Applying proposition 3.5, lemma 3.6 and Ljusternik-schnireleman theory to the problem 3.1, we have for each $n \in \mathbb{N}$, $c_{n,t,\epsilon}$ is a critical value of $\widetilde{F}_\epsilon$ on submanifold $\mathcal{M}_t$, such that

$$0 < c_{n,t,\epsilon} \leq c_{n+1,t,\epsilon}, \quad c_{n,t,\epsilon} \longrightarrow +\infty \text{ as } n \longrightarrow +\infty.$$

Moreover, the problem (3.1) has many eigenpair sequences $\{(u_{n,t,\epsilon}, \lambda_{n,t,\epsilon})\}$ such that

$$G(\pm u_{n,t,\epsilon}) = t, \quad F_\epsilon(\pm u_{n,t,\epsilon}) = c_{n,t,\epsilon} \text{ and } \lambda_{n,t,\epsilon} = \frac{\langle F'_\epsilon(\pm u_{n,t,\epsilon}), \pm u_{n,t,\epsilon}\rangle}{\langle G'(\pm u_{n,t,\epsilon}), \pm u_{n,t,\epsilon}\rangle}.$$

Note that

$$\frac{p^+}{tp^-} c_{n,t,\epsilon} \geq \lambda_{n,t,\epsilon} \geq \frac{p^-}{tp^+} c_{n,t,\epsilon} \longrightarrow +\infty \quad as \quad n \longrightarrow +\infty. \tag{3.10}$$

Furthermore, according to (3.10), we conclude that $\lambda_{n,t,\epsilon} \longrightarrow +\infty$ as $n \longrightarrow +\infty$.
We consider the functional $F : X \longrightarrow \mathbb{R}$ defined by

$$F(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx.$$

We denote by $\widetilde{F}$ the restriction of F on $\mathcal{M}_t$, and define

$$c_{n,t} = \inf_{K\in\Sigma_{n,t}} \sup_{u\in K} \widetilde{F}(u).$$

Proposition 3.9.

$$c_{n,t,\epsilon} \longrightarrow c_{n,t}, \quad as \ \ \epsilon \longrightarrow 0.$$

Proof
Set $\epsilon = \frac{1}{k}$, $k \in \mathbb{N}^*$ and let $\alpha > 0$ such that $c_{n,t} < \alpha$. From the definition of $c_{n,t}$, there exists $K = K(\alpha) \in \Sigma_{n,t}$ such that $c_{n,t} \leq \sup_{u \in K} \widetilde{F}(u) < \alpha$.
On the other hand,

$$c_{n,t} \leq c_{n,t,\epsilon} \leq \sup_{u \in K} \widetilde{F}_\epsilon(u) \leq \sup_{u \in K} \ \widetilde{F}(u) + \epsilon \frac{p^+}{p^-} t.$$

Let $\epsilon \longrightarrow 0$, then there exists $N_\alpha > 0$ such that for all $k \geq N_\alpha$

$$\sup_{u \in K} \widetilde{F}(u) + \epsilon \frac{p^+}{p^-} t < \alpha.$$

Thus for all $\alpha > 0$, there exists $N_\alpha > 0$ such that for all $k \geq N_\alpha : c_{n,t} \leq c_{n,t,\epsilon} < \alpha$.
Hence the proof is complete.
Proof of theorem (1.1)
We prove now that $\lambda_{n,t,\epsilon} \longrightarrow \lambda_{n,t}$, where $\lambda_{n,t}$ is an eigenvalue associated with an eigenfunction $u_{n,t}$ of the problem (1.1).
Set $\epsilon = \frac{1}{k}$, $k \in \mathbb{N}^*$ and we suppose that, there exits a sequence $(u_{n,t,k})_{k \in \mathbb{N}^*}$ of solutions associated with $(\lambda_{n,t,k})_{k \in \mathbb{N}^*}$ such that $\|u_{n,t,k}\|_\epsilon = 1$.
We have $(u_{n,t,k})_{k \in \mathbb{N}}$ is bounded. For a subsequence, still denoted $(u_{n,t,k})_{k \in \mathbb{N}}$, we have $u_{n,t,k} \rightharpoonup u_{n,t}$ in X and $u_{n,t,k} \longrightarrow u_{n,t}$ in $L^{p(x)}(\Omega)$.
As the operator $A_\epsilon : \ \ X \longrightarrow X'$;

$$\langle A_\epsilon(u), v \rangle = \langle F'_\epsilon(u), v \rangle,$$

is an homeomorphism of (S_+) type and the operator $B : \ \ X \longrightarrow X'$;

$$\langle B(u), v \rangle = \langle G'(u), v \rangle,$$

is completely continuous, then we have $u_{n,t,k} \longrightarrow u_{n,t}$ in X.
If we set

$$\lambda_{n,t} = \frac{\langle F'(u_{n,t}), u_{n,t} \rangle}{\langle G'(u_{n,t}), u_{n,t} \rangle},$$

we have $\lambda_{n,t,k} \longrightarrow \lambda_{n,t}$ and $\lambda_{n,t}$ is an eigenvalue of (1.1) associated with $u_{n,t}$.
As $G(u_{n,t,\epsilon}) = t$, then we have $G(u_{n,t}) = t$ and $F(u_{n,t}) = c_{n,t}$.
The assertion $\lambda_{n,t} \longrightarrow +\infty$ can be proved in the same way as for $\lambda_{n,t,\epsilon}$.■
For all $u \in X \setminus \{0\}$, we consider the following Rayleigh quotients

$$\gamma_1(u) = \frac{\int_\Omega |\Delta u(x)|^{p(x)} dx}{\int_\Omega |u(x)|^{p(x)} dx},$$

and

$$\gamma_2(u) = \frac{\int_\Omega \frac{1}{p(x)} |\Delta u(x)|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} dx}.$$

Lemma 3.10. *The following assertions are mutually equivalent*

(1) $\lambda_* = 0$

(2) $\inf\{\gamma_1(u): \; u \in X \setminus \{0\}\} = 0$

(3) $\inf\{\gamma_2(u): \; u \in X \setminus \{0\}\} = 0$

(4) $\inf\{\lambda_{1,t}: \; t > 0\} = 0$

Proof
We have for all $x \in \Omega$

$$\frac{1}{p^+} \leq \frac{1}{p(x)} \leq \frac{1}{p^-},$$

then for all $u \in X \setminus \{0\}$

$$\frac{p^-}{p^+}\gamma_1(u) \leq \gamma_2(u) \leq \frac{p^+}{p^-}\gamma_1(u). \tag{3.11}$$

The assertions (2) and (3) are equivalent, and (4) $\Longrightarrow$ (1).
Let $\lambda_* = 0$, then there exist $(\lambda_n) \subset \Lambda$ such that $\lambda_n > 0$ and $\lambda_n \longrightarrow 0$ as $n \longrightarrow +\infty$. If u_n be the eigenfunction associated with λ_n, then $\gamma_1(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$ and (2) is hold, which implies that (1) $\Longrightarrow$ (2).
Let now (3) hold, then there exists $(v_n) \subset X$ such that $\gamma_2(v_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. If we consider $t_n = G(v_n)$ and $c_{1,t_n} = F(u_{1,t_n})$, we have

$$\lambda_{1,t_n} = \frac{\langle F'(u_{1,t_n}), u_{1,t_n}\rangle}{\langle G'(u_{1,t_n}), u_{1,t_n}\rangle}.$$

Thus

$$0 \leq \frac{F(u_{1,t_n})}{G(u_{1,t_n})} = \frac{c_{1,t_n}}{t_n} = \inf_{v \in \mathcal{M}_{t_n}} \frac{F(v)}{G(v)} = \inf_{v \in \mathcal{M}_{t_n}} \gamma_2(v) \leq \gamma_2(v_n).$$

Hence, when $n \longrightarrow +\infty$

$$\frac{c_{1,t_n}}{t_n} \longrightarrow 0 \;\; \textit{and} \;\; \gamma_2(u_{1,t_n}) \longrightarrow 0.$$

We conclude by (3.11) that $\gamma_1(u_{1,t_n}) = \lambda_{1,t_n} \longrightarrow 0$ as $n \longrightarrow +\infty$.
Then (3) $\Longrightarrow$ (4). ■

Proof of theorem 1.2
We prove that, if there exists an open set $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) < p(x)$, for all $x \in \partial U$ then, $\lambda_* = 0$.
Denote for $A \subset \overline{\Omega}$ and $\delta > 0$.

$$B(A, \delta) = \{x \in \mathbb{R}^N : \;\; dist(x, A) < \delta\}.$$

Assume that $\overline{U} \subset \Omega$. Then we have

$$\exists \epsilon_0 > 0 : \;\; p(x_0) < p(x) - 4\epsilon_0, \;\; \forall x \in \partial U,$$

$$\exists \epsilon_1 > 0 : \quad p(x_0) < p(x) - 2\epsilon_0, \quad \forall x \in B(\partial U, \epsilon_1), \tag{3.12}$$
$$\exists \epsilon_2 > 0 : \quad B(x_0, \epsilon_2) \subset U \setminus B(\partial U, \epsilon_1),$$

and

$$|p(x_0) - p(x)| < \epsilon_0, \quad \forall x \in B(x_0, \epsilon_2). \tag{3.13}$$

Take $u_0 \in C_0^\infty(\Omega)$ such that $0 \leq u_0 \leq 1$ and

$$u_0(x) = \begin{cases} 1 & \text{if } x \in U \setminus B(\partial U, \epsilon_1), \\ 0 & \text{if } x \notin U \cup B(\partial U, \epsilon_1), \end{cases} \tag{3.14}$$

Then for sufficiently small $t > 0$, we have

$$\gamma_1(tu_0) = \frac{\int_\Omega |\Delta(tu_0(x))|^{p(x)} dx}{\int_\Omega |tu_0(x)|^{p(x)} dx} \leq \frac{\int_{B(\partial U, \epsilon_1)} |\Delta(tu_0(x))|^{p(x)} dx}{\int_{B(x_0, \epsilon_2)} |tu_0(x)|^{p(x)} dx} \leq \frac{C_1}{C_2} t^{p(x_1) - p(x_2)},$$

where $C_1 = \int_{B(\partial U, \epsilon_1)} |\Delta(u_0(x))|^{p(x)} dx$ and $C_2 = \int_{B(x_0, \epsilon_2)} |u_0(x)|^{p(x)} dx$ are positive constants independent of t, $x_1 \in \overline{B(\partial U, \epsilon_1)}$ and $x_2 \in \overline{B(x_0, \epsilon_2)}$.
By using (3.12) and (3.13), we have $p(x_1) - p(x_2) > \epsilon_0$.
So $\gamma_1(tu_0) \leq \frac{C_1}{C_2} t^{\epsilon_0}$, for all $t \in]0, 1[$.
When $t \to 0$, we obtain $\inf\{\gamma_1(u) : \ u \in X \setminus \{0\}\} = 0$. This achieves the proof.
The proof of the case $p(x_0) > p(x)$ is similar.

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