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The Generalized Non-absolute type of sequence spaces

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ABSTRACT: In this paper we introduce the notion of $\lambda_{mn} - \chi^2$ and Λ^2 sequences. Further, we introduce the spaces $\left[\chi^{2q\lambda}_{f\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and $\left[\Lambda^{2q\lambda}_{f\mu}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$, which are of non-absolute type and we prove that these spaces are linearly isomorphic to the spaces χ^2 and Λ^2, respectively. Moreover, we establish some inclusion relations between these spaces.

Key Words: analytic sequence, double sequences, χ^2 space, difference sequence space,Musielak - modulus function, $p-$ metric space, Ideal; ideal convergent; fuzzy number; multiplier space; non-absolute type.

Contents

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1 \, for\, some \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

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$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11] and Tripathy have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [12] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [14] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [17] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [18]-[19], and [20] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence

$x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:
(i) For all $u, y \ge 0$,

$$uy \le M(u) + \Phi(y), (Young's\, inequality)[See\, [21]] \tag{1.2}$$

(ii) For all $u \ge 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \ge 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \le \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - (f_{mn})(u) : u \ge 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty \right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn} \left\{ inf \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [1]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized B^μ difference operator is equivalent to the following binomial representation:

$$B^\mu x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}.$$

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,
(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ $for 1 \leq p < \infty$; (or)
(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup \{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the

n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn},0))|\right) =$$
$$sup\left(\begin{vmatrix} d_{11}(x_{11,},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1},0) & d_{n2},0(x_{n2},0) & \ldots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

2. Notion of $\lambda_{mn}-$ double chi and double analytic sequences

The generalized de la Vallee-Pussin means is defined by :

$$t_{rs}(x) = \tfrac{1}{\varphi_{rs}} \textstyle\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} x_{mn},$$

where $I_{rs} = [rs - \lambda_{rs} + 1, rs]$. For the set of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallee-Poussin method.

The notion of $\lambda-$ double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \cdots \, and \, \lambda_{mn} \to \infty \, as \, m, n \to \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of x, if $B^{\mu}_{\eta}(x) \to 0 \, as \, m, n \to \infty$, where

$$\begin{aligned} B^{\mu}_{\eta}(x) &= \frac{1}{\varphi_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} (\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}) \\ & \quad ((m+n)! \, |\Delta^m x_{mn}|)^{1/m+n}, \end{aligned}$$

where $((m+n)! \, |\Delta^m x_{mn}|)^{1/m+n} = (m+n)!^{1/m+n}$
$\left(\Delta^{m-1}\lambda_{m,n}x_{mn} - \Delta^{m-1}\lambda_{m,n+1}x_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n}x_{m+1,n}\right.$
$\left. + \Delta^{m-1}\lambda_{m+1,n+1}x_{m+1,n+1}\right)^{1/m+n}$.
In particular, we say that x is a $\lambda_{mn}-$ double gai sequence if $B^{\mu}_{\eta}(x) \to 0$ as $m, n \to \infty$. Further we say that x is $\lambda_{mn}-$ double analytic sequence if $sup_{mn} \left|B^{\mu}_{\eta}(x)\right| < \infty$. We have

$lim_{m,n\to\infty}\left|B^{\mu}_{\eta}(x)-a\right|=lim_{m,n\to\infty}\left|\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}(\lambda_{m,n}-\lambda_{m,n+1}-\lambda_{m+1,n}+\lambda_{m+1,n+1})\left((m+n)!\left|\Delta^m x_{mn}\right|\right)^{1/m+n}\right|=0$. So we can say that $lim_{m,n\to\infty}\left|B^{\mu}_{\eta}(x)\right|=a$. Hence x is $\lambda_{mn}-$ convergent to a.

Lemma 2.1. *Every convergent sequence is $\lambda_{mn}-$ convergent to the same ordinary limit.*

Lemma 2.2. *If a $\lambda_{mn}-$ Musielak convergent sequence converges in the ordinary sense, then it must Musielak converge to the same $\lambda_{mn}-$ limit.*

Proof: Let $x=(x_{mn})\in w^2$ and $m,n\geq 1$. We have
$\left((m+n)!\left|\Delta^m x_{mn}\right|\right)^{1/m+n}-B^{\mu}_{\eta}(x)=\left((m+n)!\left|\Delta^m x_{mn}\right|\right)^{1/m+n}-\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left(\lambda_{m,n}-\lambda_{m,n+1}-\lambda_{m+1,n}+\lambda_{m+1,n+1}\right)$
$\left((m+n)!\left|\Delta^m x_{mn}\right|\right)^{1/m+n}=\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left(\lambda_{m,n}-\lambda_{m,n+1}-\lambda_{m+1,n}+\lambda_{m+1,n+1}\right)\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left((m+n)!\left(\Delta^{m-1}x_{mn}-\Delta^{m-1}x_{m,n+1}-\Delta^{m-1}x_{m+1,n}+\Delta^{m-1}x_{m+1,n+1}\right)\right)^{1/m+n}.$
$=\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left((m+n)!\left(\Delta^{m-1}x_{mn}-\Delta^{m-1}x_{m,n+1}-\Delta^{m-1}x_{m+1,n}+\Delta^{m-1}x_{m+1,n+1}\right)\right)^{1/m+n}\left(\lambda_{m,n}-\lambda_{m,n+1}-\lambda_{m+1,n}+\lambda_{m+1,n+1}\right).$
Therefore we have for every $x=(x_{mn})\in w^2$ that $\left((m+n)!\left|\Delta^m x_{mn}\right|\right)^{1/m+n}-B^{\mu}_{\eta}(x)=S_{mn}(x)\,(n,m\in\mathbb{N})$. where the sequence $S(x)=(S_{mn}(x))^{\infty}_{m,n=0}$ is defined by $S_{00}(x)=0$ and $S_{mn}(x)=\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left(\Delta^{m-1}x_{mn}-\Delta^{m-1}x_{m,n+1}-\Delta^{m-1}x_{m+1,n}+\Delta^{m-1}x_{m+1,n+1}\right)^{1/m+n}\left(\lambda_{m,n}-\lambda_{m,n+1}-\lambda_{m+1,n}+\lambda_{m+1,n+1}\right),(n,m\geq 1)$. □

Lemma 2.3. *A $\lambda_{mn}-$ Musielak convergent sequence $x=(x_{mn})$ converges if and only if $S(x)\in\left[\chi^2_{fB^{\mu}_{\eta}},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]^{I(F)}$*

Proof: Let $x=(x_{mn})$ be $\lambda_{mn}-$ Musielak convergent sequence. Then from Lemma 2.2 we have $x=(x_{mn})$ converges to the same $\lambda_{mn}-$ limit. We obtain $S(x)\in\left[\chi^2_{fB^{\mu}_{\eta}},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]^{I(F)}$. Conversely, let $S(x)\in\left[\chi^2_{fB^{\mu}_{\eta}},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]^{I(F)}$. We have

$$lim_{m,n\to\infty}\left((m+n)!\left|\Delta^m x_{mn}\right|\right)^{1/m+n}=lim_{m,n\to\infty}B^{\mu}_{\eta}(x).$$

From the above equation, we deduce that $\lambda_{mn}-$ convergent sequence $x=(x_{mn})$ converges. □

Lemma 2.4. *Every double analytic sequence is $\lambda_{mn}-$ double analytic.*

Lemma 2.5. *A $\lambda_{mn}-$ Musielak analytic sequence $x=(x_{mn})$ is analytic if and only if $S(x)\in\left[\Lambda^2_{fB^{\mu}_{\eta}},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]^{I(F)}$*

Proof: From Lemma 2.4 and $S_{00}(x) = 0$ and
$S_{mn}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}}$
$\left((m+n)! \left(\Delta^{m-1} x_{mn} - \Delta^{m-1} x_{m,n+1} - \Delta^{m-1} x_{m+1,n} + \Delta^{m-1} x_{m+1,n+1}\right)\right)^{1/m+n}$
$(\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}), (n, m \geq 1).$ □

3. The spaces of $\lambda_{mn}-$ double gai and double analytic sequences

In this section we introduce the sequence space
$\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and
$\left[\Lambda^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ as sets of λ_{mn} double gai and double analytic sequences:
$\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} =$
$lim_{m,n\to\infty} \left[B^{\mu}_{\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} = 0$
$\left[\Lambda^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} =$
$sup_{mn} \left[B^{\mu}_{\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} < \infty.$

Theorem 3.1. *The sequence spaces*
$\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *and*
$\left[\Lambda^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *are isomorphic to the spaces*
$\left[\chi^2_{f}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *and*
$\left[\Lambda^2_{f}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$

Proof: We only consider the case
$\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \cong$
$\left[\chi^2_{f}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and
$\left[\Lambda^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \cong$
$\left[\Lambda^2_{f}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ can be shown similarly.
Consider the transformation T defined,
$Tx = B^{\mu}_{\eta} \in \left[\chi^2_{f}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$
for every $x \in \left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. The linearity of T is obvious. It is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

To show surjective we define the sequence $x = \{x_{mn}(\lambda)\}$ by

$$B_\eta^\mu(x) = \frac{1}{\varphi_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} (\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}) \left((m+n)! \left|\Delta^m x_{mn}\right|\right)^{1/m+n} = y_{mn} \tag{3.1}$$

We can say that $B_\eta^\mu(x) = y_{mn}$ from (3.1) and $x \in \left[\chi_f^2, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$, hence $B_\eta^\mu(x) \in \left[\chi_f^2, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. We deduce from that $x \in \left[\chi^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and $Tx = y$. Hence T is surjective. We have for every $x \in \left[\chi^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ that $d(Tx,0)_{\chi^2} = d(Tx,0)_{\Lambda^2} = d(x,0)_{\left[\chi^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]^{I(F)}}$.
Hence $\left[\chi^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and $\left[\chi_f^2, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ are ismorphic. Similarly obtain other sequence spaces. □

4. Some Inclusion and Relations

Theorem 4.1. *The inclusion* $\left[\chi^2_{f\Delta_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \subset \left[\chi^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *holds*

Proof: Let $\left[\chi^2_{f\Delta_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. Then we deduce that $\frac{1}{\varphi_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} (\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1})$ $\left((m+n)! \left|\Delta^m x_{mn}\right|\right)^{1/m+n} \leq \frac{1}{\varphi_{rs}} lim_{m,n\to\infty} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}}$ $(\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}) \left((m+n)! \left|\Delta^m x_{mn}\right|\right)^{1/m+n} =$ $lim_{m,n\to\infty} \left((m+n)! \left|\Delta^m x_{mn}\right|\right)^{1/m+n} = 0$. Hence
$x \in \left[\chi^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. □

Theorem 4.2. *The inclusion* $\left[\Lambda^2_{f\Delta_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \subset \left[\Lambda^2_{f\Delta^\lambda_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *holds.*

Proof: It is obvious. Therefore omit the proof. □

Theorem 4.3. *The inclusion* $\left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \subset \left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *hold. Furthermore, the equalities hold if and only if* $S(x) \in \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ *for every sequence* x *in the space* $\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$

Proof: Consider

$$\left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \subset \left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \tag{4.1}$$

is obvious from Lemma 2.1. Then, we have for every $x \in \left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ that $x \in \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and hence $S(x) \in \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ by Lemma 2.3. Conversely, let $x \in \left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$ Then, we have that $S(x) \in \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. Thus, it follows by Lemma 2.3 and then Lemma 2.2, that $x \in \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. We get

$$\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \subset \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} \tag{4.2}$$

From the equation (4.1) and (4.2) we get $\left[\chi^2_{f\Delta^{\lambda}_{mn}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)} = \left[\chi^2_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]^{I(F)}$. □

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