The 2D Zakharov-Kuznetsov-Burgers equation on a strip

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ABSTRACT: An initial-boundary value problem for the 2D Zakharov-Kuznetsov-Burgers equation posed on a channel-type strip was considered. The existence and uniqueness results for regular and weak solutions in weighted spaces as well as exponential decay of small solutions without restrictions on the width of a strip were proven both for regular solutions in an elevated norm and for weak solutions in the L^2-norm.

Key Words: KdV-Burgers equation , Dispersive equations, Exponential Decay

Contents

1. Introduction

We are concerned with an initial-boundary value problem (IBVP) for the two-dimensional Zakharov-Kuznetsov-Burgers (ZKB) equation

$$u_t + u_x - u_{xx} + uu_x + u_{xxx} + u_{xyy} = 0 \qquad (1.1)$$

posed on a strip modeling an infinite channel $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \in (0, B), B > 0\}$. This equation is a two-dimensional analog of the well-known Korteweg-de Vries-Burgers (KdV) equation

$$u_t + u_x - u_{xx} + uu_x + u_{xxx} = 0 \qquad (1.2)$$

which includes dissipation and dispersion and has been studied by various researchers due to its applications in Mechanics and Physics [1,2,3]. One can find extensive bibliography and sharp results on decay rates of solutions to the Cauchy problem (IVP) for (1.2) in [1]. Exponential decay of solutions to the initial problem for (1.2) with additional damping has been established in [3]. Equations (1.1) and (1.2) are typical examples of so-called dispersive equations which attract considerable attention of both pure and applied mathematicians in the past decades.

Quite recently, the interest on dispersive equations became to be extended to multi-dimensional models such as Kadomtsev-Petviashvili (KP) and Zakharov-Kuznetsov (ZK) equations [24]. As far as the ZK equation and its generalizations are concerned, the results on IVPs can be found in [5,10,17,18,19,20,23] and IBVPs were studied in [4,6,9,15,16,23]. In [15,16] was shown that IBVP for the ZK equation posed on a half-strip unbounded in x direction with the Dirichlet conditions on the boundaries possesses regular solutions which decay exponentially as $t \to \infty$ provided initial data are sufficiently small and the width of a half-strip is not too large. This means that the ZK equation may create an internal dissipative mechanism for some types of IBVPs.

The goal of our note is to prove that the ZKB equation on a strip also may create a dissipative effect without adding any artificial damping. We must mention that IBVP for the ZK equation on a strip ($x \in (0,1)$, $y \in \mathbb{R}$) has been studied in [4,22] and IBVPs on a strip ($y \in (0, L)$, $x \in \mathbb{R}$) for the ZK equation were considered in [8] and for the ZK equation with some internal damping in [7]. In the domain ($y \in (0, B)$, $x \in \mathbb{R}$, $t > 0$), the term u_x in (1.1) can be scaled out by a simple change of variables. Nevertheless, it can not be safely ignored for problems posed both on finite and semi-infinite intervals as well as on infinite in y direction bands without changes in the original domain [4,21].

The main results of our paper are the existence and uniqueness of regular and weak global-in-time solutions for (1.1) posed on a strip with the Dirichlet boundary conditions and the exponential decay rate of these solutions as well as continuous dependence on initial data.

The paper has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem. In Section 3, we prove global existence and uniqueness theorems for regular solutions in some weighted spaces and continuous dependence on initial data. In Section 4, we prove exponential decay of small regular solutions in an elevated norm corresponding to the $H^1(\mathcal{S})$-norm. In Section 5, we prove the existence, uniqueness and continuous dependence on initial data for weak solutions as well as the exponential decay rate of the $L^2(\mathcal{S})$-norm for small solutions without limitations on the width of the strip.

2. Problem and preliminaries

Let B, T, r be finite positive numbers. Define $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : \ x \in \mathbb{R}, \ y \in (0, B)\}$; $\mathcal{S}_r = \{(x, y) \in \mathbb{R}^2 : \ x \in (-r, +\infty), \ y \in (0, B)\}$ and $\mathcal{S}_T = \mathcal{S} \times (0, T)$.

Hereafter subscripts u_x, u_{xy}, etc. denote the partial derivatives, as well as ∂_x or ∂_{xy}^2 when it is convenient. Operators ∇ and Δ are the gradient and Laplacian acting over $\mathcal{S}$. By $(\cdot, \cdot)$ and $\|\cdot\|$ we denote the inner product and the norm in $L^2(\mathcal{S})$, and $\|\cdot\|_{H^k}$ stands for norms in the L^2-based Sobolev spaces. We will use also the spaces $H^s \cap L_b^2$, where $L_b^2 = L^2(e^{2bx}dx)$, see [11].

Consider the following IBVP:

$$Lu \equiv u_t - u_{xx} + uu_x + u_{xxx} + u_{xyy} = 0, \quad \text{in } \mathcal{S}_T; \tag{2.1}$$

$$u(x,0,t) = u(x,B,t) = 0, \ x \in \mathbb{R}, \ t > 0; \tag{2.2}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \mathcal{S}. \tag{2.3}$$

3. Existence of regular solutions

Approximate solutions. We will construct solutions to (2.1)-(2.3) by the Faedo-Galerkin method: let $w_j(y)$ be orthonormal in $L^2(\mathcal{S})$ eigenfunctions of the following Dirichlet problem:

$$w_{jyy} + \lambda_j w_j = 0, \ y \in (0,B); \tag{3.1}$$

$$w_j(0) = w_j(B) = 0. \tag{3.2}$$

Define approximate solutions of (2.1)-(2.3) as follows:

$$u^N(x,y,t) = \sum_{j=1}^{N} w_j(y) g_j(x,t), \tag{3.3}$$

where $g_j(x,t)$ are solutions to the following Cauchy problem for the system of N generalized Korteweg-de Vries equations:

$$\frac{\partial}{\partial t} g_j(x,t) + \frac{\partial^3}{\partial x^3} g_j(x,t) - \frac{\partial^2}{\partial x^2} g_j(x,t) - \lambda_j \frac{\partial}{\partial x} g_j(x,t)$$
$$+ \int_0^B u^N(x,y,t) u_x^N(x,y,t) w_j(y) \, dy = 0, \tag{3.4}$$

$$g_j(x,0) = \int_0^B w_j(y) u_0(x,y) \, dy, \ j = 1, ..., N. \tag{3.5}$$

The principal part of the system (3.4) is splitted with respect to index j while the semilinear part has a quadratic growth $|u^N u_x^N|$. This allows us to prove the existence of local in t regular solutions to (3.4)-(3.5) in the same manner as for the unique equation, [1,11,13]. The local existence of (3.4)-(3.5) may be proved also as a special case of the abstract Cauchy problem which includes the Cauchy problem for the KdV equation, [12].

It means that for $g_j(x,0) \in H^s$, $s \geq 3$, and for some $T_0 > 0$ the Cauchy problem (3.4)-(3.5) has a unique regular solution $g_j \in L^\infty(0,T_0; H^s(\mathcal{S}) \cap L_b^2(\mathcal{S})) \cap L^2(0,T_0; H^{s+1}(\mathcal{S}) \cap L_b^2(\mathcal{S}))$ at least for small T_0, [1,11,13]. To prove the existence of global solutions for (2.1)-(2.3), we need uniform in N global in t estimates of approximate solutions $u^N(x,y,t)$.

Estimate I. Multiply the j-th equation of (3.4) by g_j, sum up over $j = 1, ..., N$ and integrate the result with respect to x over $\mathbb{R}$ to obtain

$$\frac{d}{dt} \|u^N\|^2(t) + 2\|u_x^N\|^2(t) = 0$$

which implies

$$\|u^N\|^2(t) + 2\int_0^t \|u_x^N\|^2(s)\,ds = \|u_0^N\|^2 \quad \forall t \in (0, T). \qquad (3.6)$$

It follows from here that for N sufficiently large and $\forall t > 0$

$$\|u^N\|^2(t) + 2\int_0^t \|u_x^N\|^2(s)\,ds = \|u^N\|^2(0) \le 2\|u_0\|^2. \qquad (3.7)$$

In our calculations we will drop the index N where it is not ambiguous.

Estimate II. For some positive b, multiply the j-th equation of (3.4) by $e^{2bx}g_j$, sum up over $j = 1, ..., N$ and integrate the result with respect to x over $\mathbb{R}$. Dropping the index N, we get

$$\frac{d}{dt}(e^{2bx}, u^2)(t) + (2 + 6b)(e^{2bx}, u_x^2)(t) + 2b(e^{2bx}, u_y^2)(t)$$
$$- \frac{4b}{3}(e^{2bx}, u^3)(t) - (2b^2 + 8b^3)(e^{2bx}, u^2)(t) = 0. \qquad (3.8)$$

In our calculations, we will frequently use the following multiplicative inequalities [14]:

Proposition 3.1. *i) For all $u \in H^1(\mathbb{R}^2)$*

$$\|u\|_{L^4(\mathbb{R}^2)}^2 \le 2\|u\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}. \qquad (3.9)$$

ii) For all $u \in H^1(D)$

$$\|u\|_{L^4(D)}^2 \le C_D\|u\|_{L^2(D)}\|u\|_{H^1(D)}, \qquad (3.10)$$

where the constant C_D depends on a way of continuation of $u \in H^1(D)$ as $\tilde{u}(\mathbb{R}^2)$ such that $\tilde{u}(D) = u(D)$.

Extending $u^N(x, y, t)$ for a fixed t into exterior of $\mathcal{S}$ by 0 and exploiting the Gagliardo-Nirenberg inequality (3.9), we find

$$\frac{4b}{3}(e^{2bx}u^3)(t) \le b(e^{2bx}, u_y^2)(t) + 2b(e^{2bx}, u_x^2)(t) + 2(b^3 + \frac{8b}{9}\|u_0^N\|^2)(e^{2bx}, u^2)(t).$$

Substituting this into (3.8), we come to the inequality

$$\frac{d}{dt}(e^{2bx}, u^2)(t) + (2 + 4b)(e^{2bx}, u_x^2)(t) + b(e^{2bx}, u_y^2)(t)$$
$$\le C(b)(1 + \|u_0\|^2)(e^{2bx}, u^2)(t). \qquad (3.11)$$

By the Gronwall lemma,

$$(e^{2bx}, u^2)(t) \le C(b, T, \|u_0\|)(e^{2bx}, u_0^2).$$

Returning to (3.11) gives

$$(e^{2bx}, |u^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u^N|^2)(\tau) d\tau$$

$$\leq C(b, T, \|u_0\|)(e^{2bx}, u_0^2) \quad \forall t \in (0, T). \tag{3.12}$$

It follows from this estimate and (3.6) that uniformly in N and for any $r > 0$ and $t \in (0, T)$

$$\|u^N\|^2(t) + \int_0^t \int_0^B \int_{-r}^{+\infty} |\nabla u^N|^2 \, dx \, dy \, ds$$

$$\leq \mathbb{C}(r, b, T, \|u_0\|)(e^{2bx}, u_0^2), \tag{3.13}$$

where $\mathbb{C}$ does not depend on N.

Estimates (3.12), (3.13) make it possible to prove the existence of a weak solution to (2.1)-(2.3) passing to the limit in (3.4) as $N \to \infty$. For details of passing to the limit in the nonlinear term see [11].

Estimate III. Multiplying the j-th equation of (3.4) by $-(e^{2bx} g_{jx})_x$, and dropping the index N, we come to the equality

$$\frac{d}{dt}(e^{2bx}, u_x^2)(t) + (2 + 6b)(e^{2bx}, u_{xx}^2)(t) + 2b(e^{2bx}, u_{xy}^2)(t)$$

$$- (4b^2 + 8b^3)(e^{2bx}, u_x^2)(t) + (e^{2bx}, u_x^3)(t) - 2b(e^{2bx}u, u_x^2)(t) = 0. \tag{3.14}$$

Making use of Proposition 3.1, we estimate

$$I_1 = (e^{2bx}, u_x^3)(t) \leq \|u_x\|(t)\|e^{bx}u_x\|^2(t)_{L^4(\mathbb{S})}$$

$$\leq 2\|u_x\|(t)\|e^{bx}u_x\|(t)\|\nabla(e^{bx}u_x)\|(t)$$

$$\leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(t) + 2\big[\delta b^2 + \frac{\|u_x\|^2(t)}{2\delta}\big](e^{2bx}, u_x^2)(t).$$

Similarly,

$$I_2 = 2b(e^{2bx}, uu_x^2)(t) \leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(t)$$

$$+ \big[2b^2\delta + \frac{4b^2}{\delta}\|u_0\|^2(t)\big](e^{2bx}, u_x^2)(t).$$

Substituting I_1, I_2 into (3.14) and taking $2\delta = b$, we obtain for $\forall t \in (0, T)$:

$$(e^{2bx}, |u_x^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_x^N|^2)(s) \, ds$$

$$\leq C(b, T, \|u_0\|)(e^{2bx}, u_{0x}^2). \tag{3.15}$$

Estimate IV. Multiplying the j-th equation of (3.4) by $-2(e^{2bx}\lambda g_j)$, and dropping the index N, we come to the equality

$$\frac{d}{dt}(e^{2bx}, u_y^2)(t) + (2 + 6b)(e^{2bx}, u_{xy}^2)(t) + 2b(e^{2bx}, u_{yy}^2)(t)$$

$$- (4b^2 + 8b^3)(e^{2bx}, u_y^2)(t) + 2(1 - b)(e^{2bx}, u_x u_y^2)(t) = 0. \tag{3.16}$$

Making use of Proposition 3.1, we estimate

$$
\begin{aligned}
I = 2(1-b)(e^{2bx}, u_x u_y^2)(t) &\le 2C_D(1 \\
&+ b)\|u_x\|(t)\|e^{bx}u_y\|(t)\|(e^{bx}u_y)\|_{H^1(\mathbb{S})}(t) \\
&\le \delta(e^{2bx}, 2u_{xy}^2 + u_{yy}^2)(t) + \big[2\delta(1+b^2) \\
&+ \frac{C_D^2(1+b)^2\|u_x\|^2(t)}{\delta}\big](e^{2bx}, u_y^2)(t).
\end{aligned}
$$

Taking $\delta = b$, we transform (3.16) into the inequality

$$
\begin{aligned}
&\frac{d}{dt}(e^{2bx}, u_y^2)(t) + (2+4b)(e^{2bx}, u_{xy}^2)(t) + b(e^{2bx}, u_{yy}^2)(t) \\
&\le C(b)[1 + \|u_x\|(t)^2](e^{2bx}, u_y^2)(t).
\end{aligned}
$$

Making use of (3.7) and the Gronwall lemma, we get $\forall t \in (0, T)$:

$$
(e^{2bx}, |u_y^N|^2)(t) + \int_0^t (e^{2bx}, |u_{yy}^N|^2)(s)\, ds \le C(b, T, \|u_0\|)(e^{2bx}, u_{0y}^2).
$$

This and (3.15) imply that for all finite $r > 0$ and all $t \in (0, T)$

$$
\|u^N\|_{H^1(\mathbb{S})}^2 \le C(r, T, \|u_0\|)(e^{2bx}, |\nabla u_0|^2). \tag{3.17}
$$

Estimate V. Multiplying the j-th equation of (3.4) by $(e^{2bx}g_{jxx})_{xx}$, and dropping the index N, we come to the equality

$$
\begin{aligned}
&\frac{d}{dt}(e^{2bx}, u_{xx}^2)(t) + (2+6b)(e^{2bx}, u_{xxx}^2)(t) + 2b(e^{2bx}, u_{xxy}^2)(t) \\
&- (4b^2 + 8b^3)(e^{2bx}, u_{xx}^2)(t) - 2b(e^{2bx}, uu_{xx}^2)(t) \\
&+ 5(e^{2bx}u_x, u_{xx}^2)(t) = 0. \tag{3.18}
\end{aligned}
$$

Using (3.9), we find

$$
\begin{aligned}
I = -2b(e^{2bx}, uu_{xx}^2)(t) + 5(e^{2bx}u_x, u_{xx}^2)(t) \\
\le 2\delta(e^{2bx}, 2u_{xxx}^2 + u_{xxy}^2)(t) + \big[4b^2\delta + \frac{25}{\delta}\|u_x\|(t)^2 \\
+ \frac{4b^2}{\delta}\|u\|^2(t)\big](e^{2bx}, u_{xx}^2)(t).
\end{aligned}
$$

Taking $2\delta = b$ and substituting I into (3.18), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(e^{2bx}, u_{xx}^2)(t) + (2+4b)(e^{2bx}, u_{xxx}^2)(t) + b(e^{2bx}, u_{xxy}^2)(t) \\
&\le C(b)[1 + \|u_x\|^2(t) + \|u\|(t)^2](e^{2bx}, u_{xx}^2)(t).
\end{aligned}
$$

Taking into account (3.7), we find

$$
(e^{2bx}, |u_{xx}^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_{xx}^N|^2)(s)\,ds
$$
$$
\leq C(b, T, \|u_0\|)(e^{2bx}, u_{0xx}^2) \quad \forall t \in (0, T). \tag{3.19}
$$

Estimate VI. Differentiate (3.4) by t and multiply the result by $e^{2bx}g_{jt}$ to obtain

$$
\frac{d}{dt}(e^{2bx}, u_t^2)(t) + (2 + 6b)(e^{2bx}, u_{xt}^2)(t) + 2b(e^{2bx}, u_{ty}^2)(t)
$$
$$
- (4b^2 + 8b^3)(e^{2bx}, u_t^2)(t) + (2 - 2b)(e^{2bx}u_x, u_t^2)(t) = 0. \tag{3.20}
$$

Making use of (3.9), we estimate

$$
I = (2 - 2b)(e^{2bx}u_x, u_t^2)(t) \leq 2(2 + 2b)\|u_x\|(t)\|e^{bx}u_t\|(t)\|\nabla(e^{bx}u_t)\|(t)
$$
$$
\leq \delta(e^{2bx}, 2u_{xt}^2 + u_{ty}^2)(t) + \Big[2b^2\delta + \frac{(2 + 2b)^2\|u_x\|(t)^2}{\delta}\Big](e^{2bx}, u_t^2)(t).
$$

Taking $\delta = b$ and substituting I into (3.20), we obtain

$$
\frac{d}{dt}(e^{2bx}, u_t^2)(t) + (2 + 4b)(e^{2bx}, u_{xt}^2)(t) + b(e^{2bx}, u_{ty}^2)(t)
$$
$$
\leq C(b)[1 + \|u_x\|(t)^2](e^{2bx}, u_t^2)(t).
$$

This implies $\quad \forall t \in 0, T)$:

$$
(e^{2bx}, |u_t^N|^2)(t) + \int_0^t (e^{2bx}, |\nabla u_s^N|^2(s)\,ds
$$
$$
\leq C(b, T, \|u_0\|)(e^{2bx}, u_t^2)(0) \leq C(b, T, \|u_0\|)\|)J_0, \tag{3.21}
$$

where

$$
J_0 = \|u_0\|^2 + (e^{2bx}, u_0^2 + |\nabla u_0|^2 + |\nabla u_{0x}|^2 + u_0^2 u_{0x}^2 + |\Delta u_{0x}|^2).
$$

Estimate VII. Multiplying the j-th equation of (3.4) by $-e^{2bx}g_{jx}$, we come, dropping the index N, to the equality

$$
(e^{2bx}, [u_{xy}^2 + u_{xx}^2])(t) = +(e^{2bx}[u_t - (1 + 2b)u_{xx}], u_x)(t)
$$
$$
+ (e^{2bx}, uu_x^2)(t). \tag{3.22}
$$

Making use of (3.9), we estimate

$$
I = (e^{2bx}, uu_x^2)(t) \leq \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(t) + \Big[2b^2\delta + \frac{\|u_0\|^2}{\delta}\Big](e^{2bx}, u_x^2)(t).
$$

Taking $4\delta = 1$, using (3.15)-(3.21) and substituting I into (3.22), we get

$$(e^{2bx}, u_{xx}^{N\ 2} + u_{xy}^{N\ 2})(t) \le C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \tag{3.23}$$

Estimate VIII. We will need the following lemma :

Lemma 3.2. *Let* $u(x, y) : \mathcal{S} \to \mathbb{R}$ *be such that*

$$\int_{\mathcal{S}} e^{2bx}[u^2(x, y) + |\nabla u(x, y)|^2 + u_{xy}^2(x, y)]\, dxdy < \infty$$

and for all $x \in \mathbb{R}$ *there is some* $y_0 \in [0, B]$ *such that* $u(x, y_0) = 0$. *Then*

$$\sup_{\mathcal{S}} |e^{bx}u(x, y, t)|^2 \le \delta(1 + 2b^2)(e^{2bx}, u_y^2)(t) + 2\delta(e^{2bx}, u_{xy}^2)(t)$$

$$+ \frac{2\delta_1}{\delta}(e^{2bx}, u_x^2)(t) + \frac{1}{\delta}\Big[\frac{1}{\delta_1} + 2\delta_1 b^2\Big](e^{2bx}, u^2)(t), \tag{3.24}$$

where δ, δ_1 *are arbitrary positive numbers.*

Proof. Denote $v = e^{bx}u$. Then simple calculations give

$$\sup_{\mathcal{S}} v^2(x, y, t) \le \delta[\|v_y\|^2(t) + \|v_{xy}\|^2(t)] + \frac{1}{\delta}[\|v_x\|^2(t) + \|v\|^2(t)].$$

Returning to the function $u(x, y, t)$, we prove Lemma 3.2.   □

Multiplying the j-th equation of (3.4) by $e^{2bx}g_{jxxx}$, we come, dropping the index N, to the equality

$$(e^{2bx}, u_{xxy}^2 + u_{xxx}^2)(t) = (e^{2bx}[u_t - u_{xx}], u_{xxx})(t)$$

$$- (e^{2bx}uu_x, u_{xxx})(t) + 2b^2(e^{2bx}, u_{xy}^2)(t). \tag{3.25}$$

Using Lemma 3.2 and (3.7), we estimate

$$I = (e^{2bx}uu_x, u_{xxx})(t) \le \|u\|(t)\sup_{\mathcal{S}} |e^{bx}u_x(x, y, t)|\|e^{bx}u_{xxx}\|(t)$$

$$\le \epsilon\|u_0\|^2(e^{2bx}, u_{xxx}^2)(t) + \frac{1}{4\epsilon}\Big[\frac{1}{\delta}(1 + 2b^2)(e^{2bx}, u_x^2)(t)$$

$$+ \frac{2}{\delta}(e^{2bx}u_{xx}^2)(t) + \delta(1 + 2b^2)(e^{2bx}, u_{xy}^2)(t) + 2\delta(e^{2bx}, u_{xxy}^2)(t)\Big]. \tag{3.26}$$

Taking ϵ and δ sufficiently small, positive and substituting I into (3.25), we find

$$(e^{2bx}, |\nabla u_{xx}^N|^2)(t) \le C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \tag{3.27}$$

Consequently, it follows from the equality

$$-(e^{2bx}[u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N], u_{yy}^N)(t) = 0$$

and from

$$(e^{2bx}[u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N], u_{xyy}^N)(t) = 0$$

that

$$(e^{2bx}, |u_{yy}^N|^2 + |u_{xyy}^N|^2)(t) \leq C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \tag{3.28}$$

Jointly, estimates (3.15),(3.17), (3.19), (3.23),(3.27), (3.28) read

$$(e^{2bx}, |u^N|^2 + |\nabla u^N|^2 + |\nabla u_x^N|^2 + |\nabla u_y^N|^2 + |\nabla u_{xx}^N|^2)(t)$$
$$\leq C(b, T, \|u_0\|)J_0 \quad \forall t \in (0, T). \tag{3.29}$$

In other words,

$$e^{bx}u^N, \quad e^{bx}u_x^N \in L^\infty(0, T; H^2(\mathcal{S})) \tag{3.30}$$

and these inclusions are uniform in N.

Estimate IX. Differentiating the j-th equation of (3.4) with respect to x and multiplying the result by $e^{2bx}\partial_x^4 g_j$, we come, dropping the index N, to the equality

$$(e^{2bx}, u_{xxxy}^2 + u_{xxxx}^2)(t) = +2b^2(e^{2bx}, u_{xxy}^2)(t) - (e^{2bx}[u_{xt} - \partial_x^3 u], u_{xxxx})t)$$
$$- (e^{2bx}[u_x^2 + uu_{xx}], \partial_x^4 u)(t). \tag{3.31}$$

Making use of Lemma 3.2 and (3.29), we estimate

$$I_1 = (e^{2bx}u_x^2, \partial_x^4 u)(t) \leq \|u_x\|(t)\|e^{bx}\partial_x^4 u\|(t) \sup_{\mathcal{S}} |e^{bx}u_x(x, y, t)|$$

$$\leq \frac{\epsilon_1}{2}(e^{2bx}, |\partial_x^4 u|^2)(t) + \frac{1}{2\epsilon_1}\|u_x\|^2(t)\big[(1 + 2b^2)(e^{2bx}, u_x^2)(t)$$
$$+ 2(e^{2bx}, u_{xx}^2)(t) + (1 + 2b^2)(e^{2bx}, u_{xy}^2)(t) + 2(e^{2bx}, u_{xxy}^2)(t)\big]$$
$$\leq \frac{\epsilon_1}{2}(e^{2bx}, |\partial_x^4 u|^2)(t) + \frac{1}{2\epsilon_1}C(b, T, \|u_0\|)J_0,$$

$$I_2 = (e^{2bx}u, u_{xx}\partial_x^4 u)(t) \leq \|e^{bx}\partial_x^4 u\|(t)\|u\|(t) \sup_{\mathcal{S}} |e^{bx}u_{xx}(x, y, t|$$

$$\leq \frac{\epsilon_1}{2}\|u_0\|^2(t)(e^{2bx}, |\partial_x^4 u|^2)(t) + \frac{1}{2\epsilon_1}\{2\delta(e^{2bx}, u_{xxxy}^2)(t)$$
$$+ \delta(1 + 2b^2)(e^{2bx}, u_{xxy}^2)(t) + \frac{2}{\delta}(e^{2bx}, u_{xxx}^2)(t)$$
$$+ \frac{1}{\delta}(1 + 2b^2)(e^{2bx}, u_{xx}^2)(t)\}. \tag{3.32}$$

Applying the Young inequality, taking ϵ_1, δ sufficiently small positive, substituting I_1, I_2 into (3.31) and integrating the result, we come to the following inequality:

$$\int_0^t (e^{2bx}, |u_{xxxy}^N|^2 + |u_{xxxx}^N|^2)(s)\, ds \leq C(b, T, \|u_0\|)J_0 \ \forall t \in (0, T). \tag{3.33}$$

Estimate X. Multiplying the j-th equation of (3.4) by $-e^{2bx}\lambda^2 g_{jx}$, we come, dropping the index N, to the equality

$$(e^{2bx}, u_{xxyy}^2 + u_{xyyy}^2)(t) = -(e^{2bx}, u_{ty}, u_{xyyy}^2)(t) + (b + 2b^2)(e^{2bx}, u_{xyy}^2)(t)$$
$$- (e^{2bx} u_y u_x, u_{xyyy})(t) + (e^{2bx} u u_{xy}, u_{xyyy})(t). \tag{3.34}$$

We estimate

$$I_1 = -(e^{2bx}, u_{ty}, u_{xyyy})(t) \leq \frac{\epsilon}{2}(e^{2bx}, u_{xyyy}^2)(t) + \frac{1}{2\epsilon}(e^{2bx}, u_{yt}^2)(t),$$
$$I_2 = (e^{2bx} u_y u_x, u_{xyyy})(t) \leq \|u_x\|(t)\|e^{bx} u_{xyyy}\|(t) \sup_{\mathcal{S}} |e^{bx} u_y(x, y, t|$$

$$\leq \frac{\epsilon}{2}(e^{2bx}, u_{xyyy}^2)(t) + \frac{\|u_x\|(t)^2}{2\epsilon}\big[(1 + 2b^2)(e^{2bx}, u_y^2)(t)$$
$$+ 2(e^{2bx}, u_{xy}^2)(t) + (1 + 2b^2)(e^{2bx}, u_{yy}^2)(t) + 2(e^{2bx}, u_{xyy}^2)(t)\big],$$
$$I_3 = (e^{2bx} u u_{xy}, u_{xyyy})(t) \leq \|u\|(t)\|e^{bx} u_{xyyy}\|(t) \sup_{\mathcal{S}} |e^{bx} u_{xy}(x, y, t|$$

$$\leq \frac{\|u_0\|^2 \epsilon_1}{2}(e^{2bx}, u_{xyyy}^2)(t) + \frac{1}{2\epsilon_1}\big[2\delta(e^{2bx}, u_{xxyy}^2)(t)$$
$$+ \frac{2}{\delta}(e^{2bx}, u_{xxy}^2)(t) + \delta(1 + 2b^2)(e^{2bx}, u_{xyy}^2)(t)$$
$$+ \frac{1}{\delta}(1 + 2b^2)(e^{2bx}, u_{xy}^2)(t)\big].$$

Choosing ϵ, ϵ_1, δ sufficiently small, positive, after integration, we transform (3.34) into the form

$$\int_0^T (e^{2bx}, [|u_{xxyy}^N|^2 + |u_{xyyy}^N|^2])(t)\, dt \leq C(b, T, \|u_0\|) J_0. \tag{3.35}$$

Acting similarly, we get from the scalar product

$$(e^{2bx}[u_t^N - u_{xx}^N + u_{xxx}^N + u_{xyy}^N + u^N u_x^N], u_{yyyy}^N)(t) = 0$$

the estimate

$$\int_0^T (e^{2bx}, |u_{yyy}^N|^2)(t)\, dt \leq C(b, T, \|u_0\|) J_0. \tag{3.36}$$

Estimates (3.29), (3.30), (3.33), (3.35), (3.36) guarantee that

$$e^{bx} u^N, \quad e^{bx} u_x^N \in L^\infty(0, T; H^2(\mathcal{S}) \cap L^2(0, T; H^3(\mathcal{S})) \tag{3.37}$$

and these inclusions do not depend on N. Independence of Estimates (3.7),(3.37) of N allow us to pass to the limit in (3.4) and to prove the following result:

Theorem 3.3. *Let* $u_0(x, y) : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $u_0(x, 0) = u_0(x, B) = 0$ *and for some* $b > 0$

$$J_0 = \int_S \{u_0^2 + e^{2bx}[u_0^2 + |\nabla u_0|^2 + |\nabla u_{0x}|^2 + u_0^2 u_{0x}^2 + |\Delta u_{0x}|^2]\}\, dxdy < \infty.$$

Then there exists a regular solution to (2.1)-(2.3) $u(x,y,t)$:

$$u \in L^\infty(0,T;L^2(S)), \quad u_x \in L^2(0,T;L^2(S))$$
$$e^{bx}u,\ e^{bx}u_x \in L^\infty(0,T;H^2(S)) \cap L^2(0,T;H^3(S))$$
$$e^{bx}u_t \in L^\infty(0,T;L^2(S)) \cap L^2(0,T;H^1(S))$$

which for a.e. $t \in (0,T)$ *satisfies the identity*

$$(e^{bx}\big[u_t - u_{xx} + u_{xxx} + uu_x + u_{xyy}\big]\phi(x,y))(t) = 0, \tag{3.38}$$

where $\phi(x,y)$ *is an arbitrary function from* $L^2(S)$.

Proof. Rewrite (3.4) in the form

$$(e^{bx}\big[u_t^N - u_{xx}^N + u^N u_x^N + u_{xxx}^N + u_{xyy}^N\big], \Phi^N(y)\Psi(x))(t) = 0, \tag{3.39}$$

where $\Phi^N(y)$ is an arbitrary function from the set of linear combinations $\sum_{i=1}^N \alpha_i w_i(y)$ and $\Psi(x)$ is an arbitrary function from $H^1(\mathbb{R})$. Taking into account estimates (3.7), (3.37) and fixing Φ^N, we can easily pass to the limit as $N \to \infty$ in linear terms of (3.39). To pass to the limit in the nonlinear term, we must use (3.17) and repeat arguments of [11]. Since linear combinations $[\sum_{i=1}^N \alpha_i w_i(y)]\Psi(x)$ are dense in $L^2(S)$, we come to (3.38). This proves the existence of regular solutions to (2.1)-(2.3). $\qquad\square$

Remark 3.4. *Estimates* (3.7),(3.37) *are valid also for the limit function* $u(x,y,t)$ *and* (3.7) *obtains its sharp form:*

$$\|u\|(t)^2 + 2\int_0^t \|u_x\|(s)^2\, ds = \|u_0\|^2 \quad \forall t \in (0,T). \tag{3.40}$$

Uniqueness of a regular solution.

Theorem 3.5. *A regular solution from Theorem* 3.3 *is uniquely defined.*

Proof. Let u_1, u_2 be two distinct regular solutions of (2.1)-(2.3), then $z = u_1 - u_2$ satisfies the following initial-boundary value problem:

$$z_t - z_{xx} + z_{xxx} + z_{xyy} + \frac{1}{2}(u_1^2 - u_2^2)_x = 0 \text{ in } S_T, \tag{3.41}$$

$$z(x,0,t) = z(x,B,t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.42}$$

$$z(x,y,0) = 0. \quad (x,y) \in S. \tag{3.43}$$

Multiplying (3.41) by $2e^{bx}z$, we get

$$\frac{d}{dt}(e^{2bx}, z^2)(t) + (2 + 6b)(e^{2bx}, z_x^2)(t) - (8b^3 + 4b^2)(e^{2bx}, z^2)(t)$$
$$+ 2b(e^{2bx}, z_y^2)(t) + (e^{2bx}[u_{1x} + u_{2x}], z^2)(t)$$
$$- b(e^{2bx}(u_1 + u_2), z^2)(t) = 0. \tag{3.44}$$

We estimate

$$I_1 = (e^{2bx}(u_{1x} + u_{2x}), z^2)(t) \le \|u_{1x} + u_{2x}\|(t)\|e^{bx}z\|(t)_{L^4(\mathcal{S})}^2$$
$$\le 2\|u_{1x} + u_{2x}\|(t)\|e^{bx}z\|(t)\|\nabla(e^{bx}z)\|(t)$$
$$\le \delta(e^{2bx}, [2z_x^2 + z_y^2])(t) + [2b^2\delta + \frac{2}{\delta}(\|u_{1x}\|^2(t)$$
$$+ \|u_{2x}\|^2(t))](e^{2bx}, z^2)(t),$$
$$I_2 = b(e^{2bx}(u_1 + u_2), z^2)(t) \le b\|u_1 + u_2\|(t)\|e^{bx}z\|_{L^4(\mathcal{S})}^2$$
$$\le 2b\|u_1 + u_2\|(t)\|e^{bx}z\|(t)\|\nabla(e^{bx}z)\|(t)$$
$$\le \delta(e^{2bx}, 2z_x^2 + z_y^2)(t) + [2b^2\delta + \frac{2b^2}{\delta}(\|u_1\|^2(t) + \|u_2\|^2(t))](e^{2bx}, z^2)(t).$$

Substituting I_1, I_2 into (3.44) and taking $\delta > 0$ sufficiently small, we find

$$\frac{d}{dt}(e^{2bx}, z^2)(t) + (2 + 2b)(e^{2bx}, z_x^2)(t) + b(e^{2bx}, z_y^2)(t) \le C(b)\big[1 + \|u_1\|(t)^2$$
$$+ \|u_2\|(t)^2 + \|u_{1x}\|(t)^2 + \|u_{2x}\|(t)^2\big](e^{2bx}, z^2)(t). \tag{3.45}$$

Since
$$u_i \in L^\infty(0, T; L^2(\mathcal{S})), \quad u_{ix} \in L^2(0, T; L^2(\mathcal{S})) \quad i = 1, 2,$$

then by the Gronwall lemma,

$$(e^{2bx}, z^2)(t) = 0 \quad \forall\, t \in (0, T).$$

Hence, $u_1 = u_2$ *a.e. in* $\mathcal{S}_T$.   □

Remark 3.6. *Changing initial condition* (3.43) *for* $z(x, y, 0) = z_0(x, y) \ne 0$, *and repeating the proof of Theorem 3.4, we obtain from* (3.45) *that*

$$(e^{2bx}, z^2)(t) \le C(b, T, \|u_0\|)(e^{2bx}, z_0^2) \quad \forall t \in (0, T).$$

This means continuous dependence of regular solutions on initial data.

4. Decay of regular solutions

In this section we will prove exponential decay of regular solutions in an elevated weighted norm corresponding to the $H^1(\mathcal{S})$ norm. We start with Theorem 4.1 which is crucial for the main result.

Theorem 4.1. *Let* $b \in (0, \frac{1}{5}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}])$, $\|u_0\| \leq \frac{3\pi}{8B}$ *and* $u(x, y, t)$ *be a regular solution of* (2.1)-(2.3). *Then for all finite* $B > 0$ *the following inequality is true:*

$$\|e^{bx}u\|^2(t) \leq e^{-\chi t}\|e^{bx}u_0\|^2(0), \qquad (4.1)$$

where $\chi = \frac{1}{20}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}]\frac{\pi^2}{B^2}$.

Proof. Multiplying (2.1) by $2e^{2bx}u$, we get the equality

$$\frac{d}{dt}(e^{2bx}, u^2)(t) + (2 + 6b)(e^{2bx}, u_x^2)(t) + 2b(e^{2bx}, u_y^2)(t)$$

$$- \frac{4b}{3}(e^{2bx}, u^3)(t) - (4b^2 + 8b^3)(e^{2bx}, u^2)(t) = 0. \qquad (4.2)$$

Taking into account (3.1), we estimate

$$I = \frac{4b}{3}(e^{2bx}, u^3)(t) \leq b(e^{2bx}, u_y^2 + 2u_x^2 + 2b^2u^2)(t)$$

$$+ \frac{16b}{9}\|u_0\|^2(e^{2bx}, u^2)(t).$$

The following proposition is principal for our proof.

Proposition 4.2.

$$\int_{\mathbb{R}} \int_0^B e^{2bx}u^2(x, y, t)\, dy\, dx \leq \frac{B^2}{\pi^2} \int_{\mathbb{R}} \int_0^B e^{2bx}u_y^2(x, y, t)\, dy\, dx. \qquad (4.3)$$

Proof. Since $u(x, 0, t) = u(x, B, t) = 0$, fixing (x, t), we can use with respect to y the following Steklov inequality: if $f(y) \in H_0^1(0, \pi)$ then

$$\int_0^\pi f^2(y)\, dy \leq \int_0^\pi |f_y(y)|^2\, dy.$$

After a corresponding process of scaling we prove Proposition 4.2. $\qquad \square$

Making use of (4.3) and substituting I into (4.2), we come to the following inequality

$$\frac{d}{dt}(e^{2bx}, u^2)(t) + (2 + 4b)(e^{2bx}, u_x^2)(t)$$

$$+ \left[\frac{b\pi^2}{B^2} - 4b^2 - 10b^3 - \frac{16b}{9}\|u_0\|^2\right](e^{2bx}, u^2)(t) \leq 0$$

which can be rewritten as

$$\frac{d}{dt}(e^{2bx}, u^2)(t) + \chi(e^{2bx}, u^2)(t) \leq 0, \tag{4.4}$$

where

$$\chi = b\Big[\frac{\pi^2}{B^2} - 4b - 10b^2 - \frac{16\|u_0\|^2}{9}\Big].$$

Since we need $\chi > 0$, define

$$4b + 10b^2 = \gamma\frac{\pi^2}{B^2}, \quad \frac{16\|u_0\|^2}{9} = (1 - \gamma)^2\frac{\pi^2}{B^2}, \tag{4.5}$$

where $\gamma \in (0, 1)$. It implies $\chi = bA(\gamma)\frac{\pi^2}{B^2}$ with $A(\gamma) = \gamma(1 - \gamma)$.
It is easy to see that

$$\sup_{\gamma \in (0,1)} A(\gamma) = A(\frac{1}{2}) = \frac{1}{4}.$$

Solving (4.5), we find

$$b = \frac{1}{5}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}], \quad \|u_0\| \leq \frac{3\pi}{8B}, \quad \chi = b\frac{\pi^2}{4B^2},$$

and from (4.4) we get

$$(e^{2bx}, u^2)(t) \leq e^{-\chi t}(e^{2bx}, |u_0|^2).$$

The last inequality implies (4.1). The proof of Theorem 4.1 is complete.   □

Observe that differently from [15,16], we do not have any restrictions on the width of a strip B.

The main result of this section is the following assertion.

Theorem 4.3. *Let all the conditions of Theorem 4.1 be fulfilled. Then regular solutions of* (2.1)-(2.3) *satisfy the following inequality:*

$$(e^{2bx}, u^2 + |\nabla u|^2)(t) \leq C(b, \chi, \|u_0\|)(1 + t)e^{-\chi t}(e^{2bx}, \big[u_0^2$$
$$+ |u_0|^3 + |\nabla u_0|^2\big]) \tag{4.6}$$

or

$$\|e^{bx}u\|_{H^1(S)}^2(t) \leq C(b, \chi, \|u_0\|)(1 + t)e^{-\chi t}(e^{2bx}, u_0^2 + |u_0|^3 + |\nabla u_0|^2).$$

Proof. We start with the following lemma.

Lemma 4.4. *Regular solutions of (2.1)- (2.3) satisfy the following equality:*

$$
e^{\chi t}(e^{2bx}, |\nabla u|^2)(t) + 2 \int_0^t e^{\chi s}\{(1+3b)(e^{2bx}, u_{xx}^2)(s) + (1+4b)(e^{2bx}, u_{xy}^2)(s)
$$

$$
+ b(e^{2bx}, u_{yy}^2)(s) + \frac{b}{4}(e^{2bx}, u^4)(s)\}\, ds = \frac{e^{\chi t}}{3}(e^{2bx}, u^3)(t)
$$

$$
+ \int_0^t e^{\chi s}(\chi + 4b^2 + 8b^3)(e^{2bx}, |\nabla u|^2)(s)\, ds + 2\int_0^t e^{\chi s}\{(1+4b)(e^{2bx}u, u_x^2)(s)
$$

$$
+ 4b(e^{2bx}, uu_y^2)(s) - (\frac{4b^2 + 8b^3}{3} - \chi)(e^{2bx}, u^3)(s)\}\, ds
$$

$$
+ (e^{2bx}, |\nabla u_0|^2 - \frac{u_0^3}{3}). \tag{4.7}
$$

Proof. First we transform the scalar product

$$
- (e^{bx}\big[u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x\big],
$$
$$
\big[2(e^{bx}u_x)_x + 2e^{bx}u_{yy} + e^{bx}u^2\big])(t) = 0 \tag{4.8}
$$

into the following equality:

$$
\frac{d}{dt}(e^{2bx}, |\nabla u|^2 - \frac{u^3}{3})(t) + 2(1+3b)(e^{2bx}, u_{xx}^2)(t)
$$

$$
+ 2b(e^{2bx}, u_{yy}^2)(t) + 2(1+4b)(e^{2bx}, u_{xy}^2)(t) + \frac{b}{2}(e^{2bx}, u^4)(t)
$$

$$
= 4b^2(1+2b)(e^{2bx}, |\nabla u|^2)(t) - \frac{4b^2(1+2b)}{3}(e^{2bx}, u^3)(t)
$$

$$
+ 4b(e^{2bx}, uu_y^2)(t) + 2(1+4b)(e^{2bx}, uu_x^2)(t). \tag{4.9}
$$

To prove (4.9), we estimate separate terms in (4.8) as follows:

$$I_1 = -2(e^{bx}\big[u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x\big], (e^{bx}u_x)_x)(t)$$

$$= 2(e^{2bx}\big[u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x\big]_x, u_x)(t)$$

$$= \frac{d}{dt}(e^{2bx}, u_x^2)(t) + 2(1 + 3b)(e^{2bx}, u_{xx}^2)(t) + 2b(e^{2bx}, u_{xy}^2)(t)$$

$$- 4b^2(1 + 2b)(e^{2bx}, u_x^2)(t) + (e^{2bx}u^2, u_{xxx})(t)$$

$$- 8b(e^{2bx}, uu_x^2)(t) + \frac{8b^3}{3}(e^{2bx}, u^3)(t),$$

$$I_2 = -2(e^{bx}\big[u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x\big], e^{bx}u_{yy})(t)$$

$$= 2(e^{bx}\big[u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x\big]_y, e^{bx}u_y)(t)$$

$$= \frac{d}{dt}(e^{2bx}, u_y^2)(t) + 2(1 + 3b)(e^{2bx}, u_{xy}^2)(t) + 2b(e^{2bx}, u_{yy}^2)(t)$$

$$- 4b^2(1 + 2b)(e^{2bx}, u_y^2)(t) + (e^{2bx}u, u_{xyy})(t) - 4b(e^{2bx}, uu_y^2)(t),$$

$$I_3 = -(e^{bx}\big[u_t - u_{xx} + u_{xxx} + u_{xyy} + uu_x\big], e^{bx}u^2))(t)$$

$$= -\frac{d}{dt}(e^{2bx}, \frac{u^3}{3})(t) + \frac{4b^2}{3}(e^{2bx}, u^3)(t) + \frac{b}{2}(e^{2bx}, u^4)(t)$$

$$- 2(e^{2bx}, uu_x^2)(t) - (e^{2bx}, u_{xxx} + u_{xyy})(t).$$

Summing $I_1 + I_2 + I_3$, we obtain (4.9). In turn, multiplying it by $e^{\chi t}$ and integrating the result over $(0, t)$, we come to (4.7). The proof of Lemma 4.4 is complete. $\square$

Making use of (3.9), we estimate

$$I_4 = \frac{e^{\chi t}}{3}(e^{2bx}, u^3)(t) \le \frac{2e^{\chi t}}{3}\|u_0\|\|e^{bx}u\|(t)\|\nabla(e^{bx}u)\|(t)$$

$$\le \frac{e^{\chi t}}{2}\{(e^{2bx}, |\nabla u|^2)(t) + [\frac{b^2}{2} + \frac{4\|u_0\|^2}{9}](e^{2bx}, u^2)(t)\}.$$

Substituting I_4 into (4.7), we get

$$e^{\chi t}(e^{2bx}, |\nabla u|^2)(t) + 4\int_0^t e^{\chi s}\{(1 + 3b)(e^{2bx}, u_{xx}^2)(s) + (1 + 4b)(e^{2bx}, u_{xy}^2)(s)$$

$$+ b(e^{2bx}, u_{yy}^2)(s)\}\,ds \le 2\int_0^t e^{\chi s}(\chi + \frac{4b^2 + 8b^3}{3})(e^{2bx}, u^3)(s)\,ds$$

$$+ 2\int_0^t e^{\chi s}\{2(1 + 4b)(e^{2bx}u, u_x^2)(s) + 4b(e^{2bx}, uu_y^2)(s)\}\,ds$$

$$+ 2\int_0^t e^{\chi s}(\chi + 4b^2 + 8b^3)(e^{2bx}, |\nabla u|^2)(s)\,ds$$

$$+ [b^2 + \frac{8\|u_0\|^2}{9}]e^{\chi t}(e^{2bx}, u^2)(t) + 2(e^{2bx}, |\nabla u_0|^2 + \frac{|u_0|^3}{3}). \tag{4.10}$$

In order to estimate the right-hand side of (4.10), we will need the following

Proposition 4.5. *Let Theorem 4.1 be true. Then*

$$e^{\chi t}(e^{2bx}, u^2)(t) + \int_0^t e^{\chi s}(e^{2bx}, |\nabla u|^2)(s)\,ds$$
$$\leq C(b, \chi, \|u_0\|)(1+t)(e^{2bx}, u_0^2). \tag{4.11}$$

Proof. Consider the equality

$$\int_0^t 2e^{\chi s}(e^{2bx}[u_s - u_{xx} + u_{xxx} + u_{xyy} + uu_x], u)(s)\,ds = 0$$

which we rewrite as

$$e^{\chi t}(e^{2bx}, u^2)(t) + 2\int_0^t e^{\chi s}\{(1+3b)(e^{2bx}, u_x^2)(s) + b(e^{2bx}, u_y^2)(s)\}, ds$$
$$= \int_0^t e^{\chi s}\frac{4b}{3}(e^{2bx}, u^3)(s)ds + \int_0^t e^{\chi s}(\chi + 4b^2 + 8b^3)(e^{2bx}, u^2)(s)\,ds$$
$$+ (e^{2bx}, u_0^2). \tag{4.12}$$

By Proposition 3.1, we estimate

$$I_1 = \frac{4b}{3}(e^{2bx}, u^3)(t) \leq \frac{8b}{3}\|u\|(t)\|e^{bx}u\|(t)\|\nabla(e^{bx}u)\|(t)$$
$$\leq b(e^{2bx}, 2u_x^2 + u_y^2)(t) + [2b^3 + \frac{16b\|u_0\|^2}{9}](e^{2bx}, u^2)(t).$$

By Theorem 4.1,
$$(e^{2bx}, u^2)(t) \leq e^{-\chi t}(e^{2bx}, u_0^2).$$

Using this estimate, we substitute I_1 into (4.12) and come to the following inequality:

$$e^{\chi t}(e^{2bx}, u^2)(t) + \int_0^t e^{\chi s}\{(1+2b)(e^{2bx}, u_x^2)(s) + b(e^{2bx}, u_y^2)(s)\}, ds$$
$$\leq C(\chi, b, \|u_0\|)(1+t)(e^{2bx}, u_0^2).$$

Since $b > 0$, the proof of Proposition 4.5 is complete. □

Returning to (4.10) and using Proposition 4.5, we estimate

$$I_1 = (\chi + \frac{8b^2 + 16b^3}{3})(e^{2bx}, u^3)(s) \leq$$
$$2(e^{2bx}, |\nabla u|^2)(s) + C(\chi, b, \|u_0\|)(e^{2bx}, u^2)(s).$$

Similarly,

$$I_2 = 4(1 + 4b)(e^{2bx}, uu_x^2)(s) \le \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(s)$$
$$+ \left[2b^2\delta + \frac{16(1 + 4b)^2\|u_0\|^2}{\delta} \right](e^{2bx}, u_x^2)(s).$$

With the help of (3.10), we find

$$I_3 = 8b(e^{2bx}, uu_y^2)(s) \le 8bC_D\|u_0\|\|e^{bx}u_y\|(s)\|e^{bx}u_y\|_{H^1(\mathcal{S})}(s)$$
$$\le \delta(e^{2bx}, 2u_{xy}^2 + u_{yy}^2)(s) + \left[(2b^2 + 1)\delta + \frac{16b^2\|u_0\|^2 C_D^2}{\delta} \right](e^{2bx}, u_y^2)(s).$$

Taking $\delta = 2b$ and using Proposition 4.5, we obtain from (4.10)

$$e^{\chi t}(e^{2bx}, |\nabla u|^2)(t) \le C(b, \chi, \|u_0\|)(1 + t)(e^{2bx}, u_0^2 + |u_0|^3 + |\nabla u_0|^2).$$

Adding (4.1), we complete the proof of Theorem 4.3.

$\square$

5. Weak solutions

Here we will prove the existence, uniqueness and continuous dependence on initial data as well as exponential decay results for weak solutions of (2.1)-(2.3) when the initial function $u_0 \in L^2(\mathcal{S})$.

Theorem 5.1. *Let $u_0 \in L^2(\mathcal{S}) \cap L_b^2(\mathcal{S})$. Then for all finite positive T and B there exists at least one function*

$$u(x, y, t) \in L^\infty(0, T; L^2(\mathcal{S})), \ u_x \in L^2(0, T; L^2(\mathcal{S}))$$

such that

$$e^{bx}u \in L^\infty(0, T; L^2(\mathcal{S})) \cap L^2(0, T; H^1(\mathcal{S}))$$

. Moreover,

$$u^m \rightharpoonup \quad u \quad *-weakly \ in \ L^\infty(0, T; L^2(\mathcal{S})),$$

$$u^m \rightharpoonup u \quad weakly \ in \ L^2(0, T; H^1(\mathcal{S})),$$

where u^m are regular solutions to (2.1)-(2.3) provided by Theorem 3.3 and the following integral identity takes a place:

$$(e^{bx}u, v)(T) + \int_0^T \{-(e^{bx}u, v_t)(t) + (e^{bx}u_x, [v_{xx} + (1 + 2b)v_x$$
$$+ (b + b^2)v])(t) - \frac{1}{2}(e^{2bx}u^2, bv + v_x)(t)$$
$$+ (e^{bx}u_y, bv_y + v_{xy})(t)\} \, dt = (e^{bx}u_0, v(x, y, 0)), \tag{5.1}$$

where

$$e^{bx}v \in C(0, T; L^2(\mathcal{S})) \cap L^2(0, T; H^2(\mathcal{S})) \quad e^{bx}v_t \in L^2(0, T; L^2(\mathcal{S}))$$

is an arbitrary function.

Proof. In order to justify our calculations, we must operate with sufficiently smooth solutions $u^m(x, y, t)$. With this purpose, we consider first initial functions $u_{0m}(x, y)$, which satisfy conditions of Theorem 3.3, and obtain estimates (3.7), (3.17) for functions $u^m(x, y, t)$. This allows us to pass to the limit as $m \to \infty$ in the following identity:

$$(e^{bx}u^m, v)(T) + \int_0^T \{-(e^{bx}u^m, v_t)(t) + (e^{bx}u_x^m, [v_{xx} + (1 + 2b)v_x$$
$$+ (b + b^2)v])(t) - \frac{1}{2}(e^{2bx}|u^m|^2, bv + v_x)(t)$$
$$+ (e^{bx}u_y^m, bv_y + v_{xy})(t)\} \, dt = (e^{bx}u_{0m}, v(x, y, 0)) \tag{5.2}$$

and come to (5.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 5.2. *Obviously, regular solutions from Theorem 3.3, u^m satisfy also (5.1) sa well as (2.1)-(2.3).*

Uniqueness of a weak solution.

Theorem 5.3. *A weak solution of Theorem 5.1 is uniquely defined.*

Proof. Actually, this proof is provided by Theorem 3.5. It is sufficient to approximate the initial function $u_0 \in L^2(\mathcal{S})$ by regular functions u_{0m} in the form:

$$\lim_{m \to \infty} \|u_{0m} - u_0\| = 0,$$

where u_{om} satisfies the conditions of Theorem 3.3. This guarantees the existence of the unique regular solution to (2.1)-(2.3) and allows us to repeat all the calculations which have been done during the proof of Theorem 3.5 and to come to the following inequality:

$$\frac{d}{dt}(e^{2bx}, z_m^2)(t) + (2 + 2b)(e^{2bx}, z_{mx}^2)(t) + b(e^{2bx}, z_{my}^2)(t)$$
$$\leq C(b)\big[1 + \|u_{1m}\|(t)^2 + \|u_{2m}\|(t)^2 + \|u_{1xm}\|(t)^2 + \|u_{2xm}\|(t)^2\big](e^{2bx}, z_m^2)(t).$$

By the generalized Gronwall's lemma,

$$(e^{2bx}, z_m^2)(t) \leq exp\{\int_0^t C(b)\big[1 + \|u_{1m}\|(s)^2 + \|u_{2m}\|(s)^2 + \|u_{1xm}\|(s)^2$$
$$+ \|u_{2xm}\|(s)^2\big] \, ds\}(e^{2bx}, z_{0m}^2)(t).$$

Functions u_{1m} and u_{2m} for m sufficiently large satisfy the estimate

$$\|u_{im}\|(t)^2 + 2\int_0^t \|u_{imx}\|(s)^2 \, ds = \|u_{0m}\|^2 \leq 2\|u_0\|^2), \quad i = 1, 2.$$

Hence,

$$exp\{\int_0^t C(b)\big[1 + \|u_{1m}\|(s)^2 + \|u_{2m}\|(s)^2 + \|u_{1xm}\|(s)^2$$
$$+ \|u_{2xm}\|(s)^2\big]\, ds\} \le C(, T, \|u_0\|). \tag{5.3}$$

Since $e^{bx} z(x, y, t)$ is a weak limit of regular solutions $\{e^{bx} z_m(x, y, t)\}$, then

$$(e^{2bx}, z^2)(t) \le (e^{2bx}, z_m^2)(t) = 0.$$

This implies $u_1 \equiv u_2$ *a.e.* in $\mathcal{S}_T$. The proof of Theorem 5.3 is complete. $\square$

Remark 5.4. *Changing initial condition $z(x, y, 0) \equiv 0$ for $z(x, y, 0) = z_0(x, y) \ne 0$, and repeating the proof of Theorem 5.3, we obtain that*

$$(e^{2bx}, z^2)(t) \le C(b, T, \|u_0\|)(e^{2bx}, z_0^2) \quad \forall t \in (0, T).$$

This means continuous dependence of weak solutions on initial data.

Decay of weak solutions.

Theorem 5.5. *Let $b \in (0, \frac{1}{5}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}])$, $\|u_0\| \le \frac{3\pi}{16B}$ and $u(x, y, t)$ be a weak solution of* (2.1)-(2.3). *Then for all finite $B > 0$ the following inequality is true:*

$$\|e^{bx} u\|^2(t) \le e^{-\chi t} \|e^{bx} u_0\|^2(0), \tag{5.4}$$

where $\chi = \frac{\pi^2}{20B^2}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}]$.

Proof. Similarly to the proof of the uniqueness result for a weak solution, we approximate $u_0 \in L^2(\mathcal{S})$ by sufficiently smooth functions u_{om} in order to work with regular solutions. Acting in the same manner as by the proof of Theorem 4.1, we come to the following inequality :

$$\|e^{bx} u_m\|^2(t) \le e^{-\chi t} \|e^{bx} u_0\|^2(0), \tag{5.5}$$

where

$$\chi = \frac{\pi^2}{20B^2}[-1 + \sqrt{1 + \frac{5\pi^2}{4B^2}}].$$

Since $u(x, y, t)$ is weak limit of regular solutions $\{u_m(x, y, t)\}$ then

$$(e^{2bx}, u^2)(t) \le (e^{2bx}, u_m^2)(t) \le e^{-\chi t}(e^{2bx}, u_0^2).$$

The proof of Theorem 5.5 is complete. $\square$

We have in this Theorem a more strict condition $\|u_0\| \le \frac{3\pi}{16B}$ instead of $\|u_0\| \le \frac{3\pi}{8B}$ in the case of decay for regular solution because for weak solutions we do not have the sharp estimate (3.40), but only (3.7).

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