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Greatest Common Divisors of Shifted Balancing Numbers

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abstract: It is well known that the successive balancing numbers are relatively prime. Let for all integers a, $s_n(a)$ denote the greatest common divisor of the shifted balancing numbers of the form $s_n(a) = \gcd(B_n - a,\ B_{n+1} - 6a)$. In this study, we show that $\{s_n(\pm 1)\}$ is unbounded, whereas $\{s_n(a)\}$ is bounded for $a \neq \pm 1$.

Key Words: Balancing numbers, Lucas-balancing numbers, Balancing-like sequences, Shifted balancing numbers

Contents

1. Introduction

As usual, the n^{th} balancing number is denoted by B_n and the balancing numbers satisfy the binary recurrence $B_{n+1} = 6B_n - B_{n-1}$ with $B_0 = 0$ and $B_1 = 1$ [1]. The sequence of numbers closely associated with the balancing numbers is the Lucas-balancing numbers $\{C_n\}$ whose recurrence relation is given by $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$ and $C_1 = 3$ [15,16]. Balancing and Lucas-balancing numbers can be extended negatively, in particular $B_{-n} = -B_n$ and $C_{-n} = C_n$ [22]. Panda, in [17], explored many fascinating properties of balancing numbers, some of them are similar to the corresponding results on Fibonacci numbers, while some others are more interesting. Many exciting properties of balancing numbers and their related sequences are available in the literature. Interested readers can go through [2,3,4,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27].

In [19] Panda et.al. studied a class of binary recurrences defined by $x_{n+1} = Ax_n - Bx_{n-1}$. with $x_0 = 0$ and $x_1 = 1$ where A and B are any natural numbers.They have shown that for $B = 1$ and A not in $\{1,\ 2\}$, the sequences obtained

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from these recurrences have many important and interesting properties identical to those of balancing numbers. They named these class of sequences as balancing-like sequences. We begin with defining the sequence of generalized balancing-like sequences G_n^B with initials $G_1^B = \alpha$ and $G_2^B = \beta$ where α and β are natural numbers, as

$$G_n^B = AG_{n-1}^B - BG_{n-2}^B \text{ for } n \geq 3,$$

where A and B are natural numbers. In particular, for $A = 6$ and $B = 1$, we obtain

$$G_n^B = 6G_{n-1}^B - G_{n-2}^B \text{ for } n \geq 3,$$

and for $G_1^B = \alpha$ and $G_2^B = \beta$ we have

$$G_n^B = \beta B_{n-1} - \alpha B_{n-2},$$

which are nothing but the balancing-like sequences introduced by Panda et.al. in [19]. It is observed that, for $\alpha = 1$ and $\beta = 6$, the sequence of balancing-like numbers is nothing but the sequence of balancing numbers $\{B_n\}$. In a similar way, we introduce Lucas-balancing-like numbers,

$$G_n^C = 2\beta G_{n-1}^C - \alpha G_{n-2}^C,$$

and observe that for $\alpha = 1$ and $\beta = 3$, the sequence of Lucas-balancing numbers $\{C_n\}$ is obtained.
In [5], Chen studied about greatest common divisors of shifted Fibonacci numbers. Motivated by this, we consider a slightly different sequence of numbers which we call it as shifted balancing numbers $(B_n + a)$ by a for all integers a and let $s_n(a) = \gcd(B_n - a,\ B_{n+1} - 6a)$. In this study, the successive members of this sequence for different values of a are considered. Further, we will show that $\{s_n(\pm 1)\}$ is unbounded whereas $\{s_n(a)\}$ is bounded for $a \neq \pm 1$.

2. Preliminary results

In this section, some preliminary results concerning the greatest common divisors of balancing-like sequences are established.

Lemma 2.1. *For integers n, k and a,* $\gcd(G_n^B + aB_k,\ G_{n-1}^B + aB_{k+1}) = \gcd(G_{n-2}^B + aB_{k+2},\ G_{n-3}^B + aB_{k+3})$.

Proof: For any integers a, b, and c, as $\gcd(a, b) = \gcd(a + bc, b)$ and $\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b)$, we have

$$\begin{aligned}
\gcd(G_n^B + aB_k,\ G_{n-1}^B + aB_{k+1}) &= \gcd(-G_n^B - aB_k,\ G_{n-1}^B + aB_{k+1}) \\
&= \gcd(G_{n-2}^B + aB_{k+2},\ G_{n-1}^B + aB_{k+1}) \\
&= \gcd(G_{n-2}^B + aB_{k+2},\ -G_{n-3}^B - aB_{k+3}) \\
&= \gcd(G_{n-2}^B + aB_{k+2},\ G_{n-3}^B + aB_{k+3}).
\end{aligned}$$

which completes the proof. □

Lemma 2.2. *For integers m, k and a,*

$$\gcd(G_m^B - a,\ G_{m+1}^B - 6a) = \gcd(G_{m-2k}^B + aB_{2k-1},\ G_{m-(2k+1)}^B + aB_{2k}). \tag{2.1}$$

Proof: Simplification of the left side expression gives

$$\begin{aligned}\gcd(G_m^B - a,\ G_{m+1}^B - 6a) &= \gcd(G_m^B - a,\ -G_{m-1}^B)\\ &= \gcd(G_m^B - a,\ G_{m-1}^B)\\ &= \gcd(G_m^B + aB_{-1},\ G_{m-1}^B + aB_0).\end{aligned}$$

Because $B_{-1} = -1$ and $B_0 = 0$ and applying Lemma 2.1 k times, the result follows. □

Lemma 2.3. *Let m, k and a are integers, then*

$$\gcd(G_m^B + a,\ G_{m+1}^B + 3a) = \gcd(G_{m-2k}^B + aC_{2k},\ G_{m-(2k+1)}^B + aC_{2k+1}). \tag{2.2}$$

Proof: For integers m, k and a,

$$\begin{aligned}\gcd(G_m^B + a,\ G_{m+1}^B + 3a) &= \gcd(G_m^B + a,\ -G_{m-1}^B - 3a)\\ &= \gcd(G_m^B + a,\ G_{m-1}^B + 3a)\\ &= \gcd(G_m^B + aC_0,\ G_{m-1}^B + aC_1)\end{aligned}$$

as $C_0 = 1$ and $C_1 = 3$ and applying Lemma 2.1 k times, the result follows. □

3. Greatest common divisors of the successive members of the sequence $\{s_n(a)\}$ for different values of a

In this section, we consider the sequence $\{s_n(a)\} = \gcd(B_n - a,\ B_{n+1} - 6a)$ for different values of a and obtain some important identities.

3.1. The sequence $\{s_n(1)\}$

Theorem 3.1. *For any integer n, we have*

$$\gcd(B_{4n-1} - 1,\ B_{4n} - 6) = 2B_{2n-1} \tag{3.1}$$

$$\gcd(B_{4n} - 1,\ B_{4n+1} - 6) = B_{2n} + B_{2n-1} \tag{3.2}$$

$$\gcd(B_{4n+1} - 1,\ B_{4n+2} - 6) = 2B_{2n} \tag{3.3}$$

$$\gcd(B_{4n+2} - 1,\ B_{4n+3} - 6) = B_{2n+1} + B_{2n}. \tag{3.4}$$

Proof: For $m = 4n - 1, k = n$, and $a = 1$ in (2.1), we obtain

$$\begin{aligned}\gcd(B_{4n-1} - 1,\ B_{4n} - 6) &= \gcd(B_{2n-1} + B_{2n-1},\ B_{2n-2} + B_{2n})\\ &= \gcd(2B_{2n-1},\ 6B_{2n-1})\\ &= 2B_{2n-1}.\end{aligned}$$

This completes the proof of (3.1). Putting $m = 4n,\ k = n$, and $a = 1$ in (2.1), we have

$$\begin{aligned}\gcd(B_{4n} - 1,\ B_{4n+1} - 6) &= \gcd(B_{2n} + B_{2n-1},\ B_{2n-1} + B_{2n})\\ &= B_{2n} + B_{2n-1},\end{aligned}$$

which gives (3.2). Again for $m = 4n + 1,\ k = n$, and $a = 1$ in (2.1), we have

$$\begin{aligned}\gcd(B_{4n+1} - 1,\ B_{4n+2} - 6) &= \gcd(B_{2n+1} + B_{2n-1},\ B_{2n} + B_{2n})\\ &= \gcd(6B_{2n},\ 2B_{2n})\\ &= 2B_{2n},\end{aligned}$$

gives (3.3). Finally, setting $m = 4n + 2, k = n$, and $a = 1$ in (2.1), we have

$$\begin{aligned}\gcd(B_{4n+2} - 1,\ B_{4n+3} - 6) &= \gcd(B_{2n+2} + B_{2n-1},\ B_{2n+1} + B_{2n})\\ &= \gcd(6B_{2n+1} + 6B_{2n},\ B_{2n+1} + B_{2n})\\ &= B_{2n+1} + B_{2n},\end{aligned}$$

which gives (3.4). □

3.2. The sequence $\{s_n(2)\}$

Theorem 3.2. *For any integer n, we have*

$$\gcd(B_{4n-1} - 2,\ B_{4n} - 12) = 3 \tag{3.5}$$
$$\gcd(B_{4n} - 2,\ B_{4n+1} - 12) = 1 \tag{3.6}$$
$$\gcd(B_{4n+1} - 2,\ B_{4n+2} - 12) = 1 \tag{3.7}$$
$$\gcd(B_{4n+2} - 2,\ B_{4n+3} - 12) = 1. \tag{3.8}$$

Proof: For $m = 4n - 1, k = n$, and $a = 2$ in (2.1), we obtain

$$\begin{aligned}\gcd(B_{4n-1} - 2,\ B_{4n} - 12) &= \gcd(B_{2n-1} + 2B_{2n-1},\ B_{2n-2} + 2B_{2n})\\ &= \gcd(3B_{2n-1},\ 6B_{2n-1} + B_{2n})\\ &= \gcd(3B_{2n-1},\ B_{2n}).\end{aligned}$$

Since $\gcd(a,\ bc) = \gcd(a,\ \gcd(a, b)c)$, we have

$$\begin{aligned}\gcd(B_{4n-1} - 2,\ B_{4n} - 12) &= \gcd(3\gcd(B_{2n-1},\ B_{2n}), B_{2n})\\ &= \gcd(3,\ B_{2n})\\ &= 3,\end{aligned}$$

which is (3.5). Let $m = 4n, k = n$, and $a = 2$ in (2.1), then we have

$$\begin{aligned}\gcd(B_{4n}-2,\ B_{4n+1}-12) &= \gcd(B_{2n}+2B_{2n-1},\ B_{2n-1}+2B_{2n})\\ &= \gcd(-3B_{2n},\ B_{2n-1}+2B_{2n})\\ &= \gcd(3B_{2n},\ B_{2n-1}+2B_{2n})\\ &= \gcd(B_{2n-1},\ B_{2n-1}+2B_{2n})\\ &= \gcd(B_{2n-1},\ 2B_{2n})\\ &= \gcd(B_{2n-1},\ 2\gcd(B_{2n-1}, B_{2n}))\\ &= \gcd(B_{2n-1},\ 2)\\ &= 1,\end{aligned}$$

gives (3.6). Again putting $m = 4n+1, k = n$, and $a = 2$ in (2.1), we get

$$\begin{aligned}\gcd(B_{4n+1}-2,\ B_{4n+2}-12) &= \gcd(B_{2n+1}+2B_{2n-1},\ B_{2n}+2B_{2n})\\ &= \gcd(6B_{2n}+B_{2n-1},\ 3B_{2n})\\ &= \gcd(B_{2n-1},\ 3B_{2n})\\ &= \gcd(B_{2n-1},\ 3\gcd(B_{2n-1},\ B_{2n}))\\ &= \gcd(B_{2n-1},\ 3)\\ &= 1,\end{aligned}$$

gives (3.7). Finally setting $m = 4n+2, k = n$, and $a = 2$ in (2.1), we obtain

$$\begin{aligned}\gcd(B_{4n+2}-2,\ B_{4n+3}-12) &= \gcd(B_{2n+2}+2B_{2n-1},\ B_{2n+1}+2B_{2n})\\ &= \gcd(B_{2n+2}+2B_{2n}+2B_{2n-1}+B_{2n+1},\ B_{2n+1}+2B_{2n})\\ &= \gcd(6B_{2n+1}+6B_{2n}+B_{2n}+B_{2n-1},\ B_{2n+1}+2B_{2n})\\ &= \gcd(-6B_{2n}+B_{2n}+B_{2n-1},\ B_{2n+1}+2B_{2n})\\ &= \gcd(-5B_{2n}+B_{2n-1},\ B_{2n+1}+2B_{2n})\\ &= \gcd(3B_{2n},\ B_{2n+1}+2B_{2n})\\ &= \gcd(3\gcd(B_{2n},\ B_{2n+1}+2B_{2n}),\ B_{2n+1}+2B_{2n})\\ &= \gcd(3\gcd(B_{2n},\ B_{2n+1}),\ B_{2n+1}+2B_{2n})\\ &= \gcd(3,\ B_{2n+1}+2B_{2n})\\ &= 1,\end{aligned}$$

gives (3.8). □

3.3. The sequence $\{s_n(-1)\}$

Theorem 3.3. *For any integer n, we have*

$$\gcd(B_{4n-1}+1,\ B_{4n}+6) = B_{2n}-B_{2n-2} \tag{3.9}$$
$$\gcd(B_{4n}+1,\ B_{4n+1}+6) = B_{2n}-B_{2n-1} \tag{3.10}$$
$$\gcd(B_{4n+1}+1,\ B_{4n+2}+6) = B_{2n+1}-B_{2n-1} \tag{3.11}$$
$$\gcd(B_{4n+2}+1,\ B_{4n+3}+6) = B_{2n+1}-B_{2n}. \tag{3.12}$$

Proof: For $m = 4n-1, k = n$, and $a = -1$ in (2.1), we obtain

$$\begin{aligned}\gcd(B_{4n-1}+1,\ B_{4n}+6) &= \gcd(B_{2n-1}-B_{2n-1},\ B_{2n-2}-B_{2n})\\ &= B_{2n}-B_{2n-2},\end{aligned}$$

gives (3.9). Let $m = 4n, k = n$, and $a = -1$ in (2.1), we get

$$\gcd(B_{4n}+1,\ B_{4n+1}+6) = \gcd(B_{2n}-B_{2n-1},\ B_{2n-1}-B_{2n}) = B_{2n}-B_{2n-1},$$

which is (3.10). Putting $m = 4n+1, k = n$, and $a = -1$ in (2.1), we get

$$\begin{aligned}\gcd(B_{4n+1}+1,\ B_{4n+2}+6) &= \gcd(B_{2n+1}-B_{2n-1},\ B_{2n}-B_{2n})\\ &= B_{2n+1}-B_{2n-1},\end{aligned}$$

giving (3.11). Further, setting $m = 4n+2, k = n$, and $a = -1$ in (2.1), we obtain

$$\begin{aligned}\gcd(B_{4n+2}+1,\ B_{4n+3}+6) &= \gcd(B_{2n+2}-B_{2n-1},\ B_{2n+1}-B_{2n})\\ &= \gcd(6B_{2n+1}-6B_{2n},\ B_{2n+1}-B_{2n})\\ &= B_{2n+1}-B_{2n}\end{aligned}$$

which is (3.12). □

3.4. The sequence $\{s_n(-2)\}$

Theorem 3.4. *For any integer n, we have*

$$\gcd(B_{4n-1}+2,\ B_{4n}+12) = 1 \tag{3.13}$$
$$\gcd(B_{4n}+2,\ B_{4n+1}+12) = 1 \tag{3.14}$$
$$\gcd(B_{4n+1}+2,\ B_{4n+2}+12) = 3 \tag{3.15}$$
$$\gcd(B_{4n+2}+2,\ B_{4n+3}+12) = 1. \tag{3.16}$$

Proof: Setting $m = 4n-1, k = n$, and $a = -2$ in (2.1),

$$\begin{aligned}\gcd(B_{4n-1}+2,\ B_{4n}+12) &= \gcd(B_{2n-1}-2B_{2n-1},\ B_{2n-2}-2B_{2n})\\ &= \gcd(-B_{2n-1},\ B_{2n-2}-B_{2n})\\ &= \gcd(B_{2n-1},\ B_{2n-2}-2B_{2n}-6B_{2n-1})\\ &= \gcd(B_{2n-1},\ 3B_{2n})\\ &= \gcd(B_{2n-1},\ 3\gcd(B_{2n-1},B_{2n}))\\ &= \gcd(B_{2n-1},\ 3)\\ &= 1,\end{aligned}$$

giving (3.13). Let $m = 4n, k = n$, and $a = -2$ in (2.1). Then

$$\begin{aligned}\gcd(B_{4n}+2,\ B_{4n+1}+12) &= \gcd(B_{2n}-2B_{2n-1},\ B_{2n-1}-2B_{2n})\\ &= \gcd(-3B_{2n},\ B_{2n-1}-2B_{2n})\\ &= \gcd(3\gcd(B_{2n},\ B_{2n-1}-2B_{2n}),\ B_{2n-1}-2B_{2n})\\ &= \gcd(3\gcd(B_{2n},\ B_{2n-1}),\ B_{2n-1}-2B_{2n})\\ &= \gcd(3,\ B_{2n-1}-2B_{2n})\\ &= 1,\end{aligned}$$

gives (3.14). For $m = 4n+1, k = n$, and $a = -2$ in (2.1), we have

$$\begin{aligned}\gcd(B_{4n+1}+2,\ B_{4n+2}+12) &= \gcd(B_{2n+1}-2B_{2n-1},\ B_{2n}-2B_{2n})\\ &= \gcd(B_{2n+1}-2B_{2n-1},\ B_{2n})\\ &= \gcd(6B_{2n}-3B_{2n-1},\ B_{2n})\\ &= \gcd(-3B_{2n-1},\ B_{2n})\\ &= \gcd(3\gcd(B_{2n-1},\ B_{2n}), B_{2n})\\ &= \gcd(3,\ B_{2n})\\ &= 3,\end{aligned}$$

which is (3.15). Again setting $m = 4n+2, k = n$, and $a = -2$ in (2.1)

$$\begin{aligned}\gcd(B_{4n+2}+2,\ B_{4n+3}+12) &= \gcd(B_{2n+2}-2B_{2n-1},\ B_{2n+1}-2B_{2n})\\ &= \gcd(B_{2n+2}+2B_{2n}-2B_{2n-1}-B_{2n+1},\ B_{2n+1}-2B_{2n})\\ &= \gcd(6B_{2n+1}+B_{2n}-B_{2n-1}-6B_{2n},\ B_{2n+1}-2B_{2n})\\ &= \gcd(6B_{2n+1}-B_{2n-1}-5B_{2n},\ B_{2n+1}-2B_{2n})\\ &= \gcd(7B_{2n+1}-5B_{2n}-B_{2n+1}-B_{2n-1},\ B_{2n+1}-2B_{2n})\\ &= \gcd(7B_{2n+1}-11B_{2n},\ B_{2n+1}-2B_{2n})\\ &= \gcd(B_{2n+1}+B_{2n},\ B_{2n+1}-2B_{2n})\\ &= \gcd(B_{2n+1}+B_{2n},\ 3B_{2n+1})\\ &= \gcd(B_{2n+1}+B_{2n},\ 3\gcd(B_{2n+1}+B_{2n},\ B_{2n+1}))\\ &= \gcd(B_{2n+1}+B_{2n},\ 3)\\ &= 1,\end{aligned}$$

gives (3.16). □

Theorem 3.5. *For any integer n, we have*

$$\gcd(C_{4n-1}+1,\ C_{4n}+3) = C_{2n}+C_{2n-1} \tag{3.17}$$
$$\gcd(C_{4n}+1,\ C_{4n+1}+3) = 2C_{2n} \tag{3.18}$$
$$\gcd(C_{4n+1}+1,\ C_{4n+2}+3) = C_{2n+1}+C_{2n} \tag{3.19}$$
$$\gcd(C_{4n+2}+1,\ C_{4n+3}+3) = 2C_{2n+1}. \tag{3.20}$$

Proof: Putting $m = 4n-1, k = n$, and $a = 1$ in (2.2), we get

$$\begin{aligned}\gcd(C_{4n-1}+1,\ C_{4n}+3) &= \gcd(C_{2n-1}+C_{2n},\ C_{2n-2}+C_{2n+1})\\ &= \gcd(C_{2n-1}+C_{2n},\ 6C_{2n-1}+6C_{2n})\\ &= C_{2n-1}+C_{2n},\end{aligned}$$

which is (3.17). Let $m = 4n, k = n$, and $a = 1$ in (2.2) to obtain

$$\begin{aligned}\gcd(C_{4n}+1,\ C_{4n+1}+3) &= \gcd(C_{2n}+C_{2n},\ C_{2n-1}+C_{2n+1})\\ &= \gcd(2C_{2n},\ 6C_{2n})\\ &= 2C_{2n},\end{aligned}$$

which follows (3.18). For $m = 4n+1, k = n$, and $a = 1$ in (2.2), we get

$$\begin{aligned}\gcd(C_{4n+1}+1,\ C_{4n+2}+3) &= \gcd(C_{2n+1}+C_{2n},\ C_{2n}+C_{2n+1})\\ &= C_{2n}+C_{2n+1},\end{aligned}$$

which gives (3.19). Setting $m = 4n+2, k = n$, and $a = 1$ in (2.2), we obtain

$$\begin{aligned}\gcd(C_{4n+2}+1,\ C_{4n+3}+3) &= \gcd(C_{2n+2}+C_{2n},\ C_{2n+1}+C_{2n+1})\\ &= \gcd(6C_{2n+1},\ 2C_{2n+1})\\ &= 2C_{2n+1},\end{aligned}$$

gives (3.20). □

From the above results so far we have obtained, it is evident that $\{s_n(a)\}$ is unbounded for $a = \pm 1$. The next is to show $\{s_n(a)\}$ is bounded for $a \neq \pm 1$. For this, we prove the following two results.

Theorem 3.6. *For any integers α, β, n and a with $\alpha^2+\beta^2-6\alpha\beta-a^2 \neq 0$, we have*

$$\gcd(G^B_{4n-1}-a,\ G^B_{4n}-6a) \leq |\alpha^2+\beta^2-6\alpha\beta-a^2|.$$

Proof: For $m = 4n-1$ and $k = n$ in (2.1), we obtain

$$\begin{aligned}&\gcd(G^B_{4n-1}-a,\ G^B_{4n}-6a)\\ &= \gcd(G^B_{2n-1}+aB_{2n-1},\ G^B_{2n-2}+aB_{2n})\\ &= \gcd(\beta B_{2n-2}-\alpha B_{2n-3}+aB_{2n-1},\ \beta B_{2n-3}-\alpha B_{2n-4}+aB_{2n}).\end{aligned}$$

Using the recursion relation for B_n, let

$$f_n = \beta B_{2n-2}-\alpha B_{2n-3}+aB_{2n-1} = (\beta+6a)B_{2n-2}-(\alpha+a)B_{2n-3}$$

and

$$g_n = \beta B_{2n-3}-\alpha B_{2n-4}+aB_{2n} = (\alpha+35a)B_{2n-2}+(\beta-6\alpha-6a)B_{2n-3}.$$

Since $\gcd(f_n,\ g_n)$ divides $yf_n + zg_n$ for any integers y and z, and

$$(\alpha + a)g_n - (6\alpha + 6a - \beta)f_n = (\alpha^2 + \beta^2 - 6\alpha\beta - a^2)B_{2n-2}$$

and

$$(\beta + 6a)g_n - (\alpha + 35a)f_n = (\alpha^2 + \beta^2 - 6\alpha\beta - a^2)B_{2n-3},$$

we see that if $\alpha^2 + \beta^2 - 6\alpha\beta - a^2 \neq 0$, then the greatest common divisor of the two numbers is $|\alpha^2 + \beta^2 - 6\alpha\beta - a^2|$. Therefore $\gcd(f_n,\ g_n)$ divides $\alpha^2 + \beta^2 - 6\alpha\beta - a^2$. That is to say,

$$\gcd(G^B_{4n-1} - a,\ G^B_{4n} - 6a) \leq |\alpha^2 + \beta^2 - 6\alpha\beta - a^2|.$$

If $m = 4n + 1$ and $k = n$ in (2.1), we have, similarly

$$\gcd(G^B_{4n+1} - a,\ G^B_{4n+2} - 6a) \leq |\alpha^2 + \beta^2 - 6\alpha\beta - a^2|.$$

This ends the proof. □

Theorem 3.7. *For any integers α, β, n and a with $\alpha^2 + \beta^2 - 6\alpha\beta - a^2 \neq 0$, we have*

$$\gcd(G^B_{4n} - a,\ G^B_{4n+1} - 6a) \leq |\alpha^2 + \beta^2 - 6\alpha\beta - a^2|.$$

Proof: Again setting $m = 4n$ and $k = n$ in (2.1), we get

$$\begin{aligned}\gcd(G^B_{4n} - a,\ G^B_{4n+1} - 6a) &= \gcd(G^B_{2n} + aB_{2n-1},\ G^B_{2n-1} + aB_{2n}) \\ &= \gcd(\beta B_{2n-1} - \alpha B_{2n-2} + aB_{2n-1},\ \beta B_{2n-2} - \alpha B_{2n-3} + aB_{2n}).\end{aligned}$$

Using the recursion relation for B_n, let

$$f_n = \beta B_{2n-1} - \alpha B_{2n-2} + aB_{2n-1} = (\beta + a)B_{2n-1} - \alpha B_{2n-2}$$

and

$$g_n = \beta B_{2n-2} - \alpha B_{2n-3} + aB_{2n} = (\alpha + 6a)B_{2n-1} + (\beta - 6\alpha - a)B_{2n-2}.$$

Since $\gcd(f_n,\ g_n)$ divides $yf_n + zg_n$ for any integers y and z, and

$$\alpha g_n + (\beta - 6\alpha - a)f_n = (\alpha^2 + \beta^2 - 6\alpha\beta - a^2)B_{2n-1}$$

and

$$(\beta + a)g_n - (\alpha + 6a)f_n = (\alpha^2 + \beta^2 - 6\alpha\beta - a^2)B_{2n-2},$$

we see that if $\alpha^2 + \beta^2 - 6\alpha\beta - a^2 \neq 0$, then the greatest common divisor of the two numbers is $|\alpha^2 + \beta^2 - 6\alpha\beta - a^2|$. Therefore $\gcd(f_n, g_n)$ divides $\alpha^2 + \beta^2 - 6\alpha\beta - a^2$. That is to say,

$$\gcd(G^B_{4n} - a,\ G^B_{4n+1} - 6a) \leq |\alpha^2 + \beta^2 - 6\alpha\beta - a^2|.$$

If $m = 4n + 2$ and $k = n$ in (3), we have, similarly

$$\gcd(G^B_{4n+2} - a,\ G^B_{4n+3} - 6a) \leq |\alpha^2 + \beta^2 - 6\alpha\beta - a^2|,$$

which completes the proof. □

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