



New Types of Multifunctions in ideal Topological Spaces via e - \mathcal{J} -Open sets and $\delta\beta$ - \mathcal{J} -Open Sets

Wadei Faris AL-Omeri, M.S. Md. Noorani, A. AL-Omari and T. Noiri

ABSTRACT: The purpose of the present paper is to introduce and investigate two new classes of continuous multifunctions called upper/lower e - \mathcal{J} -continuous multifunctions and upper/lower $\delta\beta_I$ -continuous multifunctions by using the concepts of e - \mathcal{J} -open sets and $\delta\beta_I$ -open sets. The class of upper/lower e - \mathcal{J} -continuous multifunctions is contained in that of upper/lower $\delta\beta_I$ -continuous multifunctions. Several characterizations and fundamental properties concerning upper/lower e - \mathcal{J} -continuity and upper/lower $\delta\beta_I$ -continuity are obtained.

Key Words: e - \mathcal{J} -open set, $\delta\beta_I$ -open set, upper (lower) e - \mathcal{J} -continuous multifunctions, upper (lower) $\delta\beta_I$ -continuous multifunctions

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1. Introduction

It is well known that the concept of continuity play a significant role in general and ideal topology as well as all branches of mathematics and quantum physics. This concept has been extended to the setting of multifunctions from an ideal topological space into a topological space by using weaker forms of open sets. In [18] the concepts of δ -semi-continuous functions and δ -pre-continuous functions are introduced as generalizations of semi-continuous functions due to Levine [13] and precontinuous functions due to Mashhour et al. [15]. Most of these weaker forms of continuity in topological spaces have been extended to multifunction [20]. Hatir [9] introduced the notion of $\delta\beta_I$ -open sets in an ideal topological space and, by using the sets, defined $\delta\beta_I$ -continuous functions. Quit recently, Al-Omeri et al. [3] introduced and investigated the notions of e - \mathcal{J} -open sets and e - \mathcal{J} -continuous functions.

In this paper, we introduce some new classes of continuous multifunctions, namely, upper/lower e - \mathcal{J} -continuous multifunctions and upper/lower $\delta\beta_I$ -continuous

multifunctions. And we obtain several characterizations and properties of these continuous multifunction.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X , the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively.

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [21] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is said to be δ -open [23] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a δ -open set is said to be δ -closed. A point $x \in X$ is called a δ -cluster point of A if $Int(Cl(U)) \cap A \neq \emptyset$ for each open set U containing x . The set of all δ -cluster points of A is called the δ -closure of A and is denoted by $Cl_\delta(A)$ [23]. The set δ -interior of A [23] is the union of all regular open sets of X contained in A and is denoted by $Int_\delta(A)$. A is said to be δ -open if $Int_\delta(A) = A$.

The subject of ideals in topological spaces has been studied by Kuratowski [12] and Vaidyanathaswamy [22]. Jankovic and Hamlett [11] investigated further properties of ideal topological spaces. An ideal \mathcal{I} on a topological spaces (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions: $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. The applications to various fields were further investigated by Jankovic and Hamlett [11]. A topological space (X, τ) with an ideal \mathcal{I} on X is called an ideal topological space and is denoted by (X, τ, \mathcal{I}) . If $\wp(X)$ is the set of all subsets of X , a set operator $(.)^* : \wp(X) \rightarrow \wp(X)$, called a local function [24, 11] of A with respect to τ and \mathcal{I} is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$ topology, finer than τ , is defined by $Cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [11]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X .

A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be R - \mathcal{I} -open (resp. R - \mathcal{I} -closed) [25] if $A = Int(Cl^*(A))$ (resp. $A = Cl^*(Int(A))$). A point $x \in X$ is called a δ - \mathcal{I} -cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set U containing x . The family of all δ - \mathcal{I} -cluster points of A is called the δ - \mathcal{I} -closure of A and is denoted by $\delta Cl_I(A)$. The δ - \mathcal{I} -interior of A is defined by the union of all R - \mathcal{I} -open sets of X contained in A and is denoted by $\delta Int_I(A)$. A is said to be δ - \mathcal{I} -closed if $\delta Cl_I(A) = A$ [25].

A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be e - \mathcal{I} -open [3] if $A \subset Int(\delta Cl_I(A)) \cup Cl(\delta Int_I(A))$. The complement of an e - \mathcal{I} -open set is said to

be e - \mathcal{I} -closed. The intersection of all e - \mathcal{I} -closed sets containing A is called the e - \mathcal{I} -closure of A [3] and is denoted by $Cl_e^*(A)$. The union of all e - \mathcal{I} -open sets of X contained in A is called the e - \mathcal{I} -interior [3] of A and is denoted by $Int_e^*(A)$.

A subset A of an ideal topological space (X, τ, \mathcal{I}) is said to be $\delta\beta_I$ -open [9] if $A \subset Cl(Int(\delta Cl_I(A)))$. The complement of a $\delta\beta_I$ -open set is said to be $\delta\beta_I$ -closed [10]. The intersection of all $\delta\beta_I$ -closed sets containing A is called the $\delta\beta_I$ -closure of A [10] and is denoted by $\delta\beta Cl_I(A)$. The union of all $\delta\beta_I$ -open sets of X contained in A is called the $\delta\beta_I$ -interior [9] of A and is denoted by $\delta\beta Int_I(A)$. The family of all e - \mathcal{I} -open (resp. e - \mathcal{I} -closed, $\delta\beta_I$ -open, $\delta\beta_I$ -closed) sets of X containing a point $x \in X$ is denoted by $EIO(X, x)$ (resp. $EIC(X, x)$, $\delta\beta IO(X, x)$, $\delta\beta IC(X, x)$). The family of all e - \mathcal{I} -open (resp. e - \mathcal{I} -closed, $\delta\beta_I$ -open, $\delta\beta_I$ -closed) subsets of X is denoted by $EIO(X, \tau)$ (resp. $EIC(X, \tau)$, $\delta\beta IO(X, \tau)$, $\delta\beta IC(X, \tau)$).

By a multifunction $F : X \longrightarrow Y$, we mean a correspondence from each point $x \in X$ to a nonempty set $F(x)$ of Y . For a multifunction $F : X \longrightarrow Y$, the upper and lower inverse of any subset B of Y are denoted by $F^+(B)$ and $F^-(B)$, respectively, where $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \phi\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A multifunction $F : X \longrightarrow Y$ is called a surjection if $F(X) = Y$. A multifunction $F : X \longrightarrow Y$ is called upper semi-continuous (rename upper continuous) (resp. lower semi-continuous (rename lower continuous) [1] if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y .

3. e - \mathcal{I} -Continuous Multifunctions

Definition 3.1. A multifunction $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is said to be:

1. upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous) if at a point $x \in X$ if for each open set V of Y such that $F(x) \subset V$, there exists $U \in EIO(X, x)$ (resp. $U \in \delta\beta IO(X, x)$) such that $F(U) \subset V$,
2. lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous) at a point $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \phi$, there exists $U \in EIO(X, x)$ (resp. $U \in \delta\beta IO(X, x)$) such that $F(u) \cap V \neq \phi$ for every $u \in U$,
3. upper/lower e - \mathcal{I} -continuous (resp. upper/lower $\delta\beta_I$ -continuous) if F has this property at each point of X .

Definition 3.2. A subset U of an ideal topological space (X, τ, \mathcal{I}) is called an e - \mathcal{I} -neighborhood [3] (resp. $\delta\beta_I$ -neighborhood) of a point $x \in X$ if there exists an e - \mathcal{I} -open (resp. $\delta\beta_I$ -open) set A of X such that $x \in A \subset U$.

Theorem 3.3. For a multifunction $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$, the following statements are equivalent:

1. F is upper e - \mathcal{I} -continuous;
2. $F^+(V) \in EIO(X, \tau)$ for every open set V of Y ;

3. $F^-(V) \in EIC(X, \tau)$ for every closed set V of Y ;
4. $eCl^*(F^-(B)) \subset F^-(Cl(B))$ for every $B \subset Y$;
5. For each point $x \in X$ and each neighborhood V of $F(x)$, $F^+(V)$ is an e - \mathcal{J} -neighborhood of x ;
6. For each point $x \in X$ and each neighborhood V of $F(x)$, there exists an e - \mathcal{J} -neighborhood U of x such that $F(U) \subset V$;
7. $Cl(\delta Int_I(F^-(B))) \cap Int(\delta Cl_I(F^-(B))) \subset F^-(Cl(B))$ for every subset B of Y ;
8. $F^+(Int(B)) \subset Int(\delta Cl_I(F^+(B))) \cup Cl(\delta Int_I(F^+(B)))$ for every subset B of Y .

Proof: (1) \Rightarrow (2): Let V be any open set of Y and $x \in F^+(V)$, then there exists $U \in EIO(X, x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset Cl(\delta Int_I(U)) \cup Int(\delta Cl_I(U)) \subset Cl(\delta Int_I(F^+(V))) \cup Int(\delta Cl_I(F^+(V)))$. We have $F^+(V) \subset Cl(\delta Int_I(F^+(V))) \cup Int(\delta Cl_I(F^+(V)))$ and hence $F^+(V) \in EIO(X, \tau)$.

(2) \Rightarrow (3): From the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ for every subset B of Y , this proof follows immediately.

(3) \Rightarrow (4): Since $Cl(B)$ is closed in Y for any subset B of Y , $F^-(Cl(B))$ is e - \mathcal{J} -closed in X . Therefore, we obtain $eCl^*(F^-(B)) \subset F^-(Cl(B))$.

(4) \Rightarrow (3): Let V be any closed set of Y . Then we have: $eCl^*(F^-(V)) \subset F^-(Cl(V)) = F^-(V)$. This shows that $F^-(V)$ is e - \mathcal{J} -closed in X .

(2) \Rightarrow (5): Let $x \in X$ and V be a neighborhood of $F(x)$. Then there exists an open set A of Y such that $F(x) \subset A \subset V$. Therefore, we obtain $x \in F^+(A) \subset F^+(V)$. Since $F^+(A) \in EIO(X, \tau)$, Then $F^+(V)$ is an e - \mathcal{J} -neighborhood of x .

(5) \Rightarrow (6): Let $x \in X$ and V be a neighborhood of $F(x)$. Put $U = F^+(V)$, Then U is an e - \mathcal{J} -neighborhood of x and $F(U) \subset V$.

(6) \Rightarrow (1): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighborhood of $F(x)$. There exists an e - \mathcal{J} -neighborhood U of x such that $F(U) \subset V$. Therefore there exists $A \in EIO(X, \tau)$ such that $x \in A \subset U$ and hence $F(A) \subset V$.

(3) \Rightarrow (7): Since $Cl(B)$ is closed in Y for any subset B of Y , then by (3), we have $F^-(Cl(B))$ is e - \mathcal{J} -closed in X . Therefore, $Cl(\delta Int_I(F^-(B))) \cap Int(\delta Cl_I(F^-(B))) \subset Cl(\delta Int_I(F^-(Cl(B)))) \cap Int(\delta Cl_I(F^-(Cl(B)))) \subset F^-(Cl(B))$.

(7) \Rightarrow (8): By replacing $Y \setminus B$ instead of B in (7), we have $Cl(\delta Int_I(F^-(Y \setminus B))) \cap Int(\delta Cl_I(F^-(Y \setminus B))) \subset F^-(Cl(Y \setminus B))$, and $F^+(Int(B)) \subset Int(\delta Cl_I(F^+(B))) \cup Cl(\delta Int_I(F^+(B)))$.

(8) \Rightarrow (2): Let V be any open set of Y . Then, by using (8) we have $F^+(V) \in EIO(X, \tau)$ and this completes the proof. \square

Theorem 3.4. For a multifunction $F : (X, \tau, \mathcal{J}) \longrightarrow (Y, \sigma)$, the following statements are equivalent:

1. F is lower e - \mathcal{J} -continuous;
2. $F^-(V) \in EIO(X, \tau)$ for every open set V of Y ;
3. $F^+(V) \in EIC(X, \tau)$ for every closed set V of Y ;
4. $eCl^*(F^+(B)) \subset F^+(Cl(B))$ for every $B \subset Y$;
5. $F(eCl^*(A) \subset Cl(F(A))$ for every $A \subset X$;
6. $Cl(\delta Int_I(F^+(B))) \cap Int(\delta Cl_I(F^+(B))) \subset F^+(Cl(B))$ for every subset B of Y ;
7. $F^-(Int(B)) \subset Int(\delta Cl_I(F^-(B))) \cup Cl(\delta Int_I(F^-(B)))$ for every subset B of Y .

Proof: This proof is similar to that of Theorem 3.3. Thus it is omitted. \square

Theorem 3.5. For a multifunction $F : (X, \tau, \mathcal{J}) \longrightarrow (Y, \sigma)$, the following statements are equivalent:

1. F is upper $\delta\beta_I$ -continuous;
2. $F^+(V) \in \delta\beta IO(X, \tau)$ for every open set V of Y ;
3. $F^-(V) \in \delta\beta IC(X, \tau)$ for every closed set V of Y ;
4. $\delta\beta Cl_I(F^-(B)) \subset F^-(Cl(B))$ for every $B \subset Y$;
5. For each point $x \in X$ and each neighborhood V of $F(x)$, $F^+(V)$ is a $\delta\beta Cl_I$ -neighborhood of x ;
6. For each point $x \in X$ and each neighborhood V of $F(x)$, there exists a $\delta\beta Cl_I$ -neighborhood U of x such that $F(U) \subset V$;
7. $Int(Cl(\delta Int_I(F^-(B)))) \subset F^-(Cl(B))$ for every subset B of Y ;
8. $F^+(Int(B)) \subset Cl(Int(\delta Cl_I(F^+(B))))$ for every subset B of Y .

Proof: (1) \Rightarrow (2): Let V be any open set of Y and $x \in F^+(V)$, then there exists $U \in \delta\beta IO(X, x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset Cl(Int(\delta Cl_I(U))) \subset Cl(Int(\delta Cl_I(F^+(V))))$. Then we have $F^+(V) \subset Cl(Int(\delta Cl_I(F^+(V))))$ and hence $F^+(V) \in \delta\beta IO(X, \tau)$.

(2) \Rightarrow (3): From the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ for every subset B of Y , this proof follows immediately.

(3) \Rightarrow (4): Since $Cl(B)$ is closed in Y for any subset B of Y , $F^-(Cl(B))$ is $\delta\beta_I$ -closed in X . Therefore, we obtain $\delta\beta Cl_I(F^-(B)) \subset F^-(Cl(B))$.

(4) \Rightarrow (3): Let V be any closed set of Y . Then we have $\delta\beta Cl_I(F^-(V)) \subset F^-(Cl(V)) = F^-(V)$. This shows that $F^-(V)$ is $\delta\beta_I$ -closed in X .

(2) \Rightarrow (5): Let $x \in X$ and V be a neighborhood of $F(x)$. Then there exists an open

set A of Y such that $F(x) \subset A \subset V$. Therefore, we obtain $x \in F^+(A) \subset F^+(V)$. Since $F^+(A) \in \delta\beta IO(X, \tau)$, Then $F^+(V)$ is a $\delta\beta_I$ -neighborhood of x .

(5) \Rightarrow (6): Let $x \in X$ and V be a neighborhood of $F(x)$. Put $U = F^+(V)$, Then U is a $\delta\beta_I$ -neighborhood of x and $F(U) \subset V$.

(6) \Rightarrow (1): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighborhood of $F(x)$. There exists a $\delta\beta_I$ -neighborhood U of x such that $F(U) \subset V$. Therefore there exists $A \in \delta\beta IO(X, \tau)$ such that $x \in A \subset U$ and hence $F(A) \subset V$.

(3) \Rightarrow (7): Since $Cl(B)$ is closed in Y for any subset B of Y , then by (3), we have $F^-(Cl(B))$ is $\delta\beta_I$ -closed in X . Therefore, $Int(Cl(\delta Int_I(F^-(B)))) \subset Int(Cl(\delta Int_I(F^-(Cl(B))))) \subset F^-(Cl(B))$.

(7) \Rightarrow (8): By replacing $Y \setminus B$ instead of B in (7), we have $Int(Cl(\delta\beta Int_I(F^-(Y \setminus B)))) \subset F^-(Cl(Y \setminus B))$ and $F^+(Int(B)) \subset Cl(Int(\delta Cl_I(F^+(B))))$.

(8) \Rightarrow (2): Let V be any open set of Y . Then, by using (8) we have $F^+(V) \in \delta\beta IO(X, \tau)$ and this completes the proof. \square

Theorem 3.6. *For a multifunction $F : (X, \tau, \mathcal{J}) \longrightarrow (Y, \sigma)$, the following statements are equivalent:*

1. F is lower $\delta\beta_I$ -continuous;
2. $F^-(V) \in \delta\beta IO(X, \tau)$ for every open set V of Y ;
3. $F^+(V) \in \delta\beta IC(X, \tau)$ for every closed set V of Y ;
4. $\delta\beta Cl_I(F^+(B)) \subset F^+(Cl(B))$ for every $B \subset Y$;
5. $F(\delta\beta Cl_I(A)) \subset Cl(F(A))$ for every $A \subset X$;
6. $Cl(Int(\delta Cl_I(F^+(B)))) \subset F^+(Cl(B))$ for every subset B of Y ;
7. $F^-(Int(B)) \subset Cl(Int(\delta Cl_I(F^-(B))))$ for every subset B of Y .

Proof: This proof is similar to that of Theorem 3.5. Thus it is omitted. \square

Theorem 3.7. *Let $F : (X, \tau, \mathcal{J}) \longrightarrow (Y, \sigma)$ and $G : (Y, \sigma) \longrightarrow (Z, \eta)$ be multifunctions. If $F : (X, \tau, \mathcal{J}) \longrightarrow (Y, \sigma)$ is upper/lower $e\text{-}\mathcal{J}$ -continuous (resp. upper/lower $\delta\beta_I$ -continuous) and $G : (Y, \sigma) \longrightarrow (Z, \eta)$ is upper/lower continuous, then $F \circ G : X \longrightarrow Z$ is upper/lower $e\text{-}\mathcal{J}$ -continuous (resp. upper/lower $\delta\beta_I$ -continuous).*

Proof: Let V be any open subset of Z . Using the definition of $G \circ F$, we obtain $(G \circ F)^+(V) = F^+(G^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V))$). Since G is upper/lower continuous, it follows that $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is upper/lower $e\text{-}\mathcal{J}$ -continuous (resp. upper/lower $\delta\beta_I$ -continuous), it follows that $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is an $e\text{-}\mathcal{J}$ -open (resp. $\delta\beta_I$ -open) set. This shows that $G \circ F$ is an upper/lower $e\text{-}\mathcal{J}$ -continuous (resp. upper/lower $\delta\beta_I$ -continuous) multifunction. \square

Every upper e - \mathcal{I} -continuous multifunction is upper $\delta\beta_I$ -continuous but the converse need not be true as shown by the following example.

Example 3.8. Let $X = Y = \{a, b, c, d\}$. Define a topology $\tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ on X , and an ideal $\mathcal{I} = \{\phi, \{b\}\}$, and a topology $\sigma = \{\phi, Y, \{d\}, \{b, d\}, \{a, b, d\}\}$ on Y . Let $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be a multifunction defined as follows: $F(x) = \{x\}$ for each $x \in X$. Then F is upper $\delta\beta_I$ -continuous but it is not upper e - \mathcal{I} -continuous because $\{b, d\}$ is open in (Y, σ) and $F^+(\{b, d\}) = \{b, d\}$ is not e - \mathcal{I} -open in (X, τ, \mathcal{I}) .

Recall that, for a multifunction $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$, the graph multifunction $G_F : X \longrightarrow X \times Y$ is defined as follows: $G_F(x) = (\{x\} \times F(x))$ for every $x \in X$.

Lemma 3.9. [17] For a multifunction $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$, the following hold:

1. $G_F^+(A \times B) = A \cap F^+(B)$ and $G_F^-(A \times B) = A \cap F^-(B)$.

for any subsets $A \subset X$ and $B \subset Y$.

Theorem 3.10. Let $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous) if and only if $G_F : X \longrightarrow X \times Y$ is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous).

Proof: (Necessity) Suppose that $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous). Let $x \in X$ and H be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset H$. The family of $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$ and $F(x)$ is compact. Therefore, there exist a finite number of points, says, y_1, y_2, \dots, y_n in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$. Set $U = \cap\{U(y_i) : 1 \leq i \leq n\}$ and $V = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then U and V are open in X and Y , respectively, and $\{x\} \times F(x) \subset U \times V \subset H$. Since F is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous), there exists $U_o \in EIO(X, x)$ (resp. $U_o \in \delta\beta IO(X, x)$) such that $F(U_o) \subset V$. By Lemma 3.9, we have $U \cap U_o \subset U \cap F^+(V) = G_F^+(U \times V) \subset G_F^+(H)$. Therefore, we obtain $U \cap U_o \in EIO(X, x)$ this follows from Theorem 2.13 [3] (resp. $U \cap U_o \in \delta\beta IO(X, x)$) and $G_F(U \cap U_o) \subset H$. This shows that G_F is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous).

(Sufficiency). Suppose that $G_F : X \longrightarrow X \times Y$ is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous). Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in EIO(X, x)$ (resp. $U \in \delta\beta IO(X, x)$) such that $G_F(U) \subset X \times V$. Therefore, by Lemma 3.9 we have $U \subset G_F^+(X \times V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous). \square

Theorem 3.11. A multifunction $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous) if and only if $G_F : X \longrightarrow X \times Y$ is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous).

Proof: (Necessity) Suppose that $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous). Let $x \in X$ and H be any open set of $X \times Y$ such that $x \in G_F^-(H)$. Since $H \cap (\{x\} \times F(x)) \neq \emptyset$, There exists $y \in F(x)$ such that $(x, y) \in H$ and hence $(x, y) \in U \times V \subset H$ for some open sets $U \subset X$ and $V \subset Y$. Since $F(x) \cap V \neq \emptyset$, there exists $U_o \in EIO(X, x)$ (resp. $U_o \in \delta\beta IO(X, x)$) such that $U_o \subset F^-(V)$. By Lemma 3.9 we have $U \cap U_o \subset U \cap F^-(V) = G_F^-(U \times V) \subset G_F^-(H)$. Moreover, $x \in U \cap U_o \in EIO(X, \tau)$ (resp. $x \in U \cap U_o \in \delta\beta IO(X, \tau)$) and hence G_F is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous).

(Sufficiency). Suppose that G_F is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous). Let $x \in X$ and V be an open set in Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous). There exists $U \in EIO(X, x)$ (resp. $U \in \delta\beta IO(X, x)$) such that $U \subset G_F^-(X \times V)$. By Lemma 3.9 we obtain $U \subset F^-(V)$. This shows that F is lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous). \square

4. Some properties of upper (lower) e - \mathcal{I} ($\delta\beta_I$)-Continuous Multifunctions

Definition 4.1. An ideal topological space (X, τ, \mathcal{I}) is said to be e - \mathcal{I} -compact [3] (resp. $\delta\beta_I$ -compact) if every cover of X by e - \mathcal{I} -open (resp. $\delta\beta_I$ -open) sets has a finite subcover.

Theorem 4.2. Let $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be an upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous) surjective multifunction such that $F(x)$ is compact for each $x \in X$. If (X, τ, \mathcal{I}) is e - \mathcal{I} -compact (resp. $\delta\beta_I$ -compact), then (Y, σ) is compact.

Proof: Let $\{V_\lambda : \lambda \in \Delta\}$ be an open cover of Y . For each $x \in X$, $F(x)$ is compact and there exists a finite subset $\Delta(x)$ of Δ such that $F(x) \subset \cup\{V_\lambda : \lambda \in \Delta(x)\}$. Set $V(x) = \cup\{V_\lambda : \lambda \in \Delta(x)\}$. Since F is upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous), there exists $U(x) \in EIO(X, \tau)$ (resp. $U(x) \in \delta\beta IO(X, \tau)$) containing x such that $F(U(x)) \subset V(x)$. The family $\{U(x) : x \in X\}$ is an e - \mathcal{I} -open (resp. $\delta\beta_I$ -open) cover of X and there exist a finite number of points, say, x_1, x_2, \dots, x_n in X such that $X = \cup\{U(x_i) : 1 \leq i \leq n\}$. Therefore, we have $Y = F(X) = F(\cup_{i=1}^n U(x_i)) = \cup_{i=1}^n F(U(x_i)) \subset \cup_{i=1}^n V(x_i) = \cup_{i=1}^n \cup_{\lambda \in \Delta(x_i)} V_\lambda$. This shows that Y is compact. \square

Definition 4.3. Let A be a subset of an ideal topological space (X, τ, \mathcal{I}) . The e - \mathcal{I} -frontier [4] (resp. $\delta\beta_I$ -frontier) of A , denoted by $e\text{-}\mathcal{I}\text{-Fr}(A)$ (resp. $\delta\beta_I\text{-Fr}(A)$), is defined by $e\text{-}\mathcal{I}\text{-Fr}(A) = Cl_e^*(A) \cap Cl_e^*(X - A) = Cl_e^*(A) \setminus Int_e^*(A)$. (resp. $\delta\beta_I\text{-Fr}(A) = \delta\beta Cl_I(A) \cap \delta\beta Cl_I(X - A) = \delta\beta Cl_I(A) \setminus \delta\beta Int_I(A)$).

Theorem 4.4. Let $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be a multifunction. The set of all points x of X at which F is not upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous) is identical with the union of the e - \mathcal{I} -frontier (resp. $\delta\beta_I$ -frontier) of the upper inverse images of open sets containing $F(x)$.

Proof: Let $x \in X$ at which F is not upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous). Then, there exists an open set V of Y containing $F(x)$ such that $U \cap (X \setminus F^+(V)) \neq \emptyset$ for every $U \in EIO(X, x)$ (resp. $U \in \delta\beta IO(X, x)$). Therefore, $x \in Cl_e^*(X \setminus F^+(V)) = X \setminus Int_e^*(F^+(V))$ [resp. $x \in \delta\beta Cl_I(X \setminus F^+(V)) = X \setminus \delta\beta_I Int(F^+(V))$] and $x \in F^+(V)$. Thus, we obtain $x \in e\text{-}\mathcal{I}\text{-}Fr(F^+(V))$ (resp. $x \in \delta\beta_I\text{-}Fr(F^+(V))$).

Conversely, let F be upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous) at x . Suppose that V is an open set of Y containing $F(x)$ such that $x \in e\text{-}\mathcal{I}\text{-}Fr(F^+(V))$ (resp. $x \in \delta\beta_I\text{-}Fr(F^+(V))$). There exists $U \in EIO(X, x)$ (resp. $U \in \delta\beta IO(X, x)$) such that $U \in F^+(V)$. Hence $x \in Int_e^*(F^+(V))$ (resp. $\delta\beta_I Int(F^+(V))$). This is a contradiction and hence F is not upper e - \mathcal{I} -continuous (resp. upper $\delta\beta_I$ -continuous) at x . \square

Theorem 4.5. *Let $F : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ be a multifunction. The set of all points x of X at which F is not lower e - \mathcal{I} -continuous (resp. lower $\delta\beta_I$ -continuous) is identical with the union of the e - \mathcal{I} -frontier (resp. $\delta\beta_I$ -frontier) of the lower inverse images of open sets containing $F(x)$.*

Proof: The proof is shown similarly as in Theorem 4.4. \square

In the following $(D, >)$ is a directed set, (F_λ) is a net of a multifunction $F_\lambda : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ for every $\lambda \in D$ and F is a multifunction from X into Y .

Definition 4.6. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from X to Y . A multifunction $F^* : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ defined as follows: for each $x \in X$, $F^*(x) = \{y \in Y : \text{for each open neighborhood } V \text{ of } y \text{ and each } \eta \in D, \text{ there exists } \lambda \in D \text{ such that } \lambda > \eta \text{ and } V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit [5] of the net $(F_\lambda)_{\lambda \in D}$.*

Definition 4.7. *A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper e - \mathcal{I} -(resp. equally upper $\delta\beta_I$ -) continuous at $x_0 \in X$ if for every open set V_λ containing $F_\lambda(x_0)$, there exists $U \in EIO(X, x_0)$ (resp. $U \in \delta\beta IO(X, x_0)$) such that $F_\lambda(U) \subset V_\lambda$ for all $\lambda \in D$.*

Theorem 4.8. *Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from a ideal topological space (X, τ, \mathcal{I}) into a compact topological space (Y, σ) . If the following are satisfied:*

1. $\bigcup\{F_\eta(x) : \eta > \lambda\}$ is closed in Y for each $\lambda \in D$ and each $x \in X$,
2. $(F_\lambda)_{\lambda \in D}$ is equally upper e - \mathcal{I} -(resp. equally upper $\delta\beta_I$ -) continuous on X ,

then F^ upper e - \mathcal{I} -(resp. upper $\delta\beta_I$ -) continuous on X .*

Proof: From Definition 4.6 and part (1), we have $F^*(x) = \bigcap\{(\bigcup\{F_\eta(x) : \eta > \lambda\}) : \lambda \in D\}$. Since the net $(\bigcup\{F_\eta(x) : \eta > \lambda\})_{\lambda \in D}$ is a family of closed sets having the finite intersection property and Y is compact, it follows that $F^*(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let $V \in \sigma$ such that $V \neq Y$ and $F^*(x_0) \subset V$. Since $F^*(x_0) \cap (Y \setminus V) = \emptyset$, $F^*(x_0) \neq \emptyset$, and $(Y \setminus V) \neq \emptyset$, $\bigcap\{(\bigcup\{F_\eta(x_0) : \eta > \lambda\}) : \lambda \in D\} \cap (Y \setminus V) = \emptyset$, $\bigcap\{(\bigcup\{F_\eta(x_0) : \eta > \lambda\}) : \lambda \in D\} \subset V$.

$\lambda \in D\} \cap (Y \setminus V) = \phi$ and hence $\bigcap\{(\bigcup\{F_\eta(x_0) \cap (Y \setminus V) : \eta > \lambda\}) : \lambda \in D\} = \phi$. Since Y is compact and the family $\bigcap\{(\bigcup\{F_\eta(x_0) \cap (Y \setminus V) : \eta > \lambda\}) : \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that for each $\eta \in D$ with $\eta > \lambda$ we have $F_\eta(x_0) \cap (Y \setminus V) = \phi$; hence $F_\eta(x_0) \subset V$. Since the net $(F_\lambda)_{\lambda \in D}$ is equally upper e - \mathcal{J} -(resp. equally upper $\delta\beta_I$ -) continuous on X , there exists $U \in EIO(X, x_0)$ (resp. $U \in \delta\beta IO(X, x_0)$) such that $F_\eta(U) \subset V$ for each $\eta > \lambda$ hence $F_\eta(x) \cap (Y \setminus V) = \phi$ for each $x \in U$. Then we have $\bigcap\{(\bigcup\{F_\eta(x) \cap (Y \setminus V) : \eta > \lambda\}) : \lambda \in D\} = \phi$; and hence $\bigcap\{(\bigcup\{F_\eta(x) : \eta > \lambda\}) : \lambda \in D\} \cap (Y \setminus V) = \phi$. This implies that $F^*(U) \subset V$. If $V = Y$, then it is clear that for each $U \in EIO(X, x_0)$ (resp. $U \in \delta\beta IO(X, x_0)$) we have $F^*(U) \subset V$. Hence F^* is upper e - \mathcal{J} -(resp. upper $\delta\beta_I$ -) continuous at x_0 . Since x_0 is arbitrary, the proof completes. \square

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W. AL-Omeri

*School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia*

and

M.S.Md. Noorani

*School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia*

and

A. AL-Omari

*Department of Mathematics, Faculty of Science
Al AL-Bayat University, P.O.Box 130095, Mafrq25113, Jordan*

and

T. Noiri

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken 869-5142, JAPAN.