



Some Characterizations of Osculating Curves in the Lightlike Cone

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ABSTRACT: In this paper, we give the first kind and second kind osculating curves in the lightlike cone. In addition we characterize osculating curves in terms of their curvature functions.

Key Words: Osculating curve, lightlike cone.

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1. Introduction

As is well-known, semi-Riemannian manifold has played a key role in the development of general relativity. Because the physical events space is represented by a semi-Riemannian manifold. A semi-Riemannian manifold has three causal types of submanifolds; spacelike, timelike and lightlike depending on the character of the induced metric on the tangent space [1]. Due to the degeneracy of the metric, the study of lightlike submanifold have attracted the attention of many scientist.

Researchers use lightlike hypersurfaces in order to show a class of lightlike hypersurfaces came from the physically significant homogeneous spacetime manifolds of general relativity [2].

On the other hand in special relativity, a lightlike cone is the surface describing the temporal evolution of flash of light in Minkowski spacetime. Many studies have been made on curves in the lightlike cone by many mathematicians. For example, in [3], Liu studied curves in the lightlike cone and in [4], Liu and Mong gave representation formulas of curves in a Two and Three Dimensional Lightlike Cone. Furthermore, in [5], Külahcı and others (the authors) studied AW(k)- type curves in the 3-dimensional null cone.

Another research area is the characterizations of osculating curves [6,7]. In this paper we are concerned with osculating curves in the lightlike cone.

2. Curves in the lightlike cone \mathbb{Q}^{n+1}

In the following, we use the notations and concepts from [3,4] unless otherwise stated.

Let E_q^m be the m -dimensional pseudo-Euclidean space with the metric

$$\tilde{G}(x, y) = \langle X, Y \rangle = \sum_{i=1}^{m-q} x_i y_i - \sum_{j=m-q+1}^m x_j y_j$$

where $X = (x_1, x_2, \dots, x_m), Y = (y_1, y_2, \dots, y_m) \in E_q^m$, E_q^m is a flat pseudo-Riemannian manifold of signature $(m-q, q)$.

Let M be a submanifold of E_q^m . If the pseudo-Riemannian metric \tilde{G} of E_q^m induces a pseudo-Riemannian metric \tilde{G} (respectively, a Riemannian metric, a degenerate quadratic form) on M , then M is called timelike (respectively, spacelike, degenerate) submanifold of E_q^m .

Let c be a fixed point in E_q^m and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_q^n(c, r) = \{x \in E_q^{n+1} : \tilde{G}(x - c, x - c) = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$H_q^n(c, r) = \{x \in E_{q+1}^{n+1} : \tilde{G}(x - c, x - c) = -r^2\};$$

the pseudo-Riemannian null cone (quadratic cone) is defined by

$$\mathbb{Q}_q^n(c, r) = \{x \in E_q^{n+1} : \tilde{G}(x - c, x - c) = 0\}.$$

It is well known that $S_q^n(c, r)$ is a complete pseudo-Riemannian hypersurface of signature $(n-q, q)$, $q \geq 1$ in E_q^{n+1} with constant sectional curvature r^{-2} ; $H_q^n(c, r)$ is a complete pseudo-Riemannian hypersurface of signature $(n-q, q)$, $q \geq 1$ in E_{q+1}^{n+1} with constant sectional curvature $-r^{-2}$; $\mathbb{Q}_q^n(c)$ is a degenerate hypersurface in E_q^{n+1} . The spaces E_q^n , $S_q^n(c, r)$, $H_q^n(c, r)$ and $\mathbb{Q}_q^n(c)$ are called pseudo-Riemannian space form. The point c is called the center of $S_q^n(c, r)$, $H_q^n(c, r)$ and $\mathbb{Q}_q^n(c)$. When $c = 0$ and $q = 1$, we simply denote $\mathbb{Q}_1^n(0)$ by \mathbb{Q}^n and call it the lightlike cone (simply the light cone).

Let E_1^{n+2} be the $(n+2)$ -dimensional Minkowski space and \mathbb{Q}^{n+1} the null cone in E_1^{n+2} . A vector $\alpha \neq 0$ in E_1^{n+2} is called spacelike, timelike or null(lightlike), if $\langle \alpha, \alpha \rangle > 0$, $\langle \alpha, \alpha \rangle < 0$ or $\langle \alpha, \alpha \rangle = 0$, respectively. A frame field $\{e_1, e_2, \dots, e_{n+1}, e_{n+2}\}$ on E_1^{n+2} is called on asymptotic orthonormal frame field, if

$$\begin{aligned} \langle e_{n+1}, e_{n+1} \rangle &= \langle e_{n+2}, e_{n+2} \rangle = 0, \langle e_{n+1}, e_{n+2} \rangle = 1, \\ \langle e_{n+1}, e_i \rangle &= \langle e_{n+2}, e_i \rangle = 0, \langle e_i, e_j \rangle = \delta_{ij}, \quad i, j = 1, 2, \dots, n. \end{aligned}$$

We assume that the curve $x : I \rightarrow \mathbb{Q}^{n+1} \subset E_1^{n+1}$, $t \rightarrow x(t) \in \mathbb{Q}^{n+1}$, is a regular curve in \mathbb{Q}^{n+1} . In the following, we always assume that the curve is regular and $x'(t) = \frac{dx(t)}{dt}$, for all $t \in I \subset \mathbb{R}$.

Definition 2.1. A curve $x(t)$ in E_1^{n+2} is called a Frenet curve, if for all $t \in I$, the vector fields $x(t), x'(t), x''(t), \dots, x^{(n)}(t), x^{(n+1)}(t)$ are linearly independent and the vector fields $x(t), x'(t), x''(t), \dots, x^{(n)}(t), x^{(n+1)}(t), x^{(n+2)}(t)$ are linearly dependent, where $x^{(n)}(t) = \frac{d^n x(t)}{dt^n}$. Since $\langle x, x \rangle = 0$ and $\langle x, dx \rangle = 0$, $dx(t)$ is spacelike. Then the induced arc lenght(or simply the arc lenght) s of the curve $x(t)$ can be defined by

$$ds^2 = \langle dx(t), dx(t) \rangle$$

If we take the arc lenght s of the curve $x(t)$ as the parameter and denote $x(t) = x(t(s))$, then $x'(s) = \frac{dx}{ds}$ is a spacelike unit tangent vector field of the curve $x(s)$. Now we choose the vector $y(s)$, the spacelike normal space V^{n-1} of the curve $x(s)$ such that they satisfy the following conditions:

$$\begin{aligned} \langle x(s), y(s) \rangle &= 1, \\ \langle x(s), x(s) \rangle &= \langle y(s), y(s) \rangle = \langle x'(s), y(s) \rangle = 0, \\ V^{n-1} &= \{span_R\{x, y, x'\}\}^\perp, \\ Span_R\{x, y, x', V^{n-1}\} &= E_1^{n+2}, \end{aligned}$$

From the above explanations, we have the following remark.

Remark 2.2. From [4], for any asymptotic orthonormal frame $\{x, \alpha, \beta, y\}$ of the curve $x : I \rightarrow \mathbb{Q}^3 \subset E_1^4$ with

$$\begin{aligned} \langle x, x \rangle &= \langle y, y \rangle = \langle x, \alpha \rangle = \langle x, \beta \rangle = \langle y, \alpha \rangle = \langle y, \beta \rangle = \langle \alpha, \beta \rangle = 0, \\ \langle x, y \rangle &= \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1 \end{aligned} \quad (2.1.1)$$

the Frenet formulas read

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) + \lambda(s)\beta(s) - y(s) \\ \beta'(s) &= \tau(s)x(s) - \lambda(s)\alpha(s) \\ y'(s) &= -\kappa(s)\alpha(s) - \tau(s)\beta(s). \end{aligned} \quad (2.1.2)$$

Recall that orbitrary curve $x(s)$ in $\mathbb{Q}^3 \subset E_1^4$ is called osculating curve of the first or second kind if its position vector (with respect to some chosen origion) always lies in the orthogonal complement y^\perp or β^\perp , respectively [6], where $y^\perp = span\{\beta, \alpha, y\}$, and $\beta^\perp = span\{y, \alpha, x\}$.

Consequently the position vector of the osculating curve of the first and second kind satisfies the equations respectively

$$x(s) = \theta(s)\alpha(s) + \mu(s)\beta(s) + \gamma(s)y(s) \quad (2.1.3)$$

$$x(s) = \theta(s)x(s) + \mu(s)\alpha(s) + \gamma(s)y(s) \quad (2.1.4)$$

some differentiable functions $\theta(s)$, $\mu(s)$ and $\gamma(s)$. Since $\langle x(s), y(s) \rangle = 1$, the equation (2.1.3) easily implies a contradiction. Hence we can say that there isn't a osculating curve of the first kind in the lightlike cone $\mathbb{Q}^3 \subset E_1^4$.

3. Osculating Curves of the Second Kind in the Lightlike Cone \mathbb{Q}^3

In this section, we characterize osculating curves of the second kind in the lightlike cone $\mathbb{Q}^3 \subset E_1^4$ by using the components of their position vectors and the curvature functions.

Theorem 3.1. *Let $x(s)$ be a unit speed curve with the non-zero cone curvature functions $\kappa(s)$, $\tau(s)$, $\lambda(s)$ and if $\kappa \neq \frac{1}{2}(\frac{\tau}{\lambda})^2$, then $x(s)$ is congruent to an osculating curve of the second kind if and only if*

$$\{A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} (\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)\}' + A.\frac{\tau(s)}{\lambda(s)}.\kappa(s)e^{\int \frac{\tau(s)}{\lambda(s)} ds} = 0,$$

where $A \in \mathbb{R}$.

Proof: Assume that $x(s)$ is an osculating curve of the second kind with the cone curvature functions $\kappa(s)$, $\tau(s)$ and $\lambda(s)$ in the lightlike cone \mathbb{Q}^3 . Differentiating (2.1.4) with respect to s and using (2.1.2), we get

$$\begin{aligned} \alpha(s) &= (\theta'(s) + \mu(s)\kappa(s))x + (-\mu(s) + \gamma'(s))y + (\theta(s) + \mu'(s) - \gamma(s)\kappa(s))\alpha \\ &\quad + (\mu(s)\lambda(s) - \tau(s)\gamma(s))\beta. \end{aligned} \quad (3.1.1)$$

Exposing the inner product y, x, α, β of the both side of (3.1.1), respectively, we have

$$\begin{aligned} \theta'(s) + \mu(s)\kappa(s) &= 0 \\ -\mu(s) + \gamma'(s) &= 0 \\ \theta(s) + \mu'(s) - \gamma(s)\kappa(s) &= 1 \\ \mu(s)\lambda(s) - \tau(s)\gamma(s) &= 0. \end{aligned} \quad (3.1.2)$$

Using (3.1.2) and making necessary calculations, we get, for $A \in \mathbb{R}$

$$\begin{aligned} \gamma(s) &= A.e^{\int \frac{\tau(s)}{\lambda(s)} ds}, \\ \mu(s) &= \frac{\tau(s)}{\lambda(s)} A.e^{\int \frac{\tau(s)}{\lambda(s)} ds}, \\ \theta(s) &= 1 + A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} (\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2). \end{aligned} \quad (3.1.3)$$

Thus by using (3.1.3), $x(s)$ can be written as osculating curve of the second kind as follows:

$$\{A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} (\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)\}' + A.\frac{\tau(s)\kappa(s)}{\lambda(s)} e^{\int \frac{\tau(s)}{\lambda(s)} ds} = 0 \quad (3.1.4)$$

Conversely, let $x(s)$ be unit speed curve with the cone curvatures $\kappa(s)$, $\tau(s)$, $\lambda(s)$ and assume that $x(s)$ holds (3.1.4). From (2.1.4), we can write

$$\begin{aligned} Y(s) &= x(s) - (1 + A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} .(\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)).x \\ &\quad - \frac{\tau(s)}{\lambda(s)}.A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} .\alpha - A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} .y, \end{aligned}$$

and using (2.1.2) and (3.1.4), we can find $Y'(s) = 0$. Hence $Y(s) = \text{constant}$. Thus $x(s)$ is congruent to an osculating curve of the second kind. The proof is completed. \square

In particular, assume that the curvature functions $\kappa(s)$ and $\tau(s)$, $\lambda(s)$ of rectifying curve $x(s)$ in Q^3 are constant and different from zero and let be $\kappa \neq \frac{1}{2}(\frac{\tau}{\lambda})^2$. Then equation (3.1.4) easily implies a contradiction. Hence we can say the following theorem.

Theorem 3.2. *There is not an osculating curve lying fully in the lightlike cone Q^3 if $\kappa \neq \frac{1}{2}(\frac{\tau}{\lambda})^2$, so that the curvature functions are non-zero.*

But if any two of the curvature functions are constant, we can think the following statement:

Theorem 3.3. *Let $x(s)$ be unit speed curve in the lightlike cone Q^3 , with curvatures $\kappa(s)$, $\tau(s)$ and $\lambda(s)$. Then $x(s)$ is congruent to an osculating curve if and only if*

a) *If $\kappa(s) \neq \text{constant}$, $\tau, \lambda = \text{constant}$, for $A \in \mathbb{R}_0^+$,*

$$\kappa(s) = \frac{\tau^2}{2\lambda^2} + D.e^{-\frac{2\tau}{\lambda}s}, \quad D \in \mathbb{R}.$$

b) *If $\tau(s) \neq \text{constant}$, $\kappa, \lambda = \text{constant}$, for $A \in \mathbb{R}_0^+$,*

$$\frac{\tau''(s)}{\lambda} + \frac{3.\tau(s)\tau'(s)}{\lambda^2} + \frac{1}{\lambda^3}.\tau^3(s) - \frac{2\kappa}{\lambda}.\tau(s) = 0.$$

c) *If $\lambda(s) \neq \text{constant}$, $\kappa, \tau = \text{constant}$, for $A \in \mathbb{R}_0^+$,*

$$\lambda''(s)\lambda(s) - 2.(\lambda'(s))^2 + \tau.\lambda'(s) + 2.\kappa.\lambda^2(s) - \tau^2 = 0.$$

Proof: a) If $\kappa(s) \neq \text{constant}$, $\tau, \lambda = \text{constant}$, then from (3.1.4), for $A \in \mathbb{R}_0^+$, we can get

$$\frac{\tau}{\lambda} \cdot (\kappa(s) - (\frac{\tau}{\lambda})' - (\frac{\tau}{\lambda})^2) + (\kappa'(s) - (\frac{\tau}{\lambda})'' - (\frac{\tau^2}{\lambda^2})') + \frac{\tau\kappa(s)}{\lambda} = 0.$$

$$\kappa'(s) + \frac{2\tau}{\lambda}\kappa(s) - \frac{\tau^3}{\lambda^3} = 0.$$

By solving this equation, we can have

$$\kappa(s) = \frac{\tau^2}{2\lambda^2} + D.e^{-\frac{2\tau}{\lambda}s}, \quad D \in \mathbb{R}.$$

b) If $\tau(s) \neq \text{constant}$, $\kappa, \lambda = \text{constant}$, from (3.1.4), for $A \in \mathbb{R}_0^+$, we can obtain

$$\frac{\tau(s)}{\lambda} (\kappa - (\frac{\tau(s)}{\lambda})' - (\frac{\tau(s)}{\lambda})^2) + (\kappa' - (\frac{\tau(s)}{\lambda})'' - (\frac{\tau^2(s)}{\lambda^2})') + \frac{\tau(s)\kappa}{\lambda} = 0.$$

$$\frac{\tau''(s)}{\lambda} + \frac{3\tau(s)\tau'(s)}{\lambda^2} + \frac{1}{\lambda^3} \cdot \tau^3(s) - \frac{2\tau(s)\kappa}{\lambda} = 0.$$

c) If $\lambda(s) \neq \text{constant}$, $\kappa, \tau = \text{constant}$, then from (3.1.4), for $A \in \mathbb{R}_0^+$, we can get

$$\frac{\tau}{\lambda(s)} (\kappa - (\frac{\tau}{\lambda(s)})' - (\frac{\tau}{\lambda(s)})^2) + (\kappa' - (\frac{\tau}{\lambda(s)})'' - (\frac{\tau^2}{\lambda^2(s)})') + \frac{\tau\kappa}{\lambda(s)} = 0.$$

$$\lambda''(s)\lambda(s) - 2(\lambda'(s))^2 + \tau\lambda'(s) + 2\kappa\lambda^2(s) - \tau^2 = 0.$$

□

Theorem 3.4. If $x(s)$ be unit speed osculating curve in the lightlike cone \mathbb{Q}^3 so that curvatures $\kappa(s)$, $\tau(s)$, $\lambda(s)$ are non-zero. The following statements holds:

a) the distance function $\rho^2(x(s)) = 2A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} + d^2$, $A, d \in \mathbb{R}_0^+$.

Only if $\tau = 0$, the distance function ρ is constant and

b) the second binormal component, the principal normal component and the tangential component of the position vector of the curve is as follows respectively:

$$\begin{aligned}\langle x(s), y(s) \rangle &= \theta(s) = 1 + A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} \left(\kappa(s) - \left(\frac{\tau(s)}{\lambda(s)} \right)' - \left(\frac{\tau(s)}{\lambda(s)} \right)^2 \right), \\ \langle x(s), \alpha(s) \rangle &= \mu(s) = \frac{\tau(s)}{\lambda(s)} . A.e^{\int \frac{\tau(s)}{\lambda(s)} ds}, \\ \langle x(s), x(s) \rangle &= \gamma(s) = A.e^{\int \frac{\tau(s)}{\lambda(s)} ds}.\end{aligned}$$

In addition, if $\tau = 0$, tangential component of the position vector of the curve is constant.

c) the normal component of the position vector of the curve of the second kind is $\|x^N(s)\| = \mu(s) = 2\gamma - 2\gamma\theta + d^2$.

d) the curvatures $\kappa(s)$, $\tau(s)$ and $\lambda(s)$ satisfy the following equality

$$z(s) = A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} = F.e^{\int \frac{\eta'(s)}{\kappa(s) + \eta(s)} ds} = F.e^{\int \frac{(\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)' }{\kappa(s) + (\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)} ds},$$

where $A, F \in \mathbb{R}$, $\eta(s) = \{\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2\}$.

Proof: (a) Let $x(s)$ be unit speed osculating curve of the second kind with curvatures $\kappa(s)$, $\tau(s)$, $\lambda(s)$. The position vector of $x(s)$ holds (2.1.4). Also the functions $\theta(s)$, $\mu(s)$, $\gamma(s)$ holds (3.1.3).

$$\rho^2(x(s)) = \|x(s)\| = \langle x(s), x(s) \rangle = 2\gamma\theta + \mu^2. \quad (3.1.5)$$

By multiplying μ the both side of third equation in (3.1.2) and making necessary calculations, we can have

$$\mu^2 = 2\gamma - 2\gamma\theta + d^2, \quad d \in \mathbb{R}. \quad (3.1.6)$$

From (3.1.5),

$$\rho^2(x(s)) = 2\gamma + d^2. \quad (3.1.7)$$

Therefore, we have

$$\rho^2(x(s)) = 2A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} + d^2, \quad A, d \in \mathbb{R}_0^+$$

If $\tau = 0$, we have $\gamma(s) = A$. Hence $\rho(x(s)) = d = \text{constant}$.

b) From (2.1.4) and (3.1.3), we can write the following, for $A \in \mathbb{R}$,

$$\begin{aligned}\langle x(s), y(s) \rangle &= \theta(s) = 1 + A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} \left\{ \kappa(s) - \left(\frac{\tau(s)}{\lambda(s)} \right)' - \left(\frac{\tau(s)}{\lambda(s)} \right)^2 \right\} \\ \langle x(s), \alpha(s) \rangle &= \mu(s) = \frac{\tau(s)}{\lambda(s)} . A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} \\ \langle x(s), x(s) \rangle &= \gamma(s) = A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} .\end{aligned}$$

Conversely, if $\langle x(s), y(s) \rangle = \theta(s)$ holds, let's differentiate $\langle x'(s), y(s) \rangle + \langle x(s), y'(s) \rangle = \theta'(s)$, we have

$$\langle a(s), y(s) \rangle + \langle x(s), -\kappa(s)\alpha(s) - \tau(s)\beta(s) \rangle = \theta(s)'.$$

Furthermore, since $\langle x(s), \alpha(s) \rangle = \mu(s)$ and $\theta(s)' = -\kappa(s)\mu(s)$, we get

$$\tau(s) < x(s), \beta(s) > = 0 .$$

Since $\tau(s) \neq 0$, $\langle x(s), \beta(s) \rangle = 0$. This indicates that $x(s)$ is an osculating curve of the second kind.

Conversely, if $\langle x(s), \alpha(s) \rangle = \mu(s)$ holds, let's differentiate $\langle \alpha(s), \alpha(s) \rangle + \langle x(s), \alpha'(s) \rangle = 1 + \gamma(s)\kappa(s) - \theta(s)$. Furthermore since $\langle x(s), \alpha(s) \rangle = \mu(s)$, $\langle x(s), x(s) \rangle = \gamma(s)$, $\langle x(s), y(s) \rangle = \theta(s)$. So we can get

$$\lambda(s)\langle x(s), \beta(s) \rangle = 0. \quad (3.1.8)$$

Since $\lambda(s) \neq 0$, $\langle x(s), \beta(s) \rangle = 0$. Thus $x(s)$ is an osculating curve of the second kind.

Differentiating $\langle x(s), x(s) \rangle = \gamma(s)$ with respect to s , we can have

$$\langle x'(s), x(s) \rangle + \langle x(s), x'(s) \rangle = \gamma'(s) \implies 2\langle x(s), \alpha(s) \rangle = \gamma'(s) = \mu(s),$$

again differentiating with respect to s and making necessary calculations, we can get

$$\lambda(s)\langle x(s), \beta(s) \rangle = 0.$$

Since $\lambda(s) \neq 0$, $\langle x(s), \beta(s) \rangle = 0$. Thus $x(s)$ is an osculating curve of the second kind. This proves (b).

c) From the equation $x(s) = \theta(s)x(s) + \mu(s)\alpha(s) + \gamma(s)y(s)$, we can write

$$x^N(s) = \mu(s)\alpha(s) + \gamma(s)y(s).$$

Since $\|x^N(s)\|^2 = \langle x^N(s), x^N(s) \rangle = \mu^2(s)$ and from (3.1.6), we can get

$$\mu^2 = 2\gamma - 2\gamma\theta + d^2.$$

Since $\theta(s) = 1$, we can have $\mu^2 = d^2$. Thus $\|x^N(s)\| = d = \text{constant}$.
Conversely, let $\|x^N(s)\| = 2\gamma - 2\gamma\theta + d^2$. Then we can write

$$\langle x^N(s), x^N(s) \rangle = \langle \mu(s)\alpha(s) + \gamma(s)y(s), \mu(s)\alpha(s) + \gamma(s)y(s) \rangle = \mu^2(s). \quad (3.1.9)$$

Hence we can get

$$\|x^N(s)\| = \mu = \langle x(s), \alpha(s) \rangle. \quad (3.1.10)$$

Differentiating (3.1.10) with respect to s and since $\langle x(s), x(s) \rangle = \gamma(s)$, $\langle x(s), y(s) \rangle = \theta(s)$, we can get

$$\begin{aligned} \langle x(s), \alpha(s) \rangle' &= \mu' \\ \langle \alpha(s), \alpha(s) \rangle + \langle x(s), \kappa(s)x(s) + \lambda(s)\beta(s) - y(s) \rangle &= \mu' \\ 1 + \kappa(s)\langle x(s), x(s) \rangle - \langle x(s), y(s) \rangle + \lambda(s)\langle x(s), \beta(s) \rangle &= 1 + \gamma(s)\kappa(s) - \theta \\ \lambda(s)\langle x(s), \beta(s) \rangle &= 0, \end{aligned}$$

since $\lambda(s) \neq 0$, $\langle x(s), \beta(s) \rangle = 0$. Thus $x(s)$ is an osculating curve of the second kind. This proves (c).

d) Let $x(s)$ be an osculating curve of the second kind with curvatures $\kappa(s)$, $\tau(s)$, $\lambda(s)$. Since the position vector of $x(s)$ holds (2.1.4), we write for $A \in \mathbb{R}$

$$\{A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} (\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)\}' + A.\frac{\tau(s)}{\lambda(s)}\kappa(s)e^{\int \frac{\tau(s)}{\lambda(s)} ds} = 0.$$

Let $z(s) = A.e^{\int \frac{\tau(s)}{\lambda(s)} ds}$. Thus $z'(s) = A.\frac{\tau(s)}{\lambda(s)}e^{\int \frac{\tau(s)}{\lambda(s)} ds}$ and assume that $\eta(s) = \{\kappa(s) - (\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2\}$. So we can write the following equation

$$(z(s).\eta(s))' + z'(s)\kappa(s) = 0.$$

By solving this equation, we can get for $F \in \mathbb{R}$,

$$z(s) = F.e^{\int \frac{\eta'(s)}{\kappa(s) + \eta(s)} ds}.$$

Thus,

$$\begin{aligned}
z(s) &= A.e^{\int \frac{\tau(s)}{\lambda(s)} ds} = F.e^{\int \frac{\eta'(s)}{\kappa(s)+\eta(s)} ds} \\
&= F.e^{\int \frac{(\kappa(s)-(\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)' }{\kappa(s)+(\kappa(s)-(\frac{\tau(s)}{\lambda(s)})' - (\frac{\tau(s)}{\lambda(s)})^2)} ds}.
\end{aligned} \tag{3.1.12}$$

Conversely, if (d) holds, for $\kappa(s)$, $\tau(s)$, $\lambda(s) \neq 0$, let $x(s)$ be curve that holds the equation (3.1.12). Differentiating twice of the equation (3.1.12), we can obtain (3.1.4). Hence from Theorem 3.1, $x(s)$ is an osculating curve of the second kind. \square

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