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(-1,1) Metabelian rings *

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abstract: The strucure of the set of all non-nilpotent (-1,1) metabelian ring is studied. An additive basis of a free (-1,1) metabelian rings is constructed. It is proved that any identity in a non-nilpotent 2, 3-torsion free (-1,1) metabelian ring of degree greater than or equal to 6 is consequence of four defining identity of $\mathcal{M}$ where $\mathcal{M}$ is the metabelian (-1,1) ring.

Key Words: Non-nilpotent, variety of (-1,1) rings, free metabelian rings, (-1,1) rings.

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1. Introduction

The first example of solvable but not nilpotent alternative and (-1, 1) rings were constructed by Dorofeev [4], [5]. He also gave an example of a finite dimensional right alternative right nilpotent algebra which is not nilpotent.
Varieties of two-step solvable nearly associative algebras were studied by many authors [2,6,7,8,9]. Thus Medvedev [9] proved that the varieties of metabelian alternative, Jordan Mal'tsev and type (-1, 1) algebras are specht. Pchelintsev [6] obtained a series results on the structure of lattices of varieties of nearly associative metabelian algebras.
In this paper, we study (-1, 1) metabelian rings. They are contained in the class of algebras of type (γ, δ). In this class of ring the square of an ideal is also an ideal and hence called 2- variety. A 2, 3- torsion free ring of type (γ, δ) if satisfies the identities
$(x, x, x) = 0,$

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$(x, y, z) + \gamma(y, z, x) + \delta(z, y, x) = 0,$
$(x, y, z) - \gamma(x, z, y) + (1 - \delta)(y, z, x) = 0,$

where $\gamma^2 - \delta^2 + \delta - 1 = 0$, and $(x, y, z) = (xy)z - x(yz)$ is the associator of elements x, y and z.
This paper includes the five sections. In sec 2 we prove that the simplest consequences of the defining relations. In sec 3 and 4, operator of the length 3 and 4 are processed. In sec 5 the function $\{x, y, z\} = (yx)z + (zx)y$ is introduced, its properties are studied, and auxiliary identities necessary for constructing additive bases in free rings are proved. In sec 6, a basis of a free (-1, 1) metabelian rings is constructed and the following main results is proved.

Theorem 1.1. *Any identity of degree ≥6 in a non-nilpotent sub variety of the variety $\mathcal{M}$ of metabelian (-1, 1) ring which is 2 and 3- torsion free is a consequence of the defining identities of $\mathcal{M}$.*

Using the terminology of [7], we obtain the following corollary.

Corollary 1.2. *The topological rank of the variety of metabelian rings of type (-1, 1) is equal to 2.*

2. Consequences of the defining relations

An algebra is said to be metabelian if the identity $(xy)(zt) = 0$ holds in this algebra. follows, by a ring we always mean a metabelain ring of type (-1, 1) and use the identity $(xy)(zt) = 0$.
A ring R is said to be a (-1, 1) ring if it satisfies the following identities

$$(x, y, z) + (y, z, x) + (z, x, y) = 0, \tag{1}$$

and

$$(x, y, z) + (x, z, y) = 0. \tag{2}$$

for all $x, y, z \in R$.
In a nonassociative ring R we define a commutator $[x, y] = xy - yx$, and the anticommutator is defined as $x \circ y = xy + yx$ for all $x, y \in R$. The Jordan identity is defined as the product is commutative, that is $xy = yx$ and the lie identity is defined as the the product is anti commutative that is $xy = -yx$. Throughout this paper we use Lie and Jordan identities toget the results. Now applying (2) in (1) setting $z = ab$ and using metabelian condition we obtain

$$0 = -x(y(ab)) + y(x(ab)) + ((ab)x)y.$$

Hence $R_xR_y = L_yL_x + L_xL_y$.
Thus

$$R_xR_y = L_x \circ L_y. \tag{3}$$

Again setting $x = ab$ in (2) and using the metabelain condition, anti-commutativity and Jordan identity we obtain

$0 = ((ab)y)z - (ab)(yz) + ((ab)z)y - (ab)(zy)$

Thus

$$(xy)z = (xz)y. \tag{4}$$

The above relation in operator form is

$$R_yR_z - R_zR_y = 0implies[R_y, R_z] = 0. \tag{5}$$

Similarly setting $y = ab$ in (2) we obtain

$(xy)z = x(yz)x(zy)$. This relation in operator form is $L_xR_z = R_zL_x - L_zL_x$
Thus

$$L_xR_z = (R_z - L_z)L_x. \tag{6}$$

Lemma 2.1. *Any operator word in a metabelian ring (-1, 1) can be represented as a linear combination of words of the form $T_aL_b....L_c$ having the same composition.*

Proof: The Lemma is proved by applying the identities from (3) to (6) by an obvious induction on the length of an operator word. □

3. Processing of operator words of length 3

Lemma 3.1. $R_x(L_y \circ L_z) = (L_x \circ L_y)R_z = R_xR_yR_z$*, and all these functions are symmetric with respect to the variables x, y, and z.*

Proof: From the relation (3) the Lemma is proved. □

Lemma 3.2. *The following relation holds:* $(L_xL_zL_y + L_yL_xL_z) + L_z(L_x \circ L_y) = 0.$

Proof: We calculate the product $R_xR_yR_z$ by using Lemma 2.1:
$R_xR_yR_z = (L_x \circ L_y)R_z = L_xL_yR_z + L_yL_xR_z$
$= L_x((R_z - L_z)L_y) + L_y((R_z - L_z)L_x)$(by (6))
$=((R_z - L_z)L_x)L_y + ((R_z - L_z)L_y)L_x - (L_xL_zL_y + L_yL_zL_x)$ (by (6))
$= R_z(L_x \circ L_y) - L_z(L_x \circ L_y) - (L_xL_zL_y + L_yL_zL_x)$
$R_xR_yR_z = R_zR_xR_y - L_z(L_x \circ L_y) - (L_xL_zL_y + L_yL_zL_x)$(by (3))
Thus $L_z(L_x \circ L_y) + (L_xL_zL_y + L_yL_zL_x) = 0.$

□

Lemma 3.3. $\Sigma L_{x\sigma}L_{y\sigma}L_{z\sigma} = 0$*, where σ ranges over the symmetric group S_3.*

Proof: $L_zL_xL_y + L_zL_yL_x + L_xL_zL_y + L_yL_zL_x = 0$. Setting $x = y = z$ in the Lemma 3.2, we obtain
$4L_x^3 = 0$. (as the ring is 2- torsion free)
$L_x^3 = 0.$ □

The linearization of this identity proves Lemma 3.2.

Lemma 3.4. *The following relation holds* $(L_x \circ L_y)L_z = R_x R_y L_z = 0$.

Proof: Lemma 3.2 implies $(L_x L_z L_y + L_y L_z L_x) + L_z(L_x \circ L_y) = 0$.
This together with Lemma 3.3 gives
$0 = L_x L_y L_z + L_y L_x L_z$
$= (L_x L_y + L_y L_x)L_z$
$= (L_x \circ L_y)L_z$.
Now this equality and relation (3) gives $R_x R_y L_z = 0$. □

Lemma 3.5. *The following relation holds:*
$(R_a L_y L_b + R_b L_y L_a) + R_y(L_a \circ L_b) - L_y(L_a \circ L_b) = 0$.

Proof: The following identity which holds in any arbitrary ring is obtain by multiplying out associators and performing cancellations generally known as Tiechmuller identity.
$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$.
Taking the defining identities in to account, we obtain
$x(x, y, x) + (x, x, y)x - (x^2, y, x) + (x, xy, x) - (x, x, yx)$
$= x((yx)x) + (x, y, x^2) + (x^2, y, x) - (x(xy))x - ((xy)x)x - (x(x(yx)) - (x^2, x, y) + x(x^2 y)$
$= x((xy)x - x(yx)) + (x^2 y)x - (x(xy))x) - (x^2 y)x + x^2(yx) + (x(xy))x - ((xy)x) - x^2(yx)) + x(x(yx))$
$= x((yx)x) + (x, y, x^2) - (x(xy))x - x(x(yx)) - x(x(yx)) + x(x^2 y) - ((xy)x)x$
$= x((yx)x) - x(x^2 y)x + x^3 y - (x(xy))x - ((xy)x)x - x(x(yx)) + x(x^2 y)$
$= x((yx)x) + x^3 y - (x(xy))x - ((xy)x)x - x(x(yx))$
Thus we have the identity

$$x((yx)x) + x^3 y - (x(xy))x - ((xy)x)x - x(x(yx)) = 0 \tag{7}$$

Let us linearlize this identity with respect to x and assuming that $a, b \in R$ and $c \in R^2$ we obtain
$a((yc)b) + b((yc)a) + ((ca)b + (cb)a + (ac)b + (bc)a)y - (a(cy))b - (b((cy))a - ((cy)a)b - ((cy)b)a - a(b(yc)) - b(a(yc)) = 0$.
The operator form of this identity is
$0 = L_y(R_a L_b + R_b L_a) + (R_a R_b + R_b L_a)R_y + (L_a R_b + L_b R_a)R_y - R_y(L_b R_b + L_b R_a) - R_y(R_a R_b + R_b R_a) - L_y(L_a L_b + L_b L_a)$.
$= L_y(R_a L_b + R_b L_a) + (R_a \circ R_b)R_y + (L_a R_b + L_b R_a)R_y - R_y(L_a R_b + L_b R_a) - R_y(R_a \circ R_b) - L_y(L_a \circ L_b)$.
Applying equation (5) we reduce this equality to the form
$0 = L_y(R_a L_b + R_b L_a) + (L_a R_b + L_b R_a)R_y - R_y(L_a R_b + L_b R_a) - L_y(L_a \circ L_b)$.
Let us transform its right-hand side by using (6) and (3)
$0 = ((R_a - L_a)L_y)L_b + ((R_b - L_b)L_y)L_a + L_a(L_b \circ L_y) + L_b(L_a \circ L_y) - R_y(R_b - L_b)L_a - R_y(R_a - L_a)L_b - L_y(L_a \circ L_b)$.
Removing parentheses and performing cancellations, we obtain
$0 = R_a L_y L_b - L_a L_y L_b + R_b L_y L_a - L_b L_y L_a + L_a L_b L_y + L_a L_y L_b + L_b L_a L_y + L_b L_y L_a -$

$R_yR_bL_a + R_yL_bL_a - R_yR_aL_b + R_yL_aL_b - L_yL_aL_b - L_yL_bL_a$.
$= R_aL_yL_b + R_bL_yL_a + ((L_a \circ L_b)L_y - R_yR_bL_a - R_yR_aL_b) + R_yL_aL_b + R_yL_bL_a +$
$R_yL_aL_b - L_yL_aL_b - L_yL_bL_a$.
This relation together with Lemma 3.4 gives the required result.

□

Lemma 3.6. *The relation $R_y(L_x \circ L_y) = 0$ holds.*

Proof: By Lemma 3.5 we have

$$0 = (R_aL_bL_y + R_bL_yL_a) + R_y(L_a \circ L_b) - L_y(L_a \circ L_b). \tag{8}$$

Permuting each of the symbols (by) and (ay) in this equality, we obtain

$$0 = (R_aL_yL_b + R_yL_bL_a) + R_b(L_a \circ L_y) - L_b(L_a \circ L_y). \tag{9}$$

and

$$0 = (R_yL_bL_a + R_bL_aL_y) + R_a(L_y \circ L_b) - L_a(L_b \circ L_y). \tag{10}$$

Adding of the inequalities (8), (9) and (10) we obtain
$0 = 2(R_y(L_a \circ L_b) + R_b(L_a \circ L_y) + R_a(L_b \circ L_y)) - (L_y(L_a \circ L_b) + L_b(L_a \circ L_y) +$
$L_a(L_y \circ L_b))$.
This equality and Lemmas 3.1 and 3.3 imply
$0 = 2(3R_a(L_b \circ L_y))$
$= 6R_a(L_b \circ L_y)$, whence
$R_a(L_b \circ L_y) = 0$ because of 2, 3 -torsion free. □

Lemma 3.7. *The following relation holds: $R_aL_xL_b + R_bL_xL_a = L_x(L_a \circ L_b)$.*

Proof: The relation is obtained from Lemma 3.5 and Lemma 3.6:

$$0 = (R_xL_bL_a + R_aL_bL_x) - L_b(L_x \circ L_a). \tag{11}$$

Cyclically permuting the symbols x, b and a, we obtain
That is

$$0 = (R_bL_aL_x + R_xL_aL_b) - L_a(L_b \circ L_x) \tag{12}$$

□

Lemma 3.8. $(R_aL_b + R_bL_a)L_x = L_aL_xL_b + L_bL_xL_a$.

Proof:
In Lemma 3.7
The sum of (11) and (12) equalities gives
$0 = (R_xL_bL_a + R_aL_bL_x) - L_b(L_x \circ L_a)$.
$= (R_bL_aL_x + R_xL_aL_b) - L_a(L_b \circ L_x)$.
Thus $R_x(L_bL_a + R_aL_b) + (R_aL_b + R_bL_a)L_x = L_a(L_b \circ L_x) + L_b(L_x \circ L_a)$.

That is $R_x(L_a \circ L_b) + (R_aL_b + R_bL_a)L_x = L_a(L_b \circ L_x) + L_b(L_x \circ L_a)$.
From Lemmas (3.4) and (3.6) we get $(R_aL_b + R_bL_a)L_x = L_aL_bL_x + L_aL_xL_b + L_bL_xL_a + L_bL_aL_x$
$= L_aL_xL_b + L_bL_xL_a$. □

Lemma 3.9. *The relation* $R_xR_yT_z = 0$ *holds.*

Proof: This Lemma follows from equation (5) and Lemma (3.4) and (3.6). □

Lemma 3.10. *In any metabelian (-1, 1) ring the following relations are valid:*
(a)$R_x(L_y \circ L_z) = (L_x \circ L_y)L_z = R_xR_yT_z$;
(b) $(R_aL_(b) + R_bL_a)L_x = L_aL_xL_b + L_bL_xL_a$;
(c) $R_aL_xL_b + R_bL_xL_a = L_x(L_a \circ L_b)$.

Corollary 3.11. *The function* $T_aL_xL_y....L_zL_b$ *is skew-symmetric with respect to* $x, y, , z$

4. Processing of operator words of length 4

Lemma 4.1. *The relation* $(R_aL_b + R_bL_a)L_xL_y = 0$ *holds.*

Proof: Applying Lemmas (3.8) and (3.4) ,we have
$(R_aL_b + R_bL_a)L_xL_y = L_aL_xL_b + L_bL_xL_a$.
That is $((R_aL_b + R_bL_a)L_x)L_y = (L_aL_xL_b + L_bL_xL_a)L_y$
$= L_aL_xL_bL_y + L_bL_xL_aL_y$.
□

Lemma 4.2. *The relation* $L_aL_b(L_x \circ L_y) = 0$ *holds.*

Proof: From Lemmas (3.8) and (3.4) , we have
$0 = R_xL_yL_b + R_yL_xL_b - L_xL_bL_y - L_yL_bL_x = R_xL_yL_b + R_yL_xL_b + L_b(L_x \circ L_y)$
Multiplying this equality by L_a on the left and transforming the result by using identity (6) and Lemmas (3.4) and (3.7), we obtain
$-L_aL_b(L_x \circ L_y) = L_aR_xL_yL_b + L_aR_yL_xL_b$
$= (R_x - L_x)L_aL_yL_b + (R_y - L_y)L_aL_xL_b$
$= R_xL_aL_yL_b - L_xL_aL_yL_b + R_yL_aL_xL_b - L_yL_aL_xL_b$
$= (R_xL_aL_y + R_yL_aL_x)L_b - L_xL_aL_yL_b - L_yL_aL_xL_b$
$= L_aL_xL_yL_b + L_aL_yL_xL_b - L_xL_aL_yL_b - L_yL_aL_xL_b$ (*by Lemma* 3.7)
$= L_aL_xL_yL_b + L_aL_yL_xL_b + L_aL_xL_yL_b + L_aL_yL_xL_b$ (*by Lemma* 3.4)
$= L_a(L_x \circ L_y)L_b + L_a(L_x \circ L_y)L_b$
$= 2L_a(L_x \circ L_y)L_b = 0.$ (*by Lemma* 3.4)

5. Auxiliary Identities

Suppose that $\mathcal{M}$ is an arbitrary variety of metabelian algebras of type (-1, 1), R is a free ring in the variety $\mathcal{M}$ and $X = \{ x_1, x_2... \}$ is a set of free generators of R. For elements $x, y \in R^2$, we write
$x \approx y(n)$ and $x \equiv y(n)$ if,for any $a_1, a_2...a_n \in R$,
$(x - y)L(a_1)L(a_2)...L(a_n) = 0$ and
$(x - y)T(a_1)T(a_2)...T(a_n) = 0$, respectively. □

Lemma 5.1. $[[x, y], z] \equiv$ *0(2).*

Proof: Identity (1) implies
$0 = (zx)y - z(xy) - (xz)y + x(zy) + (yz)x - y(zx)$.
That is $z(xy) = (zx)y - (xz)y + x(zy) + (yz)x - y(zx)$.
$= (zx)y + y(xz) - x(yz) + (yz)x - y(zx)$.
$=(zx)y + (xz)y - x(zy) + (yz)x - y(zx)$.
Multiplying both sides of the last equality by R_tL_u, we obtain
$(xy)L_zR_tL_u = ((zx)y + y(xz) - x(zy) + (yz)x - y(zx))R_tL_u$
$= (zx)R_yR_tL_u + (xz)R_yR_tL_u - (zy)L_xR_tL_u + (yz)R_xR_tL_u - (zx)L_yR_tL_u$
$= -(zy)L_xR_tL_u - (zx)L_yR_tL_u$ (by Lemma 3.9)
Thus$(xy)L_zR_tL_u = -(zy)L_xR_tL_u - (zx)L_yR_tL_u$.Since the right hand side of this equality is symmetric with respect to x and y we have
$(xy)L_zR_tL_u = (yx)L_zR_tL_u$, or

$$[x, y]L_zR_tL_u = 0. \tag{13}$$

Thus

$$[x, y](R_t - L_t)L_zL_u, by identity(2.6) \tag{14}$$

$$[[x, y], t]L_zL_u = 0. \tag{15}$$

Lemma (3.9) gives$[x, y]R_zR_tL_u = 0$
Subtracting (11) from this equality, we obtain
$[x, y](R_z - L_z)R_tL_u = 0$ and
$[[x, y], z]R_tL_u = 0$.
This relation, equality (11), and Lemma (3.1) imply the required assertion. □

Let us introduce the auxiliary function $\{ x, y, z \} = (yx)z + (zx)y$ obviously, it is symmetric with respect to y and z i.e., $\{ x, y, z \} = \{ x, z, y \}$.

Lemma 5.2. *The following assertions are valid*
(a) $\{ a, b, x \} L_y + \{ a, b, y \} L_x + \{a, x, y\} L_b \approx$ *0 (2);*
(b) $\{ a, b, x \} \approx \{ b, a, x \} L_y$ *(2);*
(c) $\{ a, b, x \} \approx$ *0 (2).*

Proof: In the proof of this lemma, $u \approx v$ means $u \approx v$ (2).
(a) Note that Lemma 4.2 can be written in the form $R_aL_b + R_bL_a \approx 0$. Therefore

$\{ a, b, x \} L_y + \{ a, b, y \} L_x = (ba)R_xL_y + (xa)R_bL_y + (ba)R_yL_x + (ya)R_bL_x$
$\approx (xa)R_bL_y + (ya)R_bL_x \approx -(xa)R_bR_y - (ya)R_bL_x$
$\approx$ - $\{ a, x, y \} L_b$.
(b) Taking into account identities (1) and (2), and applying the Jordan product we obtain
$(ba)x + (xa)b = (ba)x - b(ax) + b(ax) + (xa)b - x(ab) + x(ab)$
$= (b, a, x) + (x, a, b) + b(ax) + x(ab)$
$= -(a, x, b) + b(ax) + x(ab)$
$= -(ax)b + a(xb) + b(ax) + x(ab)$.
Hence $(ba)x + (xa)b = -[b, ax] - a(xb) - x(ab) = 0$.
The application of the operator$L_uR_yL_v$ to this relation yields
$0 = ((ba)R_x + (xa)R_b - [b, ax] - (xb)L_a - (ab)L_x)L_uR_yL_v$
$= ((ba)R_x + (xa)R_b - (xb)L_a - (ab)L_x)L_uR_yL_v$ (by (11))
$= ((ba)R_x + (xa)R_b - (xb)L_a - (ab)L_x)(R_y - L_y)L_uL_v$ (by (6))
$= (-(xb)(R_y - L_y)L_a - (ab)(R_y - L_y)L_x - (ba)R_xL_y + (xb)L_aL_y + (ab)L_xL_y)L_uL_v$
$= (-(xb)R_yL_a + (xb)L_yL_a - (ab)R_yL_x + (ab)L_yL_x - (ba)R_xL_y - (xa)R_bL_y + (xb)L_aL_y + (ab)L_xL_y)L_uL_y$
$= (-(ba)R_xL_y - (xa)R_bLy - (xb)R_yL_a - (ab)R_yL_x)(L_yL_a + L_aL_y) + (ab)(L_yL_x + L_xL_y)L_uL_y$
$= (-(ba)R_xL_y - (xa)R_bLy - (xb)R_yL_a - (ab)R_yLx + (xb)R_yR_a + (ab)R_yR_x)L_uL_x$ by Lemma (3.4)
$= -(ba)R_xL_y - (xa)R_bL_y - (xb)R_yL_a - (ab)R_yL_x \approx 0$
$= (ba)R_xL_y + (xa)R_bL_y \approx (xb)R_yL_a - (ab)R_yL_x$
$= \{a, b, x\}L_y \approx \{b, a, x\}L_y$ (by Lemma (4.1))

(c) Assertions (a) and (b) imply $(\{ b, a, x \} L_y + \{ b, a, y \} L_x) + \{ a, x, y \} L_b \approx 0$. Applying (8) to the expression in parentheses, we obtain
- $\{ b, a, x \} L_y + \{ a, x, y \} L_b \approx 0$,
that is $\{ a, x, y \} L_b \approx \{ b, x, y \} L_a$.
Therefore, the function $\{ a, b, x \} L_yL_uL_v$ is symmetric with respect to the variables a, b, x, and y. By virtue of (a), we have $3\{ a, b, x \} L_y \approx 0$, and the assumption on the Torsion free gives the required relation (c). □

Lemma 5.3. *If a ring R satisfies the relation $R[R^2, R] \equiv 0(n)$ for some $n \geqslant 2$ then R is nilpotent.*

Proof: First $[R^2, R]R \subseteq R[R^2, R] + [[R^2, R], R] \equiv 0(n)$, which means that $[R^2, R] \equiv 0(n + 1)$ as is known, the identity
$[xy, z] = x[y, z] + [x, z]y + (x, y, z) + (z, x, y) - (x, z, y)$ holds in an arbitrary ring. By virtue of identity (2) , the expression in parenthesis vanishes,
$-(z, x, y) = x[y, z] + [x, z]y - [xy, z] + (x, y, z) - (x, z, y)$
$(z, x, y) = x[y, z] + [x, z]y - [xy, z]$.
Setting $y = w \in R^2$ in the last equality, we obtain $(z, x, w) = [xw, z] - x[w, z] \equiv 0(n + 1)$ whence $wL_xL_z \equiv 0(n + 1)$. In particular, $wR_cL_xL_z \equiv 0(n + 1)$. Finally, Lemma 2.1 implies $R^2 \equiv 0(n + 4)$. □

Corollary 5.4. *suppose that, in ring R*
$[a,b]\ L(x_1)L(x_2)...L(x_n)=0$ *or* $[a,b]\ R(x_1)L(x_2)...L(x_n)=0$
for any elements $a,b,x_1...x_n (n\geq 3)$*. Then the ring R is nilpotent.*

Proof: Let $[a,b]L(x_1)L(x_2)...L(x_n)=0$. By virtue of (11) we have
$[a,b]L(x_1)R(x_2)L(x_3)...L(x_n)=0$.
These two equalities and Lemma 2.1 imply $[a,b]L(x_1)\equiv 0(n-1)$, i.e., $R[R,R]\equiv 0(n)$ by Lemma 5.3 the ring R is nilpotent. □

Lemma 5.5. *If in a ring R, $0=(R_a-L_a)L_b...L_c$ for any element $a,b...c$, then R is nilpotent.*

Proof: Relation (6) (that is,$L_xR_z-R_zL_x+L_zL_x=0$) and the assumptions of the Lemma imply
$0=(L_xR_z-R_zL_x+L_zL_x)L_b...L_c=(L_xL_z-L_zL_x+L_zL_x)L_b...L_c=L_xL_zL_b...L_c$.
Moreover, $R_xL_zL_b...L_c=0$; thus the ring R is nilpotent by Lemma(2.1). □

6. A basis of a free metabelain (−1, 1) ring

Definition 6.1. *The regular words in a ring R on the alphabet $X_n:=\{x_1,x_2,...x_n\}$ $(n\geq 6)$ are the polylinear monomials*
(a) $(x_1x_j)R(k_1)L(k_2)...L(k_{n-2})$,
(b) $(x_jx_1)L(k_1)L(k_2)...L(k_{n-2})$,
(c) $(x_2x_1)R(3)L(4)...L(n)$,
where $T(k):=T(x_k) and k_1<k_2<...k_{n-2}$.

Lemma 6.2. *The space $P_n(R)$ of polylinear monomials in variables from $X_n(n\geq 6)$ is linearly generated by the regular words in the ring R.*

Proof: By Lemma 2.1, the space $P_n(R)$ is linearly generated by words of the form $(x_ix_j)T_aL_b...L_c$. Next, according to the corollary to Lemmas 3.10 and Lemma 4.2 words having the form $(x_ix_j)L(k_1\sigma)L(k_2\sigma...L(k_{n-2}\sigma)$ where $\sigma\in S_{n-2}, k_1<k_2<...k_{n-2}$, are linearly expressed in terms of words having the form $(x_ix_j)L(k_1)L(k_2)...L(k_{n-2})$ where $k_1<k_2<...<k_{n-2}$. By Lemmas 3.1 and 4.1 words of the form $(x_ix_j)R(k_1\sigma)L(k_2\sigma)...L(k_{n-2}\sigma)$ where $\sigma\in S_{n-2}, k_1<k_2<...<k_{n-2}$ are linearly expressed in terms of words of the form $(x_ix_j)R(k_1)L(k_2)......L(k_{n-2})$, where $k_1<k_2<...<k_{n-2}$. Identity (1) implies $0=(x_1x_i)x_j-x_1(x_ix_j)-(x_ix_1)x_j-x_i(x_1x_j)+(x_jx_i)x_1+x_j(x_ix_1)$ where

$$x_1(x_ix_j)=(x_1x_i)x_j+(x_ix_1)x_j-x_i(x_1x_j)+(x_jx_i)x_1-x_j(x_ix_1), \tag{16}$$

$$(x_ix_j)x_1=x_i(x_jx_1)+(x_ix_1)x_j-x_i(x_1x_j). \tag{17}$$

By virtue of (2), we have $(x_ix_j)x_1-x_i(x_jx_1)-(x_ix_1)x_j+x_i(x_1x_j)=0$. According to the relation (16) and (17), monomials of the form
$(x_ix_j)T(k_1)L(k_2)...L(k_{n-2})$, where $k_1<k_2<...<k_{n-2}$,
are linearly expressed in terms of words of the form

$(x_1x_j)R(k_1)L(k_2)...L(k_{n-2})$, $(x_jx_1)R(k_1)L(k_2)...L(k_{n-2})$,
$(x_1x_j)L(k_1)L(k_2)...L(k_{n-2})$,$(x_jx_1)L(k_1)L(k_2)...L(k_{n-2})$.
By (9), $[x,y]R_zL_tL_u = [x,y]L_zL_tL_u$ Therefore,
$[x_1x_j]R(k_1)L(k_2)...L(k_{n-2}) = [x_1x_j]L(k_1)L(k_2)...L(k_{n-2})$,
which gives
$(x_1x_j)R(k_1)L(k_2)...L(k_{n-2}) - (x_jx_1)R(k_1)L(k_2)...L(k_{n-2})$
$= (x_1x_j)L(k_1)L(k_2)...L(k_{n-2}) - (x_jx_1)L(k_1)L(k_2)...L(k_{n-2})$.
whence
$(x_1x_j)L(k_1)L(k_2)...L(k_{n-2}) = (x_jx_1)L(k_1)L(k_2)...L(k_{n-2})$
$-(x_1x_j)R(k_1)L(k_2)...L(k_{n-2}) + (x_jx_1)L(k_1)L(k_2)...L(k_{n-2})$.
Now, consider words of the form $(x_jx_1)R(k_1)L(k_2)...L(k_{n-2})$ where $j \geqslant 3$. We have
$= (x_jx_1)R(2)L(k_2)...L(k_{n-2}) = ((x_jx_1)x_2)L(k_2)...L(k_{n-2})$
$= -\ \{\ x_1,x_2,x_3\ \}\ L(k_2)...L(k_{n-2}) - ((x_2x_1)x_j)L(k_2)...L(k_{n-2})$
$= -\ (x_2x_1)x_j)L(k_2)...L(k_{n-2})$ (by Lemma 5.2(c))
$= -\ (x_2x_1)R(j)L(k_2)...L(k_{n-2})$
$= (-1)^j(x_2x_1)R(3)L(4)...L(n)$ (by Lemma 3.4, 4.1, and 4.2).
This means precisely that the space $P_n(R)$ is linearly generated by the regular words in the ring R. □

Lemma 6.3. *If a ring R is not nilpotent, then the regular words in this algebra are linearly independent.*

Proof: Suppose that some linear combination of regular words in R vanishes, that is,

$$\sum_j \propto_j\ (x_1x_j)R(k_1)L(k_2)...L(k_{n-2}) \quad + \quad \sum_j \beta_j(x_jx_1)L(k_1)L(k_2)...L(k_{n-2}) \\ + \quad \gamma(x_2x_1)R(3)L(4)...L(n) = 0. \qquad (18)$$

Take $v \in [R,R], w \in R^2$, and $j_0 \geq 3$. Setting $x_{j_0} = v$ and applying (10), we obtain $\beta_{j_0}(vx_1)L(k_1)L(k_2)...L(k_{n-2})$= 0.
Since the ring R is not nilpotent, the corollary of Lemma 6.2 implies β_{j_0}= 0. Therefore, all terms of the form $\beta j(x_jx_1)L(k_1)L(k_2)...L(k_{n-2})$ in (5.3) vanish.
Next, setting $x_{j_0} = w$ and using Lemma 3.4 we obtain
$\alpha_{j_0}(x_1w)R(k_1)L(k_2)...L(k_{n-2}) = 0$.

According to (6) we have
$\alpha_{j_0}w(R(k_1)L(k_1L(k_2)...L(k_{n-2}) - L(k_1)L(k_2)...L(k_{n-2})) = 0$. Since the ring R is not nilpotent, Lemma 5.5 implies $\alpha_(j_0) = 0$, and equality 6.1 takes the form $\gamma(x_2x_1)R(3)L(4)...L(n) = 0$. Setting $x_1 = w$, we obtain $\gamma(x_2w)R(3)L(4)...L(n) = 0$. Arguing the above,we conclude let $\gamma = 0$. Thus the regular words are linearly independent. Lemma 6.3 readily implies Theorem 1.1. Since not all of the metabelian (-1, 1) rings are nilpotent [2], Corollary 1.2 is valid as well. □

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