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A collocation method for solving the fractional calculus of variation problems

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ABSTRACT: In this paper, we use a family of Müntz polynomials and a computational technique based on the collocation method to solve the calculus of variation problems. This approach is utilized to reduce the solution of linear and nonlinear fractional order differential equations to the solution of a system of algebraic equations. Thus, we can obtain a good approximation even by using a smaller of collocation points.

Key Words: Müntz-Legendre polynomials, Orthogonal system, fractional differential equation, Collocation method, Caputo fractional derivative

Contents

1. Introduction

Fractional calculus which involves the study of non-integer powers of differentiation and integral operators has many applications in studying and solving the complicated problems in science and engineering [2,1]. The fractional calculus of variations as an important branch of the fractional calculus is a research area under strong recent development. For instance, the reader is referred to the recent books [4,5,3] and papers [6,7]. The calculus of variations deals with the problem of extremizing functions [8] and be is defined as follows:
All differentiable functions $y : [a, b] \to \mathbb{R}$ such that $y(a) = y_a$ and $y(b) = y_b$ that y_a and y_b are known, find minimize (or maximize) of the following functional:

$$J[y(\cdot)] = \int_a^b \mathcal{L}\big(t, y(t), {}_a^C D_t^\alpha y(t)\big) dt, \tag{1.1}$$

where ${}^{C}_{a}D^{\alpha}_{t}$ denotes the left fractional derivative operator of order α in the Caputo sense. We can solve this problem with using the following differential equation [6]

$$\frac{\partial \mathcal{L}}{\partial y} + {}^{C}_{t}D^{\alpha}_{b} \frac{\partial \mathcal{L}}{\partial\, {}^{C}_{a}D^{\alpha}_{t} y} = 0, \tag{1.2}$$

which is called the Euler-Lagrange equation. Here ${}^{C}_{t}D^{\alpha}_{b}$ denotes the right fractional derivative operator of order α which will be defined in the next section. By employing fractional derivatives into the variational integral we obtain the fractional Euler-Lagrange equation. Now, in this paper we use a kind of the Müntz-Legendre polynomials such that their fractional derivatives be Müntz-Legendre polynomials again. For this purpose, in Section 2, we express some necessary definitions and notations. In Section 3, the Müntz-Legendre polynomials are introduced. In Section 4, we solve the fractional differential equations in calculus of variations numerically by using the collocation method. Some numerical examples are given in Section 5.

2. Preliminaries and notation

In this section, we present a short overview to the fractional calculus [11,10,9]. In this sequel, we suppose $\alpha \in (0,1)$ and Γ represents the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Definition 2.1. *The fractional derivative of f in the Caputo sense is defined for $f \in C^1[0,1]$ as*

$$^{C}D^{\alpha}_{x} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t), \quad x \in [0,1]. \tag{2.1}$$

Definition 2.2. *The left and right hand sides Caputo fractional derivatives of order α are defined for $f \in C^1[0,1]$ as*

$${}^{C}_{a}D^{\alpha}_{x} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)dt, \quad x \in [0,1],$$

and

$${}^{C}_{x}D^{\alpha}_{b} f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^1 (t-x)^{-\alpha} f'(t)dt, \quad x \in [0,1],$$

respectively.

Remark 2.3. *According to the above definitions it is clear that for $a = 0$, the Caputo fractional derivative is equal to the left hand side Caputo fractional derivative.*

According to the definition of right hand side Caputo derivative [10], we get

$${}^{C}_{a}D^{\alpha}_{x} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta, t > 0. \tag{2.2}$$

Remark 2.4. *For the simplification the notation D^{α}_{*} is used instead of ${}^{C}_{a}D^{\alpha}_{x}$.*

3. Müntz polynomials

Let $\{\lambda_i,\ i \in \mathbb{N}_0\}$ be a set of real numbers which $0 \leq \lambda_0 < \lambda_1 < \cdots \to \infty$. According to Müntz's idea there are functions $s_i = \sum_{k=0}^{n} a_k x^{\lambda k}$ with real coefficients which are dense in $L^2[0,1]$ if and only if $\sum_{k=1}^{\infty} \lambda_k^{-1} = +\infty$. We consider the Müntz-Legendre polynomials that are orthogonal in the interval $(0,1)$ with respect to the weight function $w(x) = 1$.

3.1. Müntz-Legendre polynomials

Before discussion about Müntz polynomials, we need to brief review of the general form of orthogonal polynomials which are the familiar as the Jacobi polynomials. These polynomials which are denoted by $P_k^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, are widely applicable in numerical solution of differential equations. They have the following explicit form [12]

$$P_k^{(\alpha,\beta)}(x) = \sum_{m=0}^{k} \frac{(-1)^{k-m}(1+\beta)_k(1+\alpha+\beta)_{k+m}}{m!(k-m)!(1+\beta)_m(1+\alpha+\beta)_k}\left(\frac{1+x}{2}\right)^m,$$

$$(j)_0 = 1, \quad (j)_i = j(j+1)\cdots(j+i-1).$$

The Jacobi polynomials are orthogonal on $[-1,1]$, with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$. By choosing $\alpha = \beta = -1/2$ and $\alpha = \beta = 0$, the well known Chebyshev polynomials of the first kind and the Legendre polynomials are derived, respectively. They can computed by the following recurrence relation [13,14,12]

$$P_0^{(\alpha,\beta)} = 1, \quad P_1^{(\alpha,\beta)} = \frac{1}{2}[(\alpha-\beta)+(\alpha+\beta+2)x],$$
$$a_{1,k}^{\alpha,\beta} P_{k+1}^{(\alpha,\beta)}(x) = a_{2,k}^{\alpha,\beta}(x) P_k^{(\alpha,\beta)}(x) - a_{3,k}^{\alpha,\beta} P_{k-1}^{(\alpha,\beta)}(x),$$

where

$$\begin{aligned} a_{1,k}^{\alpha,\beta} &= 2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta), \\ a_{2,k}^{\alpha,\beta}(x) &= (2k+\alpha+\beta+1)[(2k+\alpha+\beta)(2k+\alpha+\beta+2)x+\alpha^2-\beta^2], \\ a_{3,k}^{\alpha,\beta} &= 2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2). \end{aligned} \tag{3.1}$$

The following formula is very useful that relates the Jacobi polynomials and their derivatives

$$\frac{d}{dx} P_k^{(\alpha,\beta)}(x) = \frac{1}{2}(k+\alpha+\beta+1) P_{k-1}^{(\alpha+1,\beta+1)}(x).$$

If the complex numbers from the set $\Lambda_n = \{\lambda_i,\ i = 0\cdots,n\}$ satisfy the condition $\Re(\lambda_k) > -1/2$, then the Müntz-Legendre polynomials on the interval $(0,1]$ are defined as follows (see [16,15,17])

$$P_n(x) := P_n(x;\Lambda_n) = \sum_{k=0}^{n} C_{n,k} x^{\lambda k}, \quad C_{n,k} = \frac{\prod_{v=0}^{n-1}(\lambda_k + \overline{\lambda_v} + 1)}{\prod_{v=0, v\neq k}^{n}(\lambda_k - \lambda_v)}. \tag{3.2}$$

For the Müntz-Legendre polynomials (3.2), the orthogonality condition is

$$(P_n, P_m) = \int_0^1 P_n(x)P_m(x)dx = \frac{\delta_{mn}}{\lambda_n + \overline{\lambda_n} + 1}.$$

Obviously $P_n(1) = 1$ and $P_n'(1) = \lambda_n + \sum_{k=0}^{n-1}(\lambda_k + \overline{\lambda_k} + 1)$ [17]. In the case $\lambda_k = k\alpha$ for positive number α, the Müntz-Legendre polynomials on the interval $[0, T]$ are defined as

$$L_n(t;\alpha) := \sum_{k=0}^{n} C_{n,k}\Big(\frac{t}{T}\Big)^{k\alpha}, \quad C_{n,k} = \frac{(-1)^{n-k}}{\alpha^n k!(n-k)!}\prod_{v=0}^{n-1}((k+v)\alpha + 1). \tag{3.3}$$

The functions $L_k(t;\alpha)$, $k = 0, 1, \cdots, n$, form an orthogonal basis for $\mathbb{M}_{n,\alpha}$, where $\mathbb{M}_{n,\alpha}$ is represented by

$$\begin{aligned}\mathbb{M}_{n,\alpha} &= \operatorname{span}\{1, t^\alpha, \cdots, t^{n\alpha}\}, \quad t \in [0, T],\\ &= \{c_0 + c_1 t^\alpha + \cdots + c_n t^{n\alpha} : c_k \in \mathbb{R}, t \in [0, T]\}.\end{aligned}$$

Theorem 3.1. *Let $\alpha > 0$ be a real number and $t \in [0, T]$. Then the following representation holds true*

$$L_n(t;\alpha) = P_n^{(0,\frac{1}{\alpha}-1)}\big(2(\frac{t}{T})^\alpha - 1\big), \tag{3.4}$$

where P is the well known Jacobi polynomial.

Proof: See [18]. □

Therefore, the Müntz-Legendre polynomials $L_n(t;\alpha)$ can be obtained by means of the following recurrence formula

$$L_0(t;\alpha) = 1, \quad L_1(t;\alpha) = \Big(\frac{1}{\alpha} + 1\Big)\Big(\frac{t}{T}\Big)^\alpha - \frac{1}{\alpha},$$

$$b_{1,n}L_{n+1}(t;\alpha) = b_{2,n}(t)L_n(t;\alpha) - b_{3,n}L_{n-1}(t;\alpha), \tag{3.5}$$

where

$$b_{1,n} = a_{1,n}^{0,\frac{1}{\alpha}-1}, \quad b_{2,n}(t) = a_{2,n}^{0,\frac{1}{\alpha}-1}\Big(2\Big(\frac{t}{T}\Big)^\alpha - 1\Big), \quad b_{3,n} = a_{3,n}^{0,\frac{1}{\alpha}-1},$$

and the coefficients $a_{\cdot,n}^{0,\frac{1}{\alpha}-1}$ are appeared in recurrence relation for Jacobi polynomials where are stated in (3.1). By applying (2.2) and (3.3), the Caputo fractional derivative of $L_n(t;\alpha)$ can be represented as

$$D_*^\alpha L_n(t;\alpha) := \sum_{k=1}^{n} D_{n,k}\Big(\frac{t}{T}\Big)^{(k-1)\alpha}, \quad D_{n,k} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+k\alpha-\alpha)T^\alpha}C_{n,k}. \tag{3.6}$$

It is important to notice that $D_*^\alpha L_n(t;\alpha) \in \mathbb{M}_{n,\alpha}$.

4. The collocation method

This section is devoted for solving a nonlinear fractional differential equation with boundary conditions that is expressed as [13,19]

$$J[y(t)] = \int_a^b \mathcal{L}\big(t, y(t),\, D_*^\alpha y(t)\big)dt, \quad t \in (0, T], \tag{4.1}$$

with the initial conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{4.2}$$

in which y_a and y_b are fixed and real numbers. Numerical evaluation of this solution is the aim of this section. At first, the solution y is approximated by $\tilde{y}_n \in \mathbb{M}_{n,\alpha}$ as the following truncated series

$$\tilde{y}_n(t) := \sum_{k=0}^{n} a_k L_k(t;\alpha), \tag{4.3}$$

where a_k is unknown coefficients and $L_k(t;\alpha)$ is Müntz-Legendre polynomials. We know that if $\tilde{y}_n \in \mathbb{M}_{n,\alpha}$ then $D_*^\alpha \tilde{y}_n$ belongs to $\mathbb{M}_{n,\alpha}$, too. Hence, we have

$$D_*^\alpha \tilde{y}_n(t) = \sum_{k=0}^{n} a_k\, D_*^\alpha L_k(t;\alpha), \tag{4.4}$$

where the fractional operator in left hand side is easily evaluated by (3.6). Substituting the above truncated series into (1.2), we get an fractional differential equation as

$$\varphi(t, a_0, a_1, \cdots, a_n) := \frac{\partial \mathcal{L}}{\partial \tilde{y}_n} + {}_t^C\!\mathcal{D}_b^\alpha \frac{\partial \mathcal{L}}{\partial D_*^\alpha \tilde{y}_n} = 0. \tag{4.5}$$

Calculating the Caputo fractional derivative in the above equation is based on Gauss-Legendre quadrature. The unknown coefficients a_k in (4.3) are obtained from collocating (4.5) in $n-1$ points along with the initial conditions

$$\tilde{y}_n(a) = y_a, \quad \tilde{y}_n(b) = y_b. \tag{4.6}$$

Now, we set the collocation points as $\theta_i,\ i = 1, \cdots, n-1$. In this case, a particularly convenient choice for the collocation points θ_i are $\theta_i = t_i^{1/\alpha},\ i = 1, \cdots, n-1$, where t_i are the well-known Chebyshev-Gauss-Lobatto points shifted to the interval $[0, T]$, i.e.,

$$t_i = \frac{T}{2} - \frac{T}{2}\cos\frac{i\pi}{n-1}, \quad i = 0, \cdots, n-1.$$

By substituting (4.3) into (4.6), we get

$$g_a(a_0, \cdots, a_n) := \sum_{k=0}^{n} a_k L_k(a; \alpha) - y_a = 0,$$
$$g_b(a_0, \cdots, a_n) := \sum_{k=0}^{n} a_k L_k(b; \alpha) - y_b = 0. \tag{4.7}$$

Also, collocating (4.5) on the mentioned nodes yields

$$\varphi(\theta_i, a_0, a_1, \cdots, a_n) = 0, \quad i = 0, \cdots, n-1. \tag{4.8}$$

The unknown coefficients $a_0, a_1, \cdots, a_n$ are obtained from the solution of system of $n+1$ algebraic equations derived from (4.7) and (4.8).

5. Numerical Examples

In this section, we apply the proposed approximation procedure in two examples.

Example 5.1. *As the first example, we consider the following minimization problem:*

$$J[y(t)] = \int_0^T ({}_0D_t^{0.5} y(t) - \frac{2}{\Gamma(2.5)} t^{1.5})^2 dt \longrightarrow \min$$
$$y(0) = 0, \quad y(1) = 1. \tag{5.1}$$

The Euler-Lagrange equation for this problem has the following form

$${}_t^C D_1^{0.5} ({}_0D_t^{0.5} \tilde{y}_n(t) - \frac{2}{\Gamma(2.5)} t^{1.5}) = 0.$$

Collocating the above fractional differential equation according to the mentioned method in previous section, yields the approximation of the solution of the problem. In Fig. 1, we can see the approximated solution obtained by our method for $\alpha = 0.5$, $n = 2, 3, 4$ and $T = 1$. For $n = 4$ we have the exact solution and we can see that by increasing n the approximated solutions converge to the exact solution.
If we solve the problem with the direct method, we can see that with $n = 30$, the solution is not exact again. In Fig. 2, the approximation solution and the exact solution with direct method are plotted. It reveals that collocation method is better than the direct method.

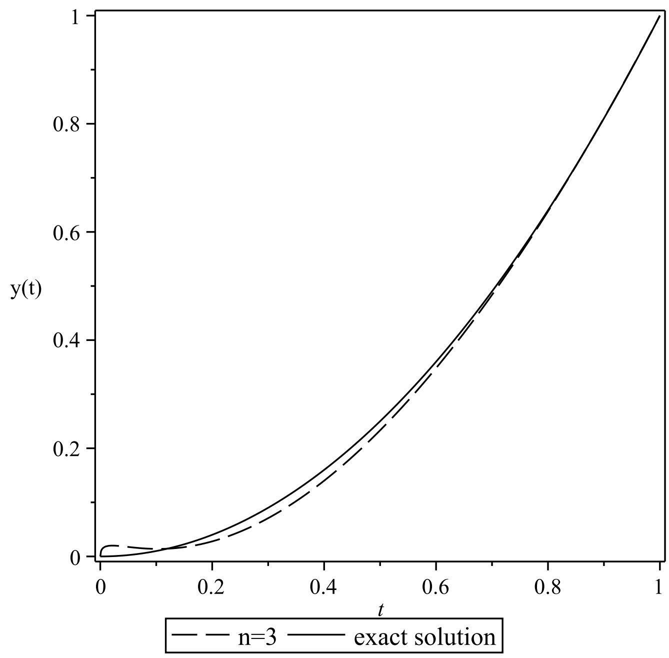

Figure 1: Analytic and approximate solutions for example (5.1) with collocation method

Example 5.2. *As the second example, we consider (4.1) with the operator L as follows*

$$L = ({}_0D_t^{0.5} y(t) - \frac{16\Gamma(6)}{\Gamma(5.5)} t^{4.5} + \frac{20\Gamma(4)}{\Gamma(3.5)} t^{2.5} - \frac{5}{\Gamma(1.5)} t^{0.5})^4, \tag{5.2}$$

and the boundary conditions are given as

$$y(0) = 0, \ y(1) = 1.$$

The exact solution of this problem is $y(t) = 16t^5 - 20t^3 + 5t$. The corresponding Euler-Lagrange equation corresponding to this problem is

$${}_t^C D_1^{0.5} ({}_0D_t^{0.5} y(t) - \frac{16\Gamma(6)}{\Gamma(5.5)} t^{4.5} + \frac{20\Gamma(4)}{\Gamma(3.5)} t^{2.5} - \frac{5}{\Gamma(1.5)} t^{0.5})^3 = 0.$$

In Fig. 3, we indicate the approximated solutions obtained for $n = 5, 6, 10$. For $n = 10$, we obtain the exact solution.

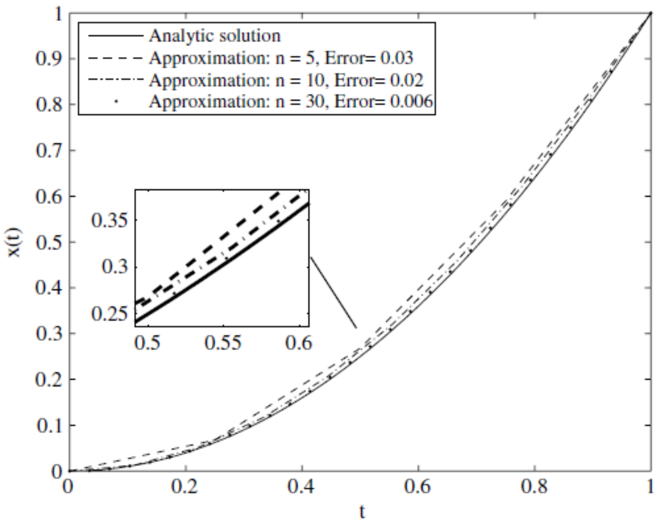

Figure 2: Analytic and approximate solutions for example (5.1) with direct method

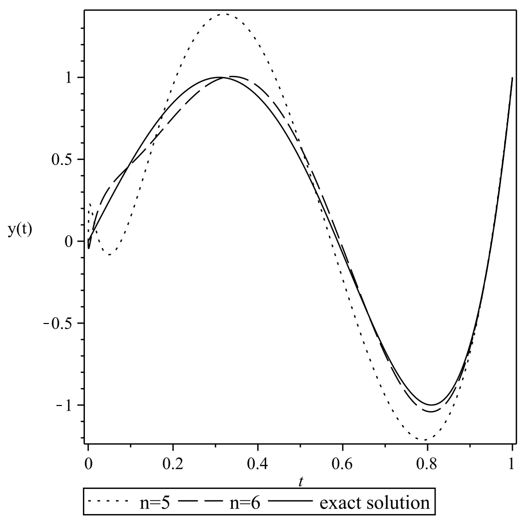

Figure 3: Analytic and approximate solutions for problem (5.2) with collocation method

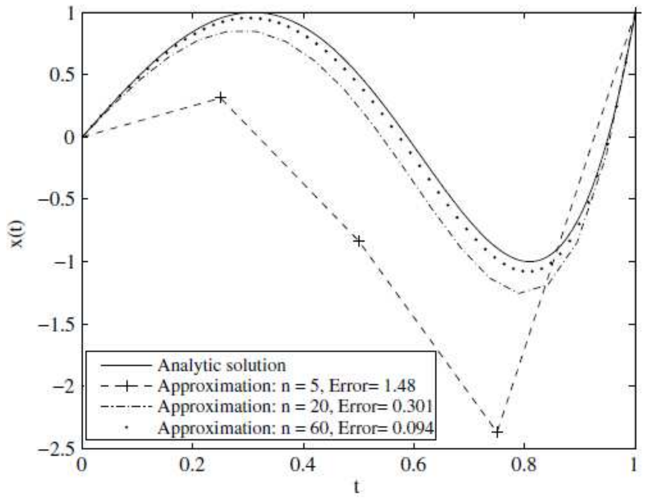

Figure 4: Analytic and approximate solutions for problem (5.2) with direct method

6. Conclusion

This paper describes an efficient method for finding the minimum of the initial value problems for fractional differential equations. We use a family of Müntz-Legendre as an approximation basis. The aim is to estimate fractional differential equations in calculus of variations based on nonclassical orthogonal polynomials with collocation method.

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