Bol. Soc. Paran. Mat. (3s.) **v. 34** 2 (2016): 35–41.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v34i2.26861

Epiconvergence method to a nonlinear value boundary problem with L^1 Data

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abstract: The aim of this work is to study the existence of solutions of a nonlinear value boundary problem with L^1 data with truncation and epiconvergence method.

Key Words: nonlinear value boundary problem; epi-convergence method; L^1 data.

Contents

1. Introduction

In this present paper, our goal is to study the problem of the existence of the weak solutions of the following value boundary problem

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Where Ω is a bounded domain in $\mathbb{R}^3$ with lipschitz boundary $\partial\Omega$, Δ_p is the p-Laplacian operator defined on the sobolev space $W_0^{1,p}(\Omega)$ to its dual $W^{-1,p'}(\Omega)$, $p > 1$, p' the conjugate of p and $f \in L^1(\Omega)$. Recently, many works is considered, in the literature of boundary problem with L^1 data, where its authors have study the existence of renormalization solutions, for more details we can refer the reader to [2,3,4]. Now we interest to an other approach to study the existence of the weak solutions of the problem (1.1). Our approach is based to the truncation and epi-convergence method.

This paper is organized in the following way. In section 2, we express the problem to study with truncation step, we define functional space for this study and we study the problem (2.1). The section 3 is reserved to the determination of the limit problem.

2000 *Mathematics Subject Classification:* 35A15, 35B40, 35D30, 35J25, 76M30.

2. Statement of the problem

In order to study the existence of solutions of the problem (1.1), we propose our method, based on two steps: truncation step and the second step is reserved to the asymptotic behaviour of solutions.

2.1. Truncation step

It is known, in [5], that there exists a sequence $(f_\varepsilon)_{\varepsilon>0}$ in $L^{p'}(\Omega)$ (with $\frac{1}{p}+\frac{1}{p'}=1$) such that $f_\varepsilon \to f$ in $L^1(\Omega)$ when ε close to 0, for more details we refer the reader to [5]. Now we consider the following problem

$$\begin{cases} -\Delta_p u &= f_\varepsilon \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

2.2. Study of problem (2.1)

Note that problem (2.1) is equivalent to the minimization problem

$$\inf_{v\in\mathbb{V}^\infty}\left\{\frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega f_\varepsilon v\right\} \tag{2.2}$$

where $\mathbb{V}^\infty$ a closed subset of $W_0^{1,p}(\Omega)$ defined by

$$\mathbb{V}^\infty = L^\infty(\Omega)\cap W_0^{1,p}(\Omega),$$

it's clear to see that $\mathbb{V}^\infty$, endowed with the norm $\|.\|_{W_0^{1,p}(\Omega)}$, is a banach and reflexive space. In the sequel, we denote $\|.\|_{\mathbb{V}^\infty} = \|.\|_{W_0^{1,p}(\Omega)}$ and we study the existence and uniqueness of solution of the problem (2.2).

Proposition 2.1. *The problem (2.2) possess an unique bounded solution u^ε in $\mathbb{V}^\infty$. Moreover there exists a subsequence of u^ε (still denoted by u^ε) and $u^* \in \mathbb{V}^\infty$ satisfy*

$$u^\varepsilon \to u^* \text{ in } \mathbb{V}^\infty$$

Proof: Let F^ε be an operator defined on $\mathbb{V}^\infty$ with real values such that

$$F^\varepsilon(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega f_\varepsilon v,\ \forall v\in\mathbb{V}^\infty \tag{2.3}$$

Let $v\in\mathbb{V}^\infty$, we have

$$\begin{aligned} pF^\varepsilon(v) &= \int_\Omega |\nabla v|^p - p\int_\Omega f_\varepsilon v, \\ &\geq \int_\Omega |\nabla v|^p - \|f_\varepsilon\|_{L^{p'}(\Omega)}\|v\|_{L^p(\Omega)}, \end{aligned}$$

thanks to Young and Poincaré inequality, there exists $C > 0$ such that

$$\begin{aligned} pF^{\varepsilon}(v) &\geq \int_{\Omega} |\nabla v|^p - C\|\nabla v\|_{L^p(\Omega)}, \\ &\geq \int_{\Omega} |\nabla v|^p - C - \frac{1}{p}\|\nabla v\|^p_{L^p(\Omega)} \\ &\geq \frac{1}{p'}\int_{\Omega} |\nabla v|^p - C. \end{aligned}$$

So

$$F^{\varepsilon}(v) \geq \frac{1}{pp'}\int_{\Omega} |\nabla v|^p - \frac{C}{p}.$$

then F^{ε} is $\mathbb{V}^{\infty}$-coercive[(1)] and we can show easily that F^{ε} is weakly lower semicontinuous on $\mathbb{V}^{\infty}$, so by applying the classical result (see [6, theorem 1.1 p.48]), and by strict convexity of F^{ε} , then the minimization problem (2.2) possess an unique solution $(u^{\varepsilon})_{\varepsilon>0} \subset \mathbb{V}^{\infty}$. From variational formulation, we have for any $v \in \mathbb{V}^{\infty}$:

$$\int_{\Omega} |\nabla u^{\varepsilon}|^{p-2}\nabla u^{\varepsilon}\nabla v = \int_{\Omega} f_{\varepsilon} v,$$

in particular for $v = u^{\varepsilon}$ and according to Poincaré, Hôlder and Young inequality (see for example [5]), there exists a constant $c > 0$ such that

$$\begin{aligned} \int_{\Omega} |\nabla u^{\varepsilon}|^p &= \int_{\Omega} f_{\varepsilon} u^{\varepsilon}, \\ &\leq C\|\nabla u^{\varepsilon}\|_{L^p(\Omega)}, \\ &\leq \frac{1}{p'}C^{p'} + \frac{1}{p}\|\nabla u^{\varepsilon}\|^p_{L^p(\Omega)}, \end{aligned}$$

thus there exists a constant $M > 0$ such that

$$\int_{\Omega} |\nabla u^{\varepsilon}|^p \leq M.$$

Hence $(u^{\varepsilon})_{\varepsilon>0}$ is a bounded sequence in $\mathbb{V}^{\infty}$.
Since $\mathbb{V}^{\infty}$ is a reflexive space, so there exists a subsequence of $(u^{\varepsilon})_{\varepsilon>0}$), still denoted by $(u^{\varepsilon})_{\varepsilon>0}$), and $u^* \in \mathbb{V}^{\infty}$ such that $u^{\varepsilon} \rightharpoonup u^*$ in $\mathbb{V}^{\infty}$, to prove the strong convergence in $\mathbb{V}^{\infty}$, we will use the fact that $-\Delta_p$ is of type (S+) for more details see for example [5]. First, we prove that

$$\limsup_{\varepsilon\to 0}\langle -\Delta u^{\varepsilon}, u^{\varepsilon} - u^*\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \leq 0.$$

[1] For more details about the coercivity definition see [6].

We have

$$\begin{aligned}\langle -\Delta u^{\varepsilon}, u^{\varepsilon}-u^{*}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} &= \int_{\Omega}\|\nabla u^{\varepsilon}\|_{L^p(\Omega)}^{p-2}\nabla u^{\varepsilon}.(\nabla u^{\varepsilon}-\nabla u^{*})\\ &= \int_{\Omega} f_{\varepsilon}(u^{\varepsilon}-u^{*})\\ &= \int_{\Omega}(f_{\varepsilon}-f)(u^{\varepsilon}-u^{*})+\int_{\Omega} f(u^{\varepsilon}-u^{*})\end{aligned}$$

Since $u^{\varepsilon} \rightharpoonup u^{*}$ in $\mathbb{V}^{\infty}$, so $u^{\varepsilon} \rightharpoonup^{*} u^{*}$ in $L^{\infty}(\Omega)$, so there exists a constant $A>0$ such that

$$\|u^{\varepsilon}-u^{*}\|_{L^{\infty}(\Omega)} \le A,$$

so there exists $B>0$ and we have

$$\begin{aligned}\langle -\Delta u^{\varepsilon}, u^{\varepsilon}-u^{*}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} &= \int_{\Omega}(f_{\varepsilon}-f)(u^{\varepsilon}-u^{*})+\int_{\Omega} f(u^{\varepsilon}-u^{*})\\ &\le B\int_{\Omega}|f_{\varepsilon}-f|+\int_{\Omega} f(u^{\varepsilon}-u^{*})\end{aligned}$$

By passing to the limit, we obtain

$$\limsup_{\varepsilon\to 0}\langle -\Delta u^{\varepsilon}, u^{\varepsilon}-u^{*}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \le 0$$

Since $-\Delta_p$ is of type $(S+)$, hence

$$u^{\varepsilon} \to u^{*} \text{ in } \mathbb{V}^{\infty}$$

□

3. Limit study

In this section, we will interest to the asymptotic behaviour of the solution u^{ε} of the problem (2.2), and this behaviour will be derived with the epi-convergence method, (see definition 4.1). Now we will determinate the epi-limit of the energy functional, linked to the minimization problem (2.2), defined by

$$\mathcal{F}^{\varepsilon}(v)=\frac{1}{p}\int_{\Omega}|\nabla v|^{p}-\int_{\Omega} f_{\varepsilon}v,\ \forall v\in\mathbb{V}^{\infty}. \tag{3.1}$$

We design by τ the strong topology on the space $\mathbb{V}^{\infty}$. In the sequel, we shall characterize, the epi-limit of the energy functional given by (3.1) in the following theorem.

Theorem 3.1. *There exists a functional* $\mathcal{F}\colon \mathbb{V}^{\infty}\to\mathbb{R}$ *such that*

$$\tau - lim_{e}\mathcal{F}^{\varepsilon}{=}\mathcal{F} \quad \text{in } \mathbb{V}^{\infty},$$

where $\mathcal{F}$ *is given by*

$$\mathcal{F}(v)=\frac{1}{p}\int_{\Omega}|\nabla v|^{p}-\int_{\Omega} fv,\ \forall v\in\mathbb{V}^{\infty}.$$

Proof: (a) We are now in position to determine the upper epi-limit.
Let $u \in \mathbb{V}^\infty$, let us $u^\varepsilon = u$, so $u^\varepsilon \to u$ in $\mathbb{V}^\infty$ when $\varepsilon \to 0$, so we obtain

$$\mathcal{F}^\varepsilon(u^\varepsilon) = \frac{1}{p}\int_\Omega |\nabla u^\varepsilon|^p - \int_\Omega f_\varepsilon u^\varepsilon.$$

Since $f_\varepsilon \to f$ in $L^1(\Omega)$ when ε close to 0, so

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega fu.$$

(b) We are now in position to determine the lower epi-limit. Let $u \in \mathbb{V}^\infty$ and $(u^\varepsilon) \subset \mathbb{V}^\infty$ such that

$$u^\varepsilon \to u \text{ in } \mathbb{V}^\infty \text{ when } \varepsilon \to 0 \tag{3.2}$$

We have

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = \liminf_{\varepsilon\to 0} \left(\frac{1}{p}\int_\Omega |\nabla u^\varepsilon|^p - \int_\Omega f_\varepsilon u^\varepsilon\right), \tag{3.3}$$

According to (3.2), we obtain

$$\lim_{\varepsilon\to 0} \frac{1}{p}\int_\Omega |\nabla u^\varepsilon|^p = \frac{1}{p}\int_\Omega |\nabla u|^p. \tag{3.4}$$

We have

$$\int_\Omega f_\varepsilon u^\varepsilon - \int_\Omega fu = \int_\Omega f_\varepsilon(u^\varepsilon - u) + \int_\Omega (f_\varepsilon - f)u.$$

Since $f_\varepsilon \to f$ in $L^1(\Omega)$ when ε close to 0, so for a small enough ε there exists a constant $c > 0$ such that $f_\varepsilon(x) \le c$, a.e $x \in \Omega$, from the Hôlder inequality we obtain

$$\begin{aligned}\left|\int_\Omega f_\varepsilon u^\varepsilon - \int_\Omega fu\right| &\le c|\Omega|^{\frac{1}{p'}}\|u^\varepsilon - u)\|_{L^p(\Omega)} + \int_\Omega |(f_\varepsilon - f)u| \\ &\le c|\Omega|^{\frac{1}{p'}}\|u^\varepsilon - u)\|_{L^p(\Omega)} + \|f_\varepsilon - f\|_{L^1(\Omega)}\|u\|_{L^\infty(\Omega)}\end{aligned}$$

so by passing to the limit when $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\left|\int_\Omega f_\varepsilon u^\varepsilon - \int_\Omega fu\right| = 0 \tag{3.5}$$

From to (3.4) and (3.5), (3.3) becomes

$$\liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f_\varepsilon u. \tag{3.6}$$

Finally $\tau - \lim_e \mathcal{F}^\varepsilon = \mathcal{F}$ in $\mathbb{V}^\infty$ □

In the sequel, we will determine the limit problem linked to (2.2), when ε approaches to zero. Thanks to the epi-convergence results, (see Annex, theorem 4.2) and the theorem 3.1, we have $\mathcal{F}^\varepsilon$ τ-epiconverges to $\mathcal{F}$ in $\mathbb{V}^\infty$.

Proposition 3.2. *There exists an unique element $u^* \in \mathbb{V}^\infty$ satisfying:*

$$\mathcal{F}(u^*) = \inf_{v\in\mathbb{V}^\infty}\{\mathcal{F}(v)\}.$$

Proof: Thanks to proposition 2.1, the solution of the problem (2.2), $(u^\varepsilon)_{\varepsilon>0}$, possess a subsequence still denoted by $(u^\varepsilon)_{\varepsilon>0}$ converges strongly to an element $u^* \in \mathbb{V}^\infty$. And thanks to a classical epi-convergence result, theorem 4.2, it follows that u^* is a solution of the following limit problem

$$\inf_{v\in\mathbb{V}^\infty}\{\mathcal{F}(v)\}. \tag{3.7}$$

According to the convexity strict of $\mathcal{F}$, so the problem (3.7) admits an unique solution in $\mathbb{V}^\infty$. □

Conclusion

We showed that the τ-cluster point of the solution of the problem (2.2) is the solution of the problem (1.1).

4. Annex

Definition 4.1 ([1, Definition 1.9])**.** *Let $(\mathbb{X},\tau)$ be a metric space and $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R}\cup\{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X},\tau)$, noted $\tau - lim_e F^\varepsilon = F$, if the following assertions are satisfied*

- *For all $x \in \mathbb{X}$, there exists x^0_ε, $x^0_\varepsilon \xrightarrow{\tau} x$ such that $\limsup_{\varepsilon\to 0} F^\varepsilon(x^0_\varepsilon) \leq F(x)$.*
- *For all $x \in \mathbb{X}$ and all x_ε with $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon) \geq F(x)$.*

This epi-convergence is a special case of the $\Gamma-$convergence introduced by De Giorgi (1979) [7]. It is well suited to the asymptotic analysis of sequences of minimization problems since one has the following fundamental result.

Theorem 4.2 ([1, theorem 1.10])**.** *Suppose that*

1. *F^ε admits a minimizer on $\mathbb{X}$,*
2. *The sequence $(\overline{u}^\varepsilon)$ is τ-relatively compact,*
3. *The sequence F^ε epi-converges to F in this topology τ.*

Then every cluster point $\overline{u}$ of the sequence $(\overline{u}^\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon'\to 0} F^{\varepsilon'}(\overline{u}^{\varepsilon'}) = F(\overline{u}),$$

if $(\overline{u}^{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\overline{u}^\varepsilon)_\varepsilon$ which converges to $\overline{u}$.

Acknowledgments

The author would like to thank the anonymous referees for interesting remarks.

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