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Generalized Locally-$\tau_g\star$-closed sets

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abstract: In this paper, we define and study a new class of generally locally closed sets called $\mathcal{I}$-locally-$\tau_g^\star$-closed sets in ideal topological spaces. We also discuss various characterizations of $\mathcal{I}$-locally-$\tau_g^\star$-closed sets in terms of g-closed sets and $\mathcal{I}_g$-closed sets.

Key Words: Ideal topological space, g-open set, g-closed set, g-local function, $(.)^\star_g$- operator, $\tau_g^\star$-open and $\tau_g^\star$-closed.

Contents

1. Introduction

According to Bourbaki [3], a locally closed set is a intersection of an open set and a closed set. In [6], Levine defined a new class of generalized open and closed sets, and discussed their characterizations in detail. In [1], Balachandran, Sundaram and Maki defined and studied generalized locally closed sets using generalized closed sets and generalized open sets. In this paper, we introduce and study a new class of $\mathcal{I}$-locally-$\tau_g^\star$-closed sets using g-local functions defined in [2] with respect to the family of generalized open sets and ideal. Also we discuss various properties of this operator in detail.

2. Preliminaries

An *ideal* $\mathcal{I}$ [5] on X is a nonempty collection of subsets of X satisfying the following: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$, and (ii) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. A topological space (X, τ) together with an ideal $\mathcal{I}$ is called an *ideal topological space* and is denoted by $(X, \tau, \mathcal{I})$. For each subset A of X, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every open set U containing $x\}$ is called the *local function* of A [5] with respect to $\mathcal{I}$ and τ. We simply write $A^\star$ instead of $A^\star(\mathcal{I}, \tau)$ in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [4] without mentioning it. Moreover, $cl^\star(A) = A \cup A^\star$ [8] defines a Kuratowski closure operator for a topology $\tau^\star$, on X which is finer than τ. A subset A of a topological space (X, τ) is said to be g-closed [6], if $cl(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a g-closed

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set is called a g-open set [6]. The collection of all g-open sets in a topological space (X, τ) is denoted by τ_g. The g-closure of A denoted by $cl_g(A)$ [2], defined as the intersection of all g-closed sets containing A and the g-interior of A denoted by $int_g(A)$, defined as the union of all g-open sets contained in A. For every $A \in \wp(X)$, $A^\star(\mathcal{I}, \tau_g) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every g-open set U containing $x\}$ is called the *g-local function* of A [2] with respect to $\mathcal{I}$ and τ_g and is denoted by $A_g^\star$. Also, $cl_g^\star(A) = A \cup A_g^\star$ [2] is a Kurotowski closure operator for a topology $\tau_g^\star = \{X - A \mid cl_g^\star(A) = A\}$ [2] on X which is finer than τ_g. A subset A of a topological space (X, τ) is said to be *locally closed* [1], if $A = U \cap V$ where U is open and V is closed. A subset A of a topological space (X, τ) is said to be *g-locally closed* [1], if $A = U \cap V$ where U is g-open and V is g-closed. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-locally-$\star$-closed* [7], if $A = U \cap V$ where U is open and V is $\star$-closed.

3. $\mathcal{I}$-locally-$\tau_g^\star$-closed sets

Definition 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}$-locally-$\tau_g^\star$-closed set if there exists a $\tau_g^\star$-open set U and a $\tau_g^\star$-closed set V such that $A = U \cap V$.*

The following Theorem 3.2 gives a characterization of $\mathcal{I}$-locally-$\tau_g^\star$-closed sets in terms of $\tau_g^\star$-open sets.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then the following are equivalent.*
(a) A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.
(b) $A = U \cap cl_g^\star(A)$ for some $\tau_g^\star$-open set U.
(c) $A_g^\star - A$ is a $\tau_g^\star$-closed set.
(d) $(X - A_g^\star) \cup A = A \cup (X - cl_g^\star(A))$ is a $\tau_g^\star$-open set.
(e) $A \subset int_g^\star(A \cup (X - A_g^\star))$.

Proof: $(a) \Rightarrow (b)$. If A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set, then there exists a $\tau_g^\star$-open set U and a $\tau_g^\star$-closed set F such that $A = U \cap F$. Clearly, $A \subset U \cap cl_g^\star(A)$. Since F is $\tau_g^\star$-closed, $cl_g^\star(A) \subset cl_g^\star(F) = F$ and so $U \cap cl_g^\star(A) \subset U \cap F = A$. Therefore, $A = U \cap cl_g^\star(A)$ for some $\tau_g^\star$-open set U.
$(b) \Rightarrow (c)$. Now $A_g^\star - A = A_g^\star \cap (X - A) = A_g^\star \cap (X - (U \cap cl_g^\star(A))) = A_g^\star \cap (X - U)$. Therefore, $A_g^\star - A$ is $\tau_g^\star$-closed.
$(c) \Rightarrow (d)$. Since $X - (A_g^\star - A) = (X - A_g^\star) \cup A$, $(X - A_g^\star) \cup A$ is $\tau_g^\star$-open. Clearly, $(X - A_g^\star) \cup A = A \cup (X - cl_g^\star(A))$.
$(d) \Rightarrow (e)$. The proof is clear.
$(e) \Rightarrow (a)$. Since $A_g^\star$ is a g-closed set, $X - A_g^\star = int_g^\star(X - A_g^\star) \subset int_g^\star(A \cup (X - A_g^\star))$. Then by hypothesis, $A \cup (X - A_g^\star) \subset int_g^\star(A \cup (X - A_g^\star))$ and so $A \cup (X - A_g^\star)$ is $\tau_g^\star$-open. Since $A = (A \cup (X - A_g^\star)) \cap cl_g^\star(A)$, A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set. □

Clearly, every open subset of an ideal topological space $(X, \tau, \mathcal{I})$ is always an $\mathcal{I}$-locally-$\tau_g^\star$-closed, since every open set is a $\tau_g^\star$-open set and X is $\tau_g^\star$-closed. The following Example 3.3 shows that the converse is not true in general. Also, every $\tau_g^\star$-closed set is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set, since X is $\tau_g^\star$-open. Example 3.4 below shows that the converse is not true in general.

Example 3.3. *Let (X, τ) be a non-discrete topology. If $\mathcal{I} = \wp(X)$, then every subset of X is $\star$-closed and so every subset of X is $\tau_g^\star$-closed and hence $\mathcal{I}$-locally-$\tau_g^\star$-closed. So there exists $\mathcal{I}$-locally-$\tau_g^\star$-closed sets which are not open.*

Example 3.4. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. If $A = \{b\}$, then $A_g^\star = \{b, c, d\} \nsubseteq A$ and so A is not a $\tau_g^\star$-closed set. Besides, since A is $\tau_g^\star$-open, A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.*

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set and $A_g^\star = X$, then A is a $\tau_g^\star$-open set.*

Proof: If A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set, then by Theorem 3.2(e), $A \subset int_g^\star(A \cup (X - A_g^\star))$. Since $A_g^\star = X$ and so $A \subset int_g^\star(A)$ which implies that A is $\tau_g^\star$-open. □

Corollary 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A_g^\star = X$, where $A \subset X$. Then A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set if and only if A is a $\tau_g^\star$-open set.*

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be an $\mathcal{I}$-locally-$\tau_g^\star$-closed subset of X, Then, the following hold.*
(a) If B is a $\tau_g^\star$-closed set, then $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.
(b) If B is a $\tau_g^\star$-open set, then $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.
(c) If B is either a g-open or a g-closed set, then $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.
(d) If B is either an open or a closed set, then $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.

Proof: Since A is$\mathcal{I}$-locally-$\tau_g^\star$-closed, there exists a $\tau_g^\star$-open set U and a $\tau_g^\star$-closed set F such that $A = U \cap F$.
(a) Let B be $\tau_g^\star$-closed. Then $A \cap B = (U \cap F) \cap B = U \cap (F \cap B)$, where $F \cap B$ is $\tau_g^\star$-closed. Hence, $A \cap B$ is $\mathcal{I}$-locally-$\tau_g^\star$-closed.
(b) If B is a $\tau_g^\star$-open set, then $A \cap B = (U \cap F) \cap B = (U \cap B) \cap F$, where $U \cap B$ is $\tau_g^\star$-open. Therefore, $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.
(c) If B is either a g-open or a g-closed set, then B is either $\tau_g^\star$-open or $\tau_g^\star$-closed. Therefore, by (a) and (b), $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.
(d) Since every open and closed set is $\tau_g^\star$-open and $\tau_g^\star$-closed respectively, the proof is clear. □

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the intersection of two $\mathcal{I}$-locally-$\tau_g^\star$-closed sets is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.*

Proof: Let A and B be $\mathcal{I}$-locally-$\tau_g^\star$-closed subsets of $(X, \tau, \mathcal{I})$. Then $A = U_1 \cap V_1$ and $B = U_2 \cap V_2$ for some $\tau_g^\star$-open sets U_1 and U_2 and $\tau_g^\star$-closed sets V_1 and V_2. Now $A \cap B = (U_1 \cap V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2)$, where $U_1 \cap U_2$ is $\tau_g^\star$-open and $V_1 \cap V_2$ is $\tau_g^\star$-closed. This implies that $A \cap B$ is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set. □

Corollary 3.9. *The family of all $\mathcal{I}$-locally-$\tau_g^\star$-closed sets in any ideal topological space $(X, \tau, \mathcal{I})$ is closed under arbitrary intersection.*

Theorem 3.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If A is a g-locally-closed set, then A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.*

Proof: If A is g-locally-closed, then there exists a g-open set U and a g-closed set V such that $A = U \cap V$. Since every g-closed set is $\tau_g^\star$-closed and every g-open set is $\tau_g^\star$-open, A is $\mathcal{I}$-locally-$\tau_g^\star$-closed. □

Theorem 3.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$ where $\mathcal{I} = \{\emptyset\}$. Then A is a g-locally closed set if and only if A is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set.*

Proof: By Theorem 3.10, every g-locally closed set is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set. Conversely, since $\mathcal{I} = \{\emptyset\}$, $A_g^\star = cl_g(A)$ which implies that $\tau_g^\star$-closed sets coincide with g-closed sets. Therefore, $\mathcal{I}$-locally-$\tau_g^\star$-closed sets coincide with g-locally-closed sets when $\mathcal{I} = \{\emptyset\}$. □

Clearly, every $\star$-closed set is an $\mathcal{I}$-locally-$\tau_g^\star$-closed set. The following Example 3.12 shows that the converse is not true in general.

Example 3.12. *Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. If $A = \{b, c, d\}$, $cl^\star(A) = X$ and so A is not $\star$-closed. But $A_g^\star = \{b, c, d\} = A$ which implies that A is $\tau_g^\star$-closed and so $\mathcal{I}$-locally-$\tau_g^\star$-closed set.*

In ideal topological spaces, locally closed sets are $\mathcal{I}$-locally-$\tau_g^\star$-closed sets, since closed sets are $\tau_g^\star$-closed sets. The following Example 3.13 shows that the converse is not true in general.

Example 3.13. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. If $\mathcal{I} = \wp(X)$, then every subset of X is $\tau_g^\star$-closed. If $A = \{b, c, d\}$, then A is $\mathcal{I}$-locally-$\tau_g^\star$-closed. Since X is the only open set containing A and A is not closed, A is not a locally closed set.*

Theorem 3.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If A is $\mathcal{I}$-locally-$\star$-closed, then A is $\mathcal{I}$-locally-$\tau_g^\star$-closed.*

Proof: If A is $\mathcal{I}$-locally-$\star$-closed, then there exists an open set U and a $\star$-closed set V such that $A = U \cap V$. Since every $\star$-closed set is $\tau_g^\star$-closed, A is $\mathcal{I}$-locally-$\tau_g^\star$-closed. □

The following Example 3.15 shows that the converse of Theorem 3.13 is not true in general.

Example 3.15. *Consider Example 3.12, if $A = \{a\}$, $cl^\star(A) = \{a, d\}$, then A is not $\star$-closed, but $cl_g^\star(A) = A$ implies that A is $\tau_g^\star$-closed. Hence A is $\mathcal{I}$-locally-$\tau_g^\star$-closed. Since X is the only open set containing A and A is not $\mathcal{I}$-locally-$\star$-closed.*

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