Properties at potential blow-up times for the incompressible Navier-Stokes equations

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ABSTRACT: In this paper we consider the Cauchy problem for the 3D Navier–Stokes equations for incompressible flows. The initial data are assumed to be smooth and rapidly decaying at infinity. A famous open problem is whether classical solutions can develop singularities in finite time. Assuming the maximal interval of existence to be finite, we give a unified discussion of various known solution properties as time approaches the blow-up time.

Key Words: Navier–Stokes equations, incompressible flows, blow–up.

Contents

1. Introduction

In this paper we consider the Cauchy problem for the 3D Navier–Stokes equations,

$$u_t + u \cdot \nabla u + \nabla p = \Delta u, \quad \nabla \cdot u = 0, \quad u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R}^3. \qquad (1.1)$$

Despite recent progress (see, for example, [1,2]), the current mathematical theory of the problem (1.1) remains fundamentally incomplete: it is known that a weak solution exists for all time $t \geq 0$ if $f \in L^2(\mathbb{R}^3)$, $\nabla \cdot f = 0$, but it is not known if weak solutions are unique. This is recognized as a major open problem since the fundamental paper of Leray [3] (for a brief account of Leray's works, see [4]). On the other hand, if f is more regular, then a unique classical solution exists in some maximal interval $0 \leq t < T_f$, but it is not known if T_f can be finite or is always infinite. In other words, it is not known if classical solutions can break down in finite time.

The Navier–Stokes equations are of fundamental importance in continuum mechanics. When one derives the equations from the balance laws of mass and momentum and from principle assumptions relating the stresses to velocity gradients, then smoothness of the solution is assumed. To make the model of the Navier–Stokes equations self–consistent, one would like to prove that singularities in the solution do not develop in finite time, from smooth initial data with finite energy. Thus far, however, this aim has not been achieved. It remains one of the fundamental open problems in nonlinear analysis, being included in the Millenium Prize Problems by the Clay Mathematics Institute [5]. In fact, it has invariably appeared in all major recent lists of the most important problems of Modern Mathematics, see e.g. [6,7,8,9,10].

In this paper we consider only classical solutions of the problem (1.1), and under our assumptions on f these will be C^∞ functions. If one normalizes the pressure so that $p(x,t) \to 0$ as $|x| \to \infty$, then the solution $(u(x,t), p(x,t))$ is unique. Its maximal interval of existence is denoted by $0 \leq t < T_f$. Assuming T_f to be finite, certain norms of $u(\cdot, t)$ will tend to infinity as $t \to T_f$, while other norms remain bounded. This issue is well studied in the literature, see e.g. [3,11,12,13,14,15,16,17,18,19], but the results are somewhat scattered. We will review some results that we consider to be very important and will also derive lower bounds for some blow–up rates. The results considered in this review all fit in our unified discussion that requires only a small selected set of relatively basic ideas. Thus, in spite of their undisputable importance, many fundamental results such as [20,21,22,23,24,25] and their recent developments will be left out.[1]

Another issue is to compare two functionals of $u(\cdot, t)$ that blow up as $t \to T_f$. Which one will blow up faster? We believe that a better understanding of this issue is important for further progress on the blow–up question and recall some simple results in Section 5.

The intent of this paper is to give rather complete proofs of some solution properties for (1.1) that must hold, as t approaches T_f, *if T_f is finite*. These results and their proofs may be helpful if one wants to construct a solution that

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[1] For a nice discussion of the celebrated Caffarelli–Kohn–Nirenberg's regularity result, see [26].

actually does blow up. They may also be helpful to show that blow–up is ultimately impossible.

For a treatment of many recent developments regarding the Navier–Stokes equations using methods of modern analysis, the reader is referred to [27,28,29]. For similar blow–up questions concerning the related Euler equations, see e.g. [30,31,32,33,34].

For simplicity of presentation (and to avoid unessential complications near $t = 0$), we put strong smoothness and spatial decay assumptions on the initial state f and require (as in [5]) that f is a divergence–free C^∞ function with all of its derivatives in $L^2(\mathbb{R}^3)$, i.e., we assume that $f \in H^n(\mathbb{R}^3)$ for all n, with $\nabla \cdot f = 0$.

As to our **notation**, we will be using the following standard definitions:

$$
\begin{aligned}
|u|^2 &= u_1^2 + u_2^2 + u_3^2 \quad \text{for} \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3, \\
|\alpha| &= \alpha_1 + \alpha_2 + \alpha_3 \quad \text{for a multi–index} \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \\
D^\alpha &= D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}, \quad D_j = \partial/\partial x_j, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \\
\| u(\cdot, t) \|_{L^q} &= \left(\sum_{i=1}^{3} \int_{\mathbb{R}^3} |u_i(x,t)|^q \, dx \right)^{1/q}, \quad 1 \le q < \infty, \quad u = (u_1, u_2, u_3), \\
\| u(\cdot, t) \| &= \| u(\cdot, t) \|_{L^2}, \\
\| u(\cdot, t) \|_\infty &= \sup \left\{ |u_i(x,t)| : x \in \mathbb{R}^3, \, 1 \le i \le 3 \right\}, \\
\| \mathcal{D}^n u(\cdot, t) \|_{L^q} &= \left(\sum_{i=1}^{3} \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \int_{\mathbb{R}^3} |D_{j_1} \cdots D_{j_n} u_i(x,t)|^q \, dx \right)^{1/q}, \quad 1 \le q < \infty, \\
\| \mathcal{D}^n u(\cdot, t) \| &= \| \mathcal{D}^n u(\cdot, t) \|_{L^2}, \\
\| \mathcal{D}^n u(\cdot, t) \|_\infty &= \sup \left\{ |D^\alpha u_i(x,t)| : x \in \mathbb{R}^3, \, 1 \le i \le 3, \, |\alpha| = n \right\}, \\
(u, v) &= \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_j(x) \, v_j(x) \, dx.
\end{aligned}
$$

(Note that $\| u(\cdot, t) \|_{L^q} \to \| u(\cdot, t) \|_\infty$, $\| \mathcal{D}^n u(\cdot, t) \|_{L^q} \to \| \mathcal{D}^n u(\cdot, t) \|_\infty$ as $q \to \infty$, for all n.[2]) We may occasionally write $\| u \|_{L^q}$ instead of $\| u(\cdot, t) \|_{L^q}$, for simplicity. Constants will be usually denoted by the letters C, c, K; we write C_λ to indicate a constant whose value may depend on a given parameter λ, etc. As usual, we often keep the same symbol for constants in spite of possible changes in their numerical values (so, for example, we write C^2 again as C, and so forth).

An outline of the paper is as follows. In Section 2, we show that a bound on the maximum norm $\| u(\cdot, t) \|_\infty$ in some interval $0 \le t < T$ implies bounds for *all*

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[2] Moreover, it is worth noticing that, under the definitions above, if an inequality of Gagliardo type $\| \mathsf{u} \|_{L^q} \le K \| \mathsf{u} \|_{L^{r_1}}^{1-\theta} \| \nabla \mathsf{u} \|_{L^{r_2}}^{\theta}$, $0 \le \theta \le 1$, holds for *scalar* functions u (and some appropriate constant K), then it will also be valid for *vector* functions u with the *same* constant K as in the scalar case. Similarly, one has $\| \mathcal{D}^n u(\cdot, t) \|_{L^q} \le \| \mathcal{D}^n u(\cdot, t) \|_{L^{q_1}}^{1-\theta} \| \mathcal{D}^n u(\cdot, t) \|_{L^{q_2}}^{\theta}$ if $1/q = (1-\theta)/q_1 + \theta/q_2$, $0 \le \theta \le 1$, and so on.

derivatives of $u(\cdot, t)$ in the same interval. This is a well known result that dates back to Leray [3] (see also [35]), but since it is the basis for all our blow–up results we will prove it here. An important implication is the following: if $\| u(\cdot, t) \|_\infty$ is bounded in $0 \leq t < T$ for some finite T, then $u(\cdot, t)$ can be continued as a C^∞ solution beyond T. This follows from well–known local constructions of solutions. (See, for example, [35] for an elementary proof. See also [16,36] for the development of a local theory under much weaker assumptions on the initial state f.) In other words, we can state the following first blow–up result:

Theorem 1.1. *If $T_f < \infty$, then* [3]
$$\sup_{0 \leq t < T_f} \| u(\cdot, t) \|_\infty = \infty. \tag{1.2}$$

In Sections 3 and 4 below, we show that boundedness of $\| u(\cdot, t) \|_{L^q}$ in some interval $0 \leq t < T$ for some $q > 3$, or that of $\| \mathcal{D}u(\cdot, t) \|_{L^q}$ if $3/2 < q \leq \infty$, implies boundedness of $\| u(\cdot, t) \|_\infty$ in the same time interval. In particular, Theorem 1.1 yields the first part of the following result:

Theorem 1.2. (i) *Let $\frac{3}{2} < q \leq \infty$. If $T_f < \infty$, then*
$$\sup_{0 \leq t < T_f} \| \mathcal{D}u(\cdot, t) \|_{L^q} = \infty. \tag{1.3}$$

(ii) *For each $\frac{3}{2} \leq q < 3$, there exists an absolute constant $c_q > 0$, independent of t and f, with the following property: if $T_f < \infty$, then*

$$\| \mathcal{D}u(\cdot, t) \|_{L^q} \geq c_q \cdot (T_f - t)^{-\frac{q - 3/2}{q}} \qquad \forall \; 0 \leq t < T_f. \tag{1.4}$$

Regarding (1.3), it will be shown in Section 4 that one actually has, for each $\frac{3}{2} < q \leq \infty$, the stronger property

$$\lim_{t \nearrow T_f} \| \mathcal{D}u(\cdot, t) \|_{L^q} = \infty \qquad (\text{if } T_f < \infty). \tag{1.5}$$

For $\frac{3}{2} < q < 3$, the limit relation (1.5) is an obvious consequence of (1.4). For $q = 2$, Theorem 1.2 was originally proved in [3]. The estimate (1.4) is only one of *many* similar lower bound results for blow-up rates of solution–size quantities $Q(u(\cdot, t))$ that have been obtained since Leray [3]. In general, these bounds result from some form of local control on $Q(u(\cdot, t))$ that is typically obtained in one of the following basic ways:

(I) lower bounds for the maximum existence time T of solution $u(\cdot, t)$ given in terms of Q;

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[3] Actually, if $T_f < \infty$ then one has $\lim_{t \nearrow T_f} \| u(\cdot, t) \|_\infty = \infty$, cf. Theorem 1.3 below.

(II) differential or integral inequalities satisfied by $Q(u(\cdot, t))$ while the solution exists;

(III) relationships with other functionals $\tilde{Q}(u)$, like Sobolev or interpolation inequalities.

As to method (I), we observe that lower bound estimates for T are a standard by–product of construction schemes in existence theory, so that blow–up estimates like (1.4) are actually very natural and widespread in the literature. Thus, for example, for the solutions $u(\cdot, t)$ of (1.1), it can be shown (see e.g. [3,13,37]) that

$$T \;\geq\; C \,\| \, \mathcal{D}f \, \|^{-4} \tag{1.6}$$

for some absolute constant $C > 0$ independent of f. It follows that, given $0 \leq t_0 < T_f$ arbitrary, we have $T_f - t_0 > C \,\| \, \mathcal{D}u(\cdot, t_0) \, \|^{-4}$, which is (1.4) in case $q = 2$. As noted in [18], this is a two–way route: had we had the estimate (1.4) for $q = 2$ in the first place, then we could have gotten (1.6) from it just as easily by merely taking $t = 0$ in (1.4).

In this review, as we will be bypassing existence theory,[4] our approach to obtaining the blow–up estimates considered here will be based on the methods (II) and (III) alone.[5] In particular, our derivation of (1.4) for $q = 2$ in Section 3 uses a well known differential inequality satisfied by the function $\| \, \mathcal{D}u(\cdot, t) \, \|^2$, which will give us that $c_2^2 \geq 2\pi\sqrt{2}$ [43]. One should note that the blow–up rate of $\| \, \mathcal{D}u(\cdot, t) \, \|$, as expressed by the estimate (1.4), is clearly consistent with the fundamental upper bound

$$\int_0^{T_f} \| \, \mathcal{D}u(\cdot, t) \, \|^2 \, dt \;\leq\; \tfrac{1}{2} \, \| \, f \, \|^2 \tag{1.7}$$

that follows from the energy equality satisfied by $u(\cdot, t)$, which we recall in Theorem 2.1. It then follows from (1.4) and (1.7) that, if $T_f < \infty$,

$$2\, c_2^2 \, T_f^{1/2} \;=\; c_2^2 \int_0^{T_f} (T_f - t)^{-1/2} \, dt \;\leq\; \int_0^{T_f} \| \, \mathcal{D}u(\cdot, t) \, \|^2 \, dt \;\leq\; \tfrac{1}{2} \, \| \, f \, \|^2,$$

so that we must have

$$c_2^4 \, \| \, \mathcal{D}f \, \|^{-4} \;\leq\; T_f \;\leq\; \frac{1}{16 \, c_2^4} \, \| \, f \, \|^4 \qquad (\text{if } T_f < \infty), \tag{1.8}$$

where, again, the first inequality follows from (1.4) by taking $t = 0$. Therefore, recalling that $c_2^2 \geq 2\pi\sqrt{2}$, finite–time blow–up at T_f is only possible if we have

$$\| \, u(\cdot, t) \, \| \, \| \, \mathcal{D}u(\cdot, t) \, \| \;\geq\; 4\pi\sqrt{2} \qquad \forall \; 0 \leq t < T_f \qquad (\text{if } T_f < \infty). \tag{1.9}$$

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[4] Some standard references for existence results are [3,13,16,19,29,35,37,38,39,40,41,42].

[5] The way to obtain lower bound estimates for blow–up rates out of nonlinear differential or integral inequalities is very simple and is shown in Lemmas 3.6 and 4.3 below, respectively.

This is only one of many *necessary* conditions for finite–time blow–up that have been found since the fundamental paper of Leray [3] (see (1.16) and (4.17) for other similar examples), the most celebrated of them being the Beale-Kato-Majda condition [30]

$$\int_0^{T_f} \| \, \nabla \times u(\cdot, t) \, \|_{L^\infty} \, dt \; = \; \infty \qquad \text{(if } T_f < \infty \text{)}, \qquad (1.10)$$

originally obtained for the Euler equations, but which also holds for the Navier–Stokes equations. Actually, the derivation of (1.10) for Navier-Stokes is much easier than for Euler, as we will show in Section 6.

The proof of Theorem 1.2 is completed by method (III) after we consider, in Section 4, the important solution norms

$$\| \, u(\cdot, t) \, \|_{L^q}, \qquad 3 < q \leq \infty.$$

If any of these norms stays bounded in some interval $0 \leq t < T$, we show that $\| \, \mathcal{D}u(\cdot, t) \, \|$ will also be bounded in that interval. As this implies boundedness for $\| \, u(\cdot, t) \, \|_\infty$ as well, it follows that

$$\sup_{0 \leq t < T_f} \| \, u(\cdot, t) \, \|_{L^q} = \infty, \qquad 3 < q \leq \infty \qquad (1.11)$$

if $T_f < \infty$. In fact, one can again derive an algebraic lower bound for the blow–up rate:

Theorem 1.3. *For each $3 \leq q \leq \infty$, there is a constant $c_q > 0$, independent of t and f, such that the following holds: if $T_f < \infty$, then*

$$\| \, u(\cdot, t) \, \|_{L^q} \geq c_q \cdot (T_f - t)^{-\kappa} \qquad \forall \; 0 \leq t < T_f, \qquad (1.12a)$$

with

$$\kappa = \frac{q-3}{2q} \quad \text{if } 3 \leq q < \infty, \qquad \kappa = \frac{1}{2} \quad \text{if } q = \infty. \qquad (1.12b)$$

In particular, from (1.12), we have $\lim\limits_{t \nearrow T_f} \| \, u(\cdot, t) \, \|_{L^q} = \infty$ if $T_f < \infty$, for each $3 < q \leq \infty$.

REMARK: The property (1.11) is also valid for the limit case $q = 3$, as shown in [44]. The proof, however, is very involved and will not be covered here. More recently, it has been shown by G. Seregin [1] the stronger result

$$\lim_{t \nearrow T_f} \| \, u(\cdot, t) \, \|_{L^3} = \infty \qquad \text{(if } T_f < \infty \text{)}. \qquad (1.13)$$

It then follows from (1.13) and the 3D Sobolev inequality $\| \, \mathsf{u} \, \|_{L^3} \leq K \, \| \, \nabla \mathsf{u} \, \|_{L^{3/2}}$ that the properties (1.3) and (1.5) above are both valid for $q = 3/2$ as well.

The estimates (1.12) were originally given in [3] and reobtained in a more general setting using semigroup ideas in [14]. They immediately imply lower bounds for the existence time of $u(\cdot, t)$ of the form

$$T \geq C_q \| f \|_{L^q}^{-\frac{2q}{q-3}}, \qquad 3 < q \leq \infty, \tag{1.14}$$

where $C_q = c_q^{\frac{2q}{q-3}}$. The estimates (1.14) are obtained directly from existence analysis in [3] (for $q = \infty$) and in [17] (for $3 < q < \infty$), thus providing another proof for (1.12). Again, our derivation of (1.12), which is carried out in Section 4, follows the method (II), first along the lines of [14] using some well established integral inequalities satisfied by the quantities $\| u(\cdot, t) \|_{L^q}$ to obtain the result, and then by deriving some less known differential inequalities that can be used for this purpose just as easily. We also obtain from the latter analysis another proof of the following result (see e.g. [3,13,14,17]) on the global existence of smooth solutions for the Navier–Stokes problem (1.1).

Theorem 1.4. *For each $3 \leq q \leq \infty$, there exists a number $\eta_q > 0$, depending only on q, such that*

$$\| f \|_{L^2}^{\frac{2q-6}{3q-6}} \| f \|_{L^q}^{\frac{q}{3q-6}} < \eta_q \implies T_f = \infty. \tag{1.15}$$

In particular, finite–time blow–up of a smooth solution $u(\cdot, t)$ can only occur if we have

$$\| u(\cdot, t) \|_{L^2}^{\frac{2q-6}{3q-6}} \| u(\cdot, t) \|_{L^q}^{\frac{q}{3q-6}} \geq \eta_q \qquad \forall\ 0 \leq t < T_f \qquad (\text{if } T_f < \infty) \tag{1.16}$$

for every $3 \leq q \leq \infty$, where $\eta_q > 0$ is the value given in (1.15) above. If blow–up happens, using the 3D Sobolev inequality

$$\| u \|_{L^{r(q)}} \leq K_q \| \nabla u \|_{L^q}, \qquad r(q) = \frac{3q}{3-q}, \qquad \frac{3}{2} \leq q < 3, \tag{1.17}$$

we obtain, from (1.12), the blow–up estimate (1.4) in Theorem 1.2 above. This illustrates the use of method (III) to derive these results. Other examples are found in Sections 4, 5 and 6 below, including the following general blow–up property for arbitrary high order derivatives of smooth solutions of (1.1).

Theorem 1.5. *Let $n \geq 2$ be an integer. If $T_f < \infty$, we have*

$$\lim_{t \nearrow T_f} \| \mathcal{D}^n u(\cdot, t) \|_{L^q} = \infty \tag{1.18}$$

for every $1 \leq q \leq \infty$.

It is also worth noticing here that, from (1.12), we clearly have

$$\int_0^{T_f} \| u(\cdot,t) \|_{L^q}^r \, dt \; = \; \infty \qquad (\text{if } T_f < \infty)$$

for any $r \geq 2q/(q-3)$, or, equivalently, for any $r \geq 2$ satisfying $2/r + 3/q \leq 1$. It is therefore natural to expect that the so-called *Prodi-Serrin condition*,

$$\int_0^T \| u(\cdot,t) \|_{L^q}^r \, dt \; < \; \infty, \qquad \frac{2}{r} + \frac{3}{q} \; \leq \; 1, \tag{1.19}$$

for some $2 \leq r < \infty$, $3 < q \leq \infty$ (arbitrary), imposed on less regular *weak* solutions, may be sufficient to guarantee strong regularity and uniqueness properties. This is indeed the case, as shown in [23,24,25,40] (see also [13,14,16,29,45,46]), but it requires a more advanced analysis and will not be discussed here. Similar observations apply to the other blow-up quantities considered in (1.5), (1.10) or (1.13), see e.g. [1,3,13,17,44].

If there is blow–up, one can ask: do certain norms blow up faster than others? The answer is yes. For example, if $3 \leq q < r \leq \infty$, we show in Section 5 that the L^r norm blows up faster than the L^q norm, with

$$\frac{\| u(\cdot,t) \|_{L^r}}{\| u(\cdot,t) \|_{L^q}} \; \geq \; c(q,r) \cdot \| f \|_{\cdot}^{-\lambda} (T_f - t)^{-\gamma}, \qquad \gamma = \frac{r-3}{r-2} \left(\frac{1}{q} - \frac{1}{r} \right) \tag{1.20}$$

for all $0 \leq t < T_f$, where $c(q,r) > 0$ depends only on q,r, and $\lambda = 2\,(r/q-1)/(r-2)$. These relations are typically obtained by the approach (III). A further result of this kind, which is related to (1.13) above, is also included in Section 5, and given a direct proof that is independent of (1.13).

In Section 6, we briefly examine some related properties for the vorticity $\omega(\cdot,t) = \nabla \times u(\cdot,t)$. Our main goal in Section 6 is to provide a short and simple proof of the Beale-Kato-Majda blow–up condition (1.10) for smooth solutions of the Navier–Stokes equations. This particular proof is not valid for the inviscid Euler equations.

Besides the famous major problems, there are still many other open questions related to our discussion and some are indicated in the text. For additional lower bound estimates and results concerning other blow–up quantities, the reader is referred to [11,18,38,47,48].

2. A bound for $\|\,u(\cdot,t)\,\|_\infty$ implies bounds for all derivatives

Under our assumptions on f stated in the introduction, the Cauchy problem (1.1) has a unique C^∞ solution $\big(u(x,t),p(x,t)\big)$, defined in some interval $0 \le t < T_f$, with $D^\alpha u(\cdot,t) \in L^2(\mathbb{R}^3)$ for all multi-indices α and all $0 \le t < T_f$. (Also, recall that we always require $p(x,t) \to 0$ as $|\,x\,| \to \infty$ to make $p(\cdot,t)$ unique.)

As in [3], we set

$$J_n^2(t) \;=\; \sum_{|\,\alpha\,|\,=\,n} \|\,D^\alpha u(\cdot,t)\,\|^2, \qquad n = 0,1,2,\dots$$

The most basic estimate for the solution of the Navier–Stokes equations is the following well known energy estimate. The proof follows from multiplying the equation for $u_i(\cdot,t)$ by $u_i(\cdot,t)$ and integrating by parts (see e.g. [13,19,29,35,39,41]).

Theorem 2.1. *We have*

$$\frac{1}{2}\,\frac{d}{dt}\,J_0^2(t) \;=\; -\,J_1^2(t), \qquad \forall\; 0 \le t < T_f, \tag{2.1a}$$

so that, in particular,

$$J_0(t) \;\le\; \|\,f\,\| \quad \text{for} \quad 0 \le t < T_f \quad \text{and} \quad \int_0^t J_1^2(s)\,ds \;\le\; \frac{1}{2}\,\|\,f\,\|^2. \tag{2.1b}$$

Note that the integral bound in (2.1b) proves (1.7). To prove the next result, we will use the 3D Sobolev inequality

$$\|\,\mathrm{v}\,\|_\infty \;\le\; C\,\|\,\mathrm{v}\,\|_{H^2} \qquad \forall\; \mathrm{v} \in H^2(\mathbb{R}^3), \tag{2.2}$$

which implies

$$\|\,D^j u(\cdot,t)\,\|_\infty \;\le\; C_j \cdot \big(J_m(t) + J_0(t)\big) \quad \text{for} \quad m \ge j+2. \tag{2.3}$$

(The bound $J_n^2(t) \le C_n\cdot\big(J_m^2(t) + J_0^2(t)\big)$ for $m > n$ follows by Fourier transform.)

Theorem 2.2. *Assume that*

$$\sup_{0 \le t < T} \|\,u(\cdot,t)\,\|_\infty =: M < \infty. \tag{2.4}$$

Then each function $J_n(t)$, $n = 1,2,\dots$, is bounded in $0 \le t < T$ by some quantity $K_n > 0$ depending only on n, M, T and $\|\,f\,\|_{H^n}$, that is, $K_n = K(n,M,T,\|\,f\,\|_{H^n})$.

Proof: Using (2.4), we will first prove that $J_1(t)$, $J_2(t)$ are bounded in $0 \le t < T$, and then make an induction argument in n. We have, for any multi–index α,

$$D^\alpha u_t + D^\alpha(u \cdot \nabla u) + \nabla D^\alpha p \;=\; \Delta D^\alpha u$$

and, because $\nabla \cdot u = 0$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}J_n^2(t) &= \sum_{|\alpha|=n}(D^\alpha u, D^\alpha u_t) \\
&\leq -\sum_{|\alpha|=n}(D^\alpha u, D^\alpha(u\cdot\nabla u)) - J_{n+1}^2(t) \\
&=: S_n(t) - J_{n+1}^2(t).
\end{aligned}
$$

It is convenient to use the short notation

$$
(D^i u, D^j u\, D^k u)
$$

for any integral

$$
\int_{\mathbb{R}^3} D^\alpha u_{\nu_1}\, D^\beta u_{\nu_2}\, D^\gamma u_{\nu_3}\, dx
$$

with $\nu_1, \nu_2, \nu_3 \in \{1,2,3\}$ and $|\alpha| = i$, $|\beta| = j$, $|\gamma| = k$.

Let $n = 2$ and consider the terms appearing in S_2,

$$
(D^\alpha u, D^\alpha(u\cdot\nabla u)), \qquad |\alpha| = 2.
$$

Thus S_2 is a sum of terms of the form

$$
(D^2 u,\, u D^3 u) \quad \text{and} \quad (D^2 u,\, Du\, D^2 u).
$$

Using integration by parts, a term $(D^2 u,\, Du\, D^2 u)$ can also be written as a sum of terms $(D^2 u,\, u D^3 u)$. Since, by (2.4),

$$
|(D^2 u,\, u D^3 u)| \leq M J_2(t) J_3(t),
$$

we obtain that

$$
\begin{aligned}
\frac{d}{dt}J_2^2(t) &\leq CM J_2(t) J_3(t) - 2J_3^2(t) \\
&\leq C^2 M^2 J_2^2(t)
\end{aligned}
$$

for some constant $C > 0$. Boundedness of $J_2(t)$ in $0 \leq t < T$ follows. Similarly, we get

$$
\begin{aligned}
\frac{d}{dt}J_1^2(t) &\leq CM J_1(t) J_2(t) - 2J_2^2(t) \\
&\leq C^2 M^2 J_1^2(t),
\end{aligned}
$$

so that the result is true for $J_1(t)$ as well.

Now, let $n \geq 2$ and assume $J_n(t)$ to be bounded in $0 \leq t < T$. We have, from (1.1),

$$
\frac{1}{2}\frac{d}{dt}J_{n+1}^2(t) \leq S_{n+1}(t) - J_{n+2}^2(t),
$$

where $S_{n+1}(t)$ is a sum of terms

$$T_j(t) = (D^{n+2}u, D^j u \, D^{n+1-j}u), \qquad 0 \le j \le n.$$

There are three cases to consider:

(i) Let $0 \le j \le n-2$. We have, by (2.3),

$$
\begin{aligned}
|T_j(t)| &\le C_n \, \| D^j u(\cdot, t) \|_\infty \, J_{n+2}(t) \, (J_{n+1}(t) + J_0(t)) \\
&\le C_n \, J_{n+2}(t) \, (J_{n+1}(t) + J_0(t)).
\end{aligned}
$$

In the latter estimate we have used (2.3) and the induction hypothesis.

(ii) Let $j = n-1$. We have, by (2.3),

$$
\begin{aligned}
|T_{n-1}(t)| &\le C_n \, \| D^{n-1} u(\cdot, t) \|_\infty \, J_{n+2}(t) \, J_2(t) \\
&\le C_n \, \| D^{n-1} u(\cdot, t) \|_\infty \, J_{n+2}(t) \\
&\le C_n \, (J_{n+1}(t) + J_0(t)) \, J_{n+2}(t).
\end{aligned}
$$

In the second estimate we have used that a bound for $J_2(t)$ is already shown.

(iii) Let $j = n$. We have, by (2.3),

$$
\begin{aligned}
|T_n(t)| &\le C_n \, J_n(t) \, J_{n+2}(t) \, \| Du(\cdot, t) \|_\infty \\
&\le C_n \, J_{n+2}(t) \, (J_3(t) + J_0(t)) \\
&\le C_n \, J_{n+2}(t) \, (J_{n+1}(t) + J_0(t)).
\end{aligned}
$$

In the last estimate we have used that $n \ge 2$.

These bounds prove that

$$\frac{d}{dt} J_{n+1}^2(t) \le C_n \, J_{n+2}(t) \, (J_{n+1}(t) + J_0(t)) - 2 J_{n+2}^2(t),$$

and boundedness of $J_{n+1}(t)$ in $0 \le t < T$ follows.   ⋄

If $\| u(\cdot, t) \|_\infty$ is bounded in some interval $0 \le t < T$, then all functions $J_n(t)$ are also bounded in $0 \le t < T$ and, using (2.3), all space derivatives of $u(\cdot, t)$ are therefore bounded in maximum norm. Estimates for the pressure and its derivatives follow from the Poisson equation satisfied by $p(\cdot, t)$,

$$-\Delta p = \sum_{i,\,j\,=\,1}^{3} D_i D_j (u_i u_j). \tag{2.5}$$

Time derivatives and mixed derivatives of u can be expressed by space derivatives, using the differential equation (1.1). Hence, if $\| u(\cdot, t) \|_\infty$ is bounded in some interval $0 \le t < T$, then all derivatives of u are bounded in the same interval, and

therefore the solution (u, p) can be continued as a C^∞ solution beyond T. This proves Theorem 1.1.

3. Blow–up of $\left\| \mathcal{D}u(\cdot, t) \right\|_{L^q}$ for $\frac{3}{2} < q \leq 2$

A physically important quantity is the vorticity, $\omega(\cdot, t) = \nabla \times u(\cdot, t)$, and the total enstrophy of the flow, given by

$$\int_{\mathbb{R}^3} | \omega(x, t) |^2 \, dx.$$

If

$$\hat{u}(k) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i \, k \cdot x} \, u(x) \, dx$$

denotes the Fourier transform of a 3D incompressible field $u(x)$, then we have

$$\hat{\omega}(k, t) = i \, k \times \hat{u}(k, t), \qquad k \cdot \hat{u}(k, t) = 0,$$

and, therefore,

$$| \hat{\omega}(k, t) | = | k | | \hat{u}(k, t) |.$$

Using Parseval's relation, one finds that

$$
\begin{aligned}
\int_{\mathbb{R}^3} | \omega(x, t) |^2 \, dx \;&=\; \int_{\mathbb{R}^3} | \hat{\omega}(k, t) |^2 \, dk \\
&=\; \sum_{i, j = 1}^{3} \int_{\mathbb{R}^3} k_i^2 \, | \hat{u}_j(k, t) |^2 \, dk \\
&=\; \left\| \mathcal{D}u(\cdot, t) \right\|^2.
\end{aligned}
$$

In this section, we establish the blow–up of $\left\| \mathcal{D}u(\cdot, t) \right\|_{L^q}$, for $\frac{3}{2} < q \leq 2$, with emphasis on $q = 2$. The remaining case $q > 2$ is covered in Section 4.

3.1. Boundedness of $\left\| \mathcal{D}u(\cdot, t) \right\|$ implies boundedness of $\left\| u(\cdot, t) \right\|_\infty$

The basic result here is the following.

Theorem 3.1. *If*

$$\sup_{0 \leq t < T} \left\| \mathcal{D}u(\cdot, t) \right\| =: C_2 < \infty, \tag{3.1a}$$

then

$$\sup_{0 \leq t < T} \left\| u(\cdot, t) \right\|_\infty \leq K \tag{3.1b}$$

for some bound K that depends only on C_2, T and $\left\| \hat{f} \right\|_{L^1}$, where $\hat{f}$ denotes

the Fourier transform of the initial state f. In particular, if $T_f < \infty$, then $\sup\limits_{0 \le t < T_f} \| \, \mathcal{D}u(\cdot, t) \, \| = \infty$.

Proof: Taking the Fourier transform of the Navier–Stokes equations, we get[6]

$$\hat{u}_t + (u \cdot \nabla u)\hat{} + (\nabla p)\hat{} = - \, | \, k \, |^2 \, \hat{u},$$

or, setting $Q(x, t) = - u \cdot \nabla u - \nabla p$,

$$\hat{u}_t = - \, | \, k \, |^2 \, \hat{u} + \hat{Q}(k, t),$$

with $\hat{u}(\cdot, 0) = \hat{f}$. Since

$$(u \cdot \nabla u)\hat{} = - \hat{Q}(k, t) - (\nabla p)\hat{}$$

is the orthogonal decomposition of the vector $(u \cdot \nabla u)\hat{}$ into a vector orthogonal to k and a vector parallel to k, it follows that $| \, \hat{Q}(k, t) \, | \le | \, (u \cdot \nabla u)\hat{} \, |$. One obtains, for each k,

$$\hat{u}(k, t) = e^{- \, | \, k \, |^2 \, t} \hat{f}(k) + \int_0^t e^{- \, | \, k \, |^2 \, (t-s)} \hat{Q}(k, s) \, ds,$$

and so,

$$| \, \hat{u}(k, t) \, | \le e^{- \, | \, k \, |^2 \, t} | \, \hat{f}(k) \, | + \int_0^t e^{- \, | \, k \, |^2 \, (t-s)} | \, (u \cdot \nabla u)\hat{}(k, s) \, | \, ds.$$

Integrating in $k \in \mathbb{R}^3$ one finds that

$$
\begin{aligned}
(2\pi)^{3/2} \, \| \, u(\cdot, t) \, \|_\infty \quad &\le \quad \int_{\mathbb{R}^3} | \, \hat{u}(k, t) \, | \, dk \\
&\le \quad \| \, \hat{f} \, \|_{L^1} + \int_0^t \int_{\mathbb{R}^3} e^{- \, | \, k \, |^2 \, (t-s)} | \, (u \cdot \nabla u)\hat{}(k, s) \, | \, dk \, ds.
\end{aligned}
$$

We then apply the Cauchy–Schwarz inequality to bound the inner integral on the right-hand side by

$$I_1^{1/2} \, I_2^{1/2},$$

where

$$I_1 = \int_{\mathbb{R}^3} e^{- \, 2 \, | \, k \, |^2 \, (t-s)} \, dk = C \cdot (t-s)^{-3/2}$$

and

$$
\begin{aligned}
I_2 \quad &= \quad \int_{\mathbb{R}^3} | \, (u \cdot \nabla u)(x, s) \, |^2 \, dx \\
&\le \quad C \cdot \| \, u(\cdot, s) \, \|_\infty^2 \, \| \, \mathcal{D}u(\cdot, s) \, \|^2 \\
&\le \quad C \cdot C_2^2 \cdot \| \, u(\cdot, s) \, \|_\infty^2,
\end{aligned}
$$

using Parseval's relation and $(3.1a)$. Thus we have shown the estimate

----

[6] For more applications of Fourier transforms to (1.1) along these lines, see Section 3.2 and [43, 49].

$$(2\pi)^{3/2} \, \| \, u(\cdot, t) \, \|_\infty \;\; \leq \;\; \| \, \hat{f} \, \|_{L^1} + C \cdot C_2 \int_0^t (t - s)^{-3/4} \, \| \, u(\cdot, s) \, \|_\infty \, ds \;\; \text{for} \;\; 0 \leq t < T$$

for some constant $C > 0$. By the singular Gronwall's lemma given in Lemma 3.2 below, boundedness of $\| \, u(\cdot, t) \, \|_\infty$ in the interval $0 \leq t < T$ follows, as claimed. $\diamond$

REMARK. By the previous argument and Lemma 3.3 below, we can see that condition $(3.1a)$ also implies

$$\sup_{0 < t < T} t^{3/4} \, \| \, u(\cdot, t) \, \|_\infty \; \leq \; K_2(T) \, \| \, f \, \|_{L^2} \tag{3.2}$$

for some bound $K_2(T) > 0$ that depends on the values of C_2, T only.

The following result is an important version of Gronwall's lemma frequently used for partial differential equations, as in the proof of Theorem 3.1 above.

Lemma 3.2. *Let $A \geq 0, B > 0, 0 < \kappa < 1$. Let $\phi \in C^0([\, 0, T\, [)$ satisfy*

$$0 \leq \phi(t) \leq A + B \int_0^t (t - s)^{-\kappa} \, \phi(s) \, ds \quad \text{for} \quad 0 \leq t < T. \tag{3.3}$$

Then $\phi(t) \leq K(T) A$ for all $0 \leq t < T$, where $K(T) > 0$ depends on B, κ, T only.

Proof: For convenience, we provide a proof for Lemma 3.2 that can be easily extended to other useful similar statements like Lemma 3.3 below. To this end, we choose $\epsilon > 0$ with

$$\int_0^\epsilon \xi^{-\kappa} \, d\xi \; = \; \frac{\epsilon^{1-\kappa}}{1 - \kappa} \; \leq \; \frac{1}{2B}$$

and, given $t \in [\, 0, T\, [$ arbitrary, we take $t_0 \in [\, 0, t\,]$ such that $\phi(t_0) = \max\limits_{0 \leq s \leq t} \phi(s)$.

CASE I: $t_0 \geq \epsilon$. Then we have

$$\begin{aligned} \phi(t_0) \;\; &\leq \;\; A + B \int_0^{t_0 - \epsilon} (t_0 - s)^{-\kappa} \, \phi(s) \, ds \; + \; B \int_{t_0 - \epsilon}^{t_0} (t_0 - s)^{-\kappa} \, \phi(s) \, ds \\ &\leq \;\; A + B \epsilon^{-\kappa} \int_0^{t_0} \phi(s) \, ds \; + \; B \, \frac{1}{2B} \, \phi(t_0), \end{aligned}$$

so that

$$\phi(t) \; \leq \; \phi(t_0) \; \leq \; 2A + 2B \epsilon^{-\kappa} \int_0^t \phi(s) \, ds.$$

CASE II: $0 \leq t_0 \leq \epsilon$. We have

$$\begin{aligned}
\phi(t_0) &\leq A + B \int_0^{t_0} (t_0 - s)^{-\kappa} \phi(s) \, ds \\
&\leq A + B \phi(t_0) \int_0^{\epsilon} \xi^{-\kappa} \, d\xi \\
&\leq A + \frac{1}{2} \phi(t_0),
\end{aligned}$$

and so,

$$\phi(t) \leq \phi(t_0) \leq 2A.$$

We have thus shown that

$$\phi(t) \leq 2A + 2B \epsilon^{-\kappa} \int_0^t \phi(s) \, ds \quad \text{for} \quad 0 \leq t < T.$$

This gives, by standard Gronwall, $\phi(t) \leq 2A \exp\left\{ 2B \epsilon^{-\kappa} T \right\}$ for all $0 \leq t < T$. $\diamond$

In a similar way, the following generalization of Lemma 3.2 can be easily obtained.

Lemma 3.3. *Let $A \geq 0$, $B_1, ..., B_n > 0$. Let $\phi \in C^0([0, T[)$ satisfy*

$$0 \leq \phi(t) \leq A + \sum_{j=1}^n B_j \int_0^t s^{-\alpha_j} (t - s)^{-\beta_j} \phi(s) \, ds \quad \text{for} \quad 0 \leq t < T, \qquad (3.4)$$

with $\alpha_j \geq 0$, $\beta_j \geq 0$ satisfying $\alpha_j + \beta_j < 1$ for all $1 \leq j \leq n$. Then $\phi(t) \leq K(T)A$ for all $0 \leq t < T$, with the quantity $K(T) > 0$ depending only on T, n and B_j, α_j, β_j, $1 \leq j \leq n$.

3.2. Blow–up of $\| \mathcal{D}u(\cdot, t) \|_{L^q}$ for $\frac{3}{2} < q \leq 2$

We now extend the proof of Theorem 3.1 by using Hölder's inequality instead of the Cauchy–Schwarz inequality and the Hausdorff–Young inequality (see e.g. [50], p. 104) instead of Parseval's relation.

Theorem 3.4. *Let $\frac{3}{2} < q \leq 2$. If*

$$\sup_{0 \leq t < T} \| \mathcal{D}u(\cdot, t) \|_{L^q} =: C_q < \infty, \qquad (3.5a)$$

then

$$\sup_{0 \leq t < T} \| u(\cdot, t) \|_{\infty} \leq K \qquad (3.5b)$$

for some bound K that depends only on q, C_q, T and $\| \hat{f} \|_{L^1}$, where $\hat{f}$ denotes the Fourier transform of the initial state f. In particular, if $T_f < \infty$, then

$$\sup_{0 \leq t < T_f} \| \mathcal{D}u(\cdot, t) \|_{L^q} = \infty. \qquad (3.6)$$

Proof: We use the same notation as in the proof of Theorem 3.1. Applying Hölder's inequality to the integral

$$I = \int_{\mathbb{R}^3} e^{-|k|^2 (t-s)} |(u \cdot \nabla u)\hat{\,}(k,s)| \, dk,$$

we obtain the bound

$$I \leq I_1^{1/q} I_2^{1/q'}, \qquad \frac{1}{q} + \frac{1}{q'} = 1,$$

with

$$I_1 = \int_{\mathbb{R}^3} e^{-q|k|^2 (t-s)} \, dk = \left(\frac{\pi}{q}\right)^{3/2} (t-s)^{-3/2}$$

and

$$\begin{aligned}
I_2^{1/q'} &= \| (u \cdot \nabla u)\hat{\,}(\cdot,s) \|_{L^{q'}} \\
&\leq 3 \| u \cdot \nabla u(\cdot,s) \|_{L^q} \\
&\leq C \| u(\cdot,s) \|_\infty \| \mathcal{D}u(\cdot,s) \|_{L^q},
\end{aligned}$$

where in the first estimate we have used the Hausdorff–Young inequality, since $q \leq 2$. We obtain that

$$(2\pi)^{3/2} \| u(\cdot,t) \|_\infty \leq \| \hat{f} \|_{L^1} + C \int_0^t (t-s)^{-\kappa} \| u(\cdot,s) \|_\infty \| \mathcal{D}u(\cdot,s) \|_{L^q} \, ds,$$

where $\kappa = \frac{3}{2q} < 1$, in view that $q > \frac{3}{2}$. By Lemma 3.2, the result now readily follows.   ◇

Remarks. (*i*) By the argument above and Lemma 3.3, we see that (3.5*a*) also gives

$$\sup_{0 < t < T} t^{3/4} \| u(\cdot,t) \|_\infty \leq K_q(T) \| f \|_{L^2} \tag{3.7}$$

for some coefficient $K_q(T) > 0$ that depends on the values of q, C_q, T only.

(*ii*) The Navier–Stokes equations on the whole $\mathbb{R}^3$ enjoy the following scaling invariance: If $(u(x,t), p(x,t))$ solves the Navier–Stokes equations, then, for every scaling parameter $\lambda > 0$, $(\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t))$ solves the same equations. The norms $\| u \|_{L^3}$ and $\| \mathcal{D}u \|_{L^{3/2}}$, which appear in the limiting values $q = 3$ in Theorem 1.3 and $q = 3/2$ in Theorem 3.4, are also invariant under such λ scalings. Better understanding of the scale invariant norms, $\| u(\cdot,t) \|_{L^3}$ and $\| \mathcal{D}u(\cdot,t) \|_{L^{3/2}}$, as $t \to T_f$, is likely to be important for further progress on the blow–up question.

(*iii*) Theorem 3.4 can also be deduced from the results of Section 4 on $\| u(\cdot,t) \|_{L^q}$, $q > 3$, using the Sobolev inequality (1.17).

3.3. A differential inequality for $\| \mathcal{D}u(\cdot,t) \|$

Here we derive the estimate (1.4) for $\| \mathcal{D}u(\cdot,t) \|$ from a nonlinear differential inequality satisfied by $\| \mathcal{D}u(\cdot,t) \|$ whether T_f is finite or not. This method dates back

to Leray [3].

Theorem 3.5. *There is an absolute constant* $0 < K < \dfrac{1}{32}$ *such that*

$$\frac{d}{dt} \| \mathcal{D}u(\cdot, t) \|^2 \leq K \| \mathcal{D}u(\cdot, t) \|^6, \qquad \forall \ 0 \leq t < T_f. \tag{3.8}$$

Proof: We have, using $\nabla \cdot u = 0$,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \mathcal{D}u(\cdot, t) \|^2 &= \sum_{j=1}^{3} (D_j u, D_j u_t) \\
&= - \sum_{j=1}^{3} (D_j u, D_j(u \cdot \nabla u)) \ - \ \| \mathcal{D}^2 u(\cdot, t) \|^2 \\
&=: \ S(t) \ - \ \| \mathcal{D}^2 u(\cdot, t) \|^2.
\end{aligned}$$

Since $(D_j u, u \cdot \nabla D_j u) = 0$, because $\nabla \cdot u = 0$, the nonlinear term $S(t)$ can be estimated by $C \| \mathcal{D}u(\cdot, t) \|_{L^3}^3$, for some constant C. Using the 3D Gagliardo–Nirenberg inequality

$$\| v \|_{L^3} \leq \Gamma \| v \|^{1/2} \| \mathcal{D}v \|^{1/2} \qquad \forall \ v \in H^1(\mathbb{R}^3)$$

for $v = D_j u$, where $\Gamma < 0.59$, and then the Young's inequality $ab \leq 1/4 \, a^4$ $+ 3/4 \, b^{4/3}$, one obtains (3.8), with $K < 1/32$, as claimed. $\diamond$

REMARK: A more involved derivation of (3.8) in [43], p. 11-14, gives that $0 < K \leq \dfrac{1}{16\pi^2}$.

The next lemma shows how nonlinear differential inequalities such as (3.8) above can be used to derive lower bound estimates in case of finite–time blow–up.

Lemma 3.6. *Let* $\mathrm{w} \in C^1([0, T_f[)$ *be a positive function satisfying a differential inequality*

$$\mathrm{w}'(t) \leq K \, \mathrm{w}(t)^\alpha \qquad \forall \ 0 \leq t < T_f \tag{3.9a}$$

for some given constants $K > 0$, $\alpha > 1$. *If* $T_f < \infty$ *and* $\sup\limits_{0 \leq t < T_f} \mathrm{w}(t) = \infty$, *then we have*

$$\mathrm{w}(t) \geq \left(\frac{1}{K \cdot (\alpha - 1)} \right)^{\frac{1}{\alpha - 1}} \cdot \left(T_f - t \right)^{-\frac{1}{\alpha - 1}} \qquad \forall \ 0 \leq t < T_f. \tag{3.9b}$$

Proof: Given $t_0 \in [0, T_f[$ arbitrary, and determining $\mathrm{v} = \mathrm{v}(t)$ by $\mathrm{v}'(t) = K\mathrm{v}(t)^\alpha$, $\mathrm{v}(t_0) = \mathrm{w}_0$, where $\mathrm{w}_0 := \mathrm{w}(t_0)$, we have $\mathrm{v}(t)$ defined for all $t_0 \leq t < t_* := t_0$ $+ 1/\big(K(\alpha - 1)\mathrm{w}_0^{\alpha-1}\big)$. Moreover, one has $\mathrm{w}(t) \leq \mathrm{v}(t)$ for all $t_0 \leq t < t_*$, with $\mathrm{v}(t)$ given by

$$\mathtt{v}(t) \;=\; \mathtt{w}_0 \cdot \Big(1 - K \cdot (\alpha - 1)\, \mathtt{w}_0^{\alpha - 1} \big(t - t_0 \big) \Big)^{-\frac{1}{\alpha - 1}}, \qquad t_0 \le t < t_*,$$

so that, in particular, $\mathtt{v}(t) \nearrow \infty$ as $t \nearrow t_*$. This gives $t_* \le T_f$, which implies (3.9b) above. $\diamond$

From (3.8), we see that $\mathtt{w}'(t) \le K\, \mathtt{w}(t)^3$ for all $0 \le t < T_f$, where $\mathtt{w}(t) := \| \mathcal{D} u(\cdot, t) \|^2$, $0 < K \le 1/(16\,\pi^2)$. Assuming $T_f < \infty$, we have $\sup_{0 \le t < T_f} \mathtt{w}(t) = \infty$ by Theorem 3.1. Therefore, by Lemma 3.6, we get the following result (which dates back to Leray [3]).

Theorem 3.7. *Assuming that $T_f < \infty$, we have*

$$\| \mathcal{D} u(\cdot, t) \| \;\ge\; \frac{c}{(T_f - t)^{1/4}} \qquad \forall\; 0 \le t < T_f, \tag{3.10}$$

for some constant $c \ge \big\{ 2\pi\sqrt{2} \big\}^{1/2} > 2.98$ (independent of f, u, T_f).

As with the other bounds for $u(\cdot, t)$ discussed in the text, the optimal (= largest, here) value of the absolute constant c in (3.10) above is not known.

4. Blow–up of $\| u(\cdot, t) \|_{L^q}$ for $3 < q \le \infty$

Using the Helmholtz projector $P_{\scriptscriptstyle H}$ (see e.g. [32,51,52]), one can write the incompressible Navier–Stokes equations as

$$u_t \;=\; \Delta u \,-\, P_{\scriptscriptstyle H}(u \cdot \nabla u), \qquad P_{\scriptscriptstyle H}(u \cdot \nabla u) \;=\; u \cdot \nabla u \,+\, \nabla p, \tag{4.1a}$$

and, if $e^{\Delta t}$ denotes the heat semigroup, then one obtains, by Duhamel's principle,

$$u(\cdot, t) \;=\; e^{\Delta t} f(\cdot) \,-\, \int_0^t e^{\Delta(t-s)} P_{\scriptscriptstyle H}(u \cdot \nabla u)(\cdot, s)\, ds. \tag{4.1b}$$

It is not difficult to show that the linear operators $P_{\scriptscriptstyle H}$ and $e^{\Delta t}$ commute, and these operators also commute with differentiation, $D_j = \partial/\partial x_j$. Using the Calderon–Zygmund theory of singular integrals (see e.g. [53], Ch. 2), one shows the fundamental property that the Helmholtz projector is bounded in L^q if $1 < q < \infty$. That is, for each $1 < q < \infty$ there exists some constant $C_q > 0$ such that

$$\| P_{\scriptscriptstyle H} v \|_{L^q} \;\le\; C_q \, \| v \|_{L^q} \qquad \forall\; v = (v_1, v_2, v_3) \in L^q(\mathbb{R}^3) \qquad (1 < q < \infty). \tag{4.2}$$

We will also need here the following well known estimate for solutions of the heat equation: given any $1 \le r \le q \le \infty$, and any multi–index α, we have, for all $t > 0$:

$$\| D^\alpha e^{\Delta t} v \|_{L^q} \;\le\; C \, \| v \|_{L^r}\, t^{-\lambda - |\alpha|/2}, \qquad \lambda = \frac{3}{2} \left(\frac{1}{r} - \frac{1}{q} \right) \tag{4.3}$$

for all $v \in L^r(\mathbb{R}^3)$, with $C > 0$ a constant depending only on the values of q, r and $|\alpha|$.

4.1. Boundedness of $\| u(\cdot, t) \|_{L^q}$ implies boundedness of $\| \mathcal{D}u(\cdot, t) \|$

The basic result here is the following.

Theorem 4.1. *Let* $3 < q \leq \infty$. *If*

$$\sup_{0 \leq t < T} \| u(\cdot, t) \|_{L^q} =: C_q < \infty, \tag{4.4a}$$

then

$$\sup_{0 \leq t < T} \| \mathcal{D}u(\cdot, t) \| \leq K_q(T) \| \mathcal{D}f \|, \tag{4.4b}$$

where $K_q(T) > 0$ *depends on the values of* q, C_q, T *only. In particular, if* $T_f < \infty$, *then*

$$\sup_{0 \leq t < T_f} \| u(\cdot, t) \|_{L^q} = \infty. \tag{4.5}$$

Proof: (Note that the result for $q = \infty$ was already shown in Section 2, and we provide another proof here.) Given $3 < q \leq \infty$, let $\frac{6}{5} < r \leq 2$ be defined by

$$\frac{1}{q} + \frac{1}{2} = \frac{1}{r}.$$

From (4.1b), we have, for each $1 \leq j \leq 3$:

$$D_j u(\cdot, t) = e^{\Delta t} D_j f - \int_0^t D_j \left[e^{\Delta(t-s)} P_H (u \cdot \nabla u)(\cdot, s) \right] ds.$$

Therefore, with

$$\kappa = \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2} \right) + \frac{1}{2},$$

the following estimates hold:

$$
\begin{aligned}
\| D_j u(\cdot, t) \| &\leq \| e^{\Delta t} D_j f \| + C \int_0^t (t - s)^{-\kappa} \| P_H (u \cdot \nabla u)(\cdot, s) \|_{L^r} ds \\
&\leq \| D_j f \| + C \int_0^t (t - s)^{-\kappa} \| u \cdot \nabla u (\cdot, s) \|_{L^r} ds \\
&\leq \| D_j f \| + C \int_0^t (t - s)^{-\kappa} \| u(\cdot, s) \|_{L^q} \| \mathcal{D}u(\cdot, s) \| ds
\end{aligned}
$$

In the first estimate, we have applied (4.3); the second estimate follows from (4.2), and the third estimate uses Hölder's inequality. We thus have

$$\| \mathcal{D}u(\cdot, t) \| \leq \sum_{j=1}^3 \| D_j f \| + C \int_0^t (t - s)^{-\kappa} \| u(\cdot, s) \|_{L^q} \| \mathcal{D}u(\cdot, s) \| ds.$$

Let us note that

$$\kappa = \frac{3}{2q} + \frac{1}{2} < 1,$$

since $q > 3$. Therefore, recalling Lemma 3.2, we see that (4.4a) implies (4.4b), as claimed. By Theorem 3.1, this gives (4.5) if T_f is finite, and the proof is now complete. ⬦

4.2. An Integral Inequality for $\| u(\cdot, t) \|_{L^q}$, $3 < q \leq \infty$

We now show a simple nonlinear integral inequality for the scalar function $\| u(\cdot, t) \|_{L^q}$ that gives some local control on the growth of $\| u(\cdot, t) \|_{L^q}$. This local control together with Theorem 4.1 imply a lower bound for $\| u(\cdot, t) \|_{L^q}$ if $T_f < \infty$, cf. Lemma 4.3 below.

Theorem 4.2. *Let $3 < q \leq \infty$ and set*

$$\kappa = \frac{3}{2q} + \frac{1}{2} < 1. \tag{4.6a}$$

Then, there is a constant $C_q > 0$ (depending only on q) such that, for any $0 \leq t_0 < T_f$, we have

$$\| u(\cdot, t) \|_{L^q} \leq \| u(\cdot, t_0) \|_{L^q} + C_q \int_{t_0}^{t} (t - s)^{-\kappa} \| u(\cdot, s) \|_{L^q}^2 \, ds, \qquad \forall \, t_0 \leq t < T_f. \tag{4.6b}$$

Proof: In the case $3 < q < \infty$, we use the following argument (adapted from [14]). Let $r = \dfrac{q}{2}$ and note that

$$\kappa = \frac{3}{2} \left(\frac{1}{r} - \frac{1}{q} \right) + \frac{1}{2}.$$

We also have $\| u_i u_j \|_{L^r} \leq \| u \|_{L^q}^2$, since $2r = q$. Using (4.1a) and Duhamel's principle, we get

$$u(\cdot, t) = e^{\Delta(t - t_0)} u(\cdot, t_0) - \sum_{j=1}^{3} \int_{t_0}^{t} e^{\Delta(t-s)} P_H \big[D_j(u_j u)(\cdot, s) \big] \, ds, \qquad t_0 \leq t < T_f,$$

which gives, by (4.2) and (4.3) above,

$$
\begin{aligned}
\| u(\cdot, t) \|_{L^q} \;\; &\leq \;\; \| u(\cdot, t_0) \|_{L^q} \; + \; C_q \sum_{j=1}^{3} \int_{t_0}^{t} (t - s)^{-\kappa} \| P_H(u_j u)(\cdot, s) \|_{L^r} \, ds \\[2mm]
&\leq \;\; \| u(\cdot, t_0) \|_{L^q} \; + \; C_q \sum_{j=1}^{3} \int_{t_0}^{t} (t - s)^{-\kappa} \| u_j u(\cdot, s) \|_{L^r} \, ds \\[2mm]
&\leq \;\; \| u(\cdot, t_0) \|_{L^q} \; + \; C_q \int_{t_0}^{t} (t - s)^{-\kappa} \| u(\cdot, s) \|_{L^q}^2 \, ds
\end{aligned}
$$

for all $t_0 \leq t < T_f$. This shows the result if $3 < q < \infty$, as claimed. The proof in the case $q = \infty$ is due to Leray and is developed in Chapters 2 and 3 of [3]. ⬦

The following lemma shows how (4.6) is used to yield the fundamental lower bound (1.12) for $\| u(\cdot, t) \|_{L^q}$, $3 < q \leq \infty$, in case of finite–time blow–up.

Lemma 4.3. *Let* $w \in C^0([0, T_f[)$ *be some positive function such that we have, for certain* $B > 0$, $\alpha > 1$, $\kappa < 1$ *constant,*

$$w(t) \; \leq \; w(t_0) \, + \, B \int_{t_0}^{t} (t - s)^{-\kappa} \, w(s)^\alpha \, ds \qquad \forall \; t_0 \leq t < T_f \quad (4.7a)$$

for each $0 \leq t_0 < T_f$. *If, in addition,* $T_f < \infty$ *and* $\sup\limits_{0 \leq t < T_f} w(t) = \infty$, *then it follows that*

$$w(t) \; > \; (\alpha - 1) \cdot \left(\frac{1 - \kappa}{B \, \alpha^\alpha} \right)^{\frac{1}{\alpha - 1}} \cdot \left(T_f - t \right)^{-\frac{1 - \kappa}{\alpha - 1}} \qquad \forall \; 0 \leq t < T_f. \qquad (4.7b)$$

Proof: Let $\lambda > 1$. Given $t_0 \in [0, T_f[$ arbitrary, by $(4.7a)$ we must have $w(t) < \lambda \, w(t_0)$ if $t > t_0$ is close to t_0. In fact, setting $\tau_* > 0$ by

$$\tau_* := \; \min \left\{ T_f, \; t_0 + \left[\frac{(1 - \kappa)(\lambda - 1)}{\lambda^\alpha B \, w(t_0)^{\alpha - 1}} \right]^{\frac{1}{1 - \kappa}} \right\},$$

we have $w(t) < \lambda \, w(t_0)$ for all $t_0 \leq t < \tau_*$. Because, if this were false, we could then find $t_1 \in \,] t_0, \tau_* [$ such that $w(t) < \lambda \, w(t_0)$ for all $t_0 \leq t < t_1$, while $w(t_1) = \lambda \, w(t_0)$. This would give, by $(4.7a)$ and the choice of τ_* above,

$$\lambda \, w(t_0) \; = \; w(t_1) \; \leq \; w(t_0) \, + \, B \int_{t_0}^{t_1} (t_1 - s)^{-\kappa} \, w(s)^\alpha \, ds$$

$$< \; w(t_0) \, + \, B \int_{t_0}^{t_1} (t_1 - s)^{-\kappa} \, \lambda^\alpha \, w(t_0)^\alpha \, ds$$

$$< \; w(t_0) \, + \, B \, \lambda^\alpha \, w(t_0)^\alpha \, \frac{(\tau_* - t_0)^{1 - \kappa}}{1 - \kappa} \; \leq \; \lambda \, w(t_0),$$

which could not be. Hence, we have $w(t) < \lambda \, w(t_0)$ for all $t_0 \leq t < \tau_*$, as claimed, and in particular w is bounded on $[t_0, \tau_*[$. Since, by assumption, w is unbounded in $[t_0, T_f[$, we must have $T_f > \tau_*$, that is,

$$w(t_0) \; > \; c(\lambda) \cdot \left(T_f - t_0 \right)^{-\frac{1 - \kappa}{\alpha - 1}}, \qquad c(\lambda) \; = \; \left(\frac{1 - \kappa}{B} \right)^{\frac{1}{\alpha - 1}} \cdot \left(\frac{\lambda - 1}{\lambda^\alpha} \right)^{\frac{1}{\alpha - 1}}$$

for $t_0 \in [0, T_f[$ arbitrary. The largest value of $c(\lambda)$ is obtained by choosing $\lambda = \alpha / (\alpha - 1)$, which yields the estimate $(4.7b)$.   ◇

From (4.6) and Lemma 4.3, we get for $3 < q \leq \infty$ the lower bound estimate

$$\| u(\cdot, t) \|_{L^q} \; \geq \; c_q \cdot \left(T_f - t \right)^{-\frac{q - 3}{2q}} \qquad \forall \; 0 \leq t < T_f \qquad (4.8a)$$

if $T_f < \infty$, where

$$c_q \;=\; \frac{q-3}{8\,q\,C_q} \quad \text{if} \;\; 3 < q < \infty, \qquad c_\infty \;=\; \frac{1}{8\,C_\infty} \quad \text{if} \;\; q = \infty,$$

(4.8*b*)

with $C_q > 0$ given in (4.6*b*) above. This proves Theorem 1.3 of Section 1 for $3 < q \leq \infty$. (Another proof for $3 \leq q < \infty$ is given in Subsection 4.3.) Using the Gagliardo inequality

$$\| \, \mathsf{u} \, \|_\infty \;\leq\; K(q) \, \| \, \mathsf{u} \, \|_{L^2}^{1-\theta} \, \| \, \nabla \mathsf{u} \, \|_{L^q}^{\theta}, \qquad \theta \;=\; \frac{3q}{5q-6} \qquad (3 < q \leq \infty), \qquad (4.9)$$

which holds for arbitrary $\mathsf{u} \in L^2(\mathbb{R}^3) \cap W^{1,q}(\mathbb{R}^3)$, we obtain, from (2.1) and (4.8),

$$\| \, \mathcal{D}u(\cdot,t) \, \|_{L^q} \;\geq\; \hat{c}_q \, \| \, f \, \|_{L^2}^{-\frac{2q-6}{3q}} \, \bigl(T_f - t\bigr)^{-\frac{5q-6}{6q}} \qquad \forall \; 0 \leq t < T_f \qquad (\text{if } T < \infty)$$

(4.10)

for each $3 < q \leq \infty$, and some constant $\hat{c}_q > 0$ that depends only on q. For $q = 3$, we can similarly obtain

$$\| \, \mathcal{D}u(\cdot,t) \, \|_{L^3} \;\geq\; c(\epsilon) \, \| \, f \, \|_{L^2}^{-4\epsilon} \, \bigl(T_f - t\bigr)^{-\frac{1}{2}+\epsilon} \qquad \forall \; 0 \leq t < T_f \qquad (\text{if } T < \infty) \quad (4.11)$$

for each $0 < \epsilon \leq 1/2$, and some constant $c(\epsilon) > 0$ depending only on ϵ, using (2.1), (4.8) and the 3D inequalities

$$\| \, \mathsf{u} \, \|_{L^r} \;\leq\; K(r) \, \| \, \mathsf{u} \, \|_{L^2}^{2/r} \, \| \, \nabla \mathsf{u} \, \|_{L^3}^{1-2/r} \qquad (3 \leq r < \infty). \qquad (4.12)$$

This completes the proof of (1.5). (For Theorem 1.2, see also (1.13), (1.17) and (4.16).)

4.3. A Differential Inequality for $\| \, u(\cdot,t) \, \|_{L^q}$, $3 < q < \infty$

We recall the fundamental estimate for the pressure $p(\cdot,t)$ obtained by the Calderon–Zygmund theory applied to the Poisson equation (2.5),

$$\| \, p(\cdot,t) \, \|_{L^r} \;\leq\; C_r \, \| \, u(\cdot,t) \, \|_{L^{2r}}^{2} \qquad \forall \; 0 \leq t < T_f \qquad (1 < r < \infty), \qquad (4.13)$$

see e.g. [32,52]. The basic result in this subsection is the following differential inequality.

Theorem 4.4. *Let $T_f \leq \infty$ and $3 < q < \infty$. Then there exists an absolute constant K_q (depending only on q) such that*

$$\frac{d}{dt} \, \| \, u(\cdot,t) \, \|_{L^q}^{q} \;\leq\; K_q \cdot \Bigl(\| \, u(\cdot,t) \, \|_{L^q}^{q} \Bigr)^{\frac{q-1}{q-3}} \qquad \forall \; 0 \leq t < T_f. \qquad (4.14)$$

Proof: Given $\delta > 0$, let $L'_\delta(\cdot)$ be a regularized sign function (see e.g. [35], p. 136), and let $\Phi_\delta(\mathfrak{u}) := L_\delta(\mathfrak{u})^q$. Multiplying the equation for $u_i(\cdot, t)$ by $\Phi'_\delta(u_i(\cdot, t))$, integrating on $\mathbb{R}^3$ and letting $\delta \to 0$, we get

$$\frac{d}{dt} \| u_i(\cdot, t) \|_{L^q}^q + q \cdot (q-1) \int_{\mathbb{R}^3} | u_i(x, t) |^{q-2} | \nabla u_i |^2 \, dx \leq$$

$$\leq q \cdot (q-1) \int_{\mathbb{R}^3} | p(x, t) | \, | u_i(x, t) |^{q-2} | \nabla u_i | \, dx$$

for all $1 \leq i \leq 3$, $0 \leq t < T_f$. Using (4.13) and Hölder's inequality, we have

$$\int_{\mathbb{R}^3} | p(x, t) | \, | u_i(x, t) |^{q-2} | \nabla u_i(x, t) | \, dx \leq$$

$$\leq C(q) \| u(\cdot, t) \|_{L^{q+2}}^2 \| u_i(\cdot, t) \|_{L^{q+2}}^{\frac{q-2}{2}} \left(\int_{\mathbb{R}^3} | u_i(x, t) |^{q-2} | \nabla u_i |^2 \, dx \right)^{\frac{1}{2}}$$

for each i, and some constant $C(q) > 0$ that depends on the value of q only. In terms of $v(x, t) = \big(v_1(x, t), v_2(x, t), v_3(x, t) \big)$ given by

$$v_i(x, t) := | u_i(x, t) |^{\frac{q}{2}}, \qquad 1 \leq i \leq 3, \qquad (4.15a)$$

we therefore have

$$\frac{d}{dt} \| v_i(\cdot, t) \|_{L^2}^2 + 4 \left(1 - \frac{1}{q} \right) \| \nabla v_i(\cdot, t) \|_{L^2}^2 \leq$$

$$\leq 2q \left(1 - \frac{1}{q} \right) C(q) \| v(\cdot, t) \|_{L^\beta}^{\frac{4}{q}} \| v_i(\cdot, t) \|_{L^\beta}^{\frac{q-2}{q}} \| \nabla v_i(\cdot, t) \|_{L^2}$$

for each i, where $\beta = 2 + 4/q$. Using the inequality

$$\| v \|_{L^\beta} \leq K(\beta) \| v \|_{L^2}^{\frac{q-1}{q+2}} \| \nabla v \|_{L^2}^{\frac{3}{q+2}} \qquad \forall \, v \in H^1(\mathbb{R}^3),$$

where the constant $K(\beta) > 0$ depends only on β, and summing on $i = 1, 2, 3$, we obtain

$$\frac{d}{dt} \| v(\cdot, t) \|_{L^2}^2 + 4 \left(1 - \frac{1}{q} \right) \| \mathcal{D} v(\cdot, t) \|_{L^2}^2 \leq C_q \| v(\cdot, t) \|_{L^2}^{\frac{q-1}{q}} \| \mathcal{D} v(\cdot, t) \|_{L^2}^{\frac{q+3}{q}}$$

(4.15b)

for all $0 \leq t < T_f$, and some constant $C_q > 0$ that depends on q only. This gives

$$\frac{d}{dt} \| v(\cdot, t) \|_{L^2}^2 \leq K_q \cdot \Big(\| v(\cdot, t) \|_{L^2}^2 \Big)^{\frac{q-1}{q-3}} \qquad \forall \, 0 \leq t < T_f$$

for some constant $K_q > 0$ depending on q only, which is equivalent to (4.14). $\diamond$

It follows from the previous proof that the estimate (4.15b) is valid more generally for any $2 < q < \infty$. Taking $q = 3$, it gives that $d/dt \, \| v(\cdot, t) \|_{L^2}^2 < 0$ if $\| v(\cdot, t) \|_{L^2}$ is appropriately small; since $\| v(\cdot, t) \|_{L^2}^2 = \| u(\cdot, t) \|_{L^3}^3$ in this case, cf. (4.15a), we conclude that $\| u(\cdot, t) \|_{L^3}$ is monotonically decreasing in t when $\| u(\cdot, 0) \|_{L^3}$ is sufficiently small, i.e.,

$$\| u(\cdot, 0) \|_{L^3} < \eta_3 \quad \Longrightarrow \quad T_f = \infty \tag{4.16}$$

for some absolute value $\eta_3 > 0$. This shows (1.12), (1.15) for $q = 3$, and also, using (1.17), the bound (1.4) for $q = 3/2$, thus completing the proof of Theorems 1.2 and 1.3 above. It also implies, by (1.13) and Gronwall's lemma, that $\| \mathcal{D}v(\cdot, t) \|_{L^2}^2$ cannot be integrable on $[0, T_f]$ if $T_f < \infty$, or, in terms of $u(\cdot, t)$, that we have

$$\sum_{i=1}^{3} \int_0^{T_f} \int_{\mathbb{R}^3} | u_i(x, t) | \, | \nabla u_i(x, t) |^2 \, dx \, dt \; = \; \infty \qquad (\text{if } T_f < \infty). \tag{4.17}$$

On the other hand, taking $q > 3$ in (4.15), we reobtain the fundamental estimate (1.12), in view of Lemma 3.6. Another important consequence is the following. Using that

$$\| \mathsf{v} \|_{L^2} \; \leq \; K(q) \, \| \mathsf{v} \|_{L^{4/q}}^{1-\delta} \, \| \nabla \mathsf{v} \|_{L^2}^{\delta}, \qquad \delta = \frac{3q - 6}{3q - 2} \qquad (2 \leq q < \infty), \tag{4.18}$$

we have, for $q > 3$:

$$\| v(\cdot, t) \|_{L^2}^{\frac{q-1}{q}} \, \| \mathcal{D}v(\cdot, t) \|_{L^2}^{\frac{q+3}{q}} \; =$$

$$= \; \| v(\cdot, t) \|_{L^2}^{\frac{2}{3q-6}} \, \| v(\cdot, t) \|_{L^2}^{\frac{3q-2}{3q-6}\left(1 - \frac{3}{q}\right)} \, \| \mathcal{D}v(\cdot, t) \|_{L^2}^{\frac{q+3}{q}}$$

$$\leq \; C(q) \, \| v(\cdot, t) \|_{L^{4/q}}^{\frac{4}{q}\frac{q-3}{3q-6}} \, \| v(\cdot, t) \|_{L^2}^{\frac{2}{3q-6}} \, \| \mathcal{D}v(\cdot, t) \|_{L^2}^{2}$$

$$= \; C(q) \, \| u(\cdot, t) \|_{L^2}^{\frac{2q-6}{3q-6}} \, \| u(\cdot, t) \|_{L^q}^{\frac{q}{3q-6}} \, \| \mathcal{D}v(\cdot, t) \|_{L^2}^{2}$$

by (4.15a) and (4.18) above. As $\| u(\cdot, t) \|_{L^2}$ never increases, this gives, because of (4.15b), that $\| v(\cdot, t) \|_{L^2} \left(= \| u(\cdot, t) \|_{L^q} \right)$ is monotonically decreasing in time whenever we have

$$\| u(\cdot, 0) \|_{L^2}^{\frac{2q-6}{3q-6}} \, \| u(\cdot, 0) \|_{L^q}^{\frac{q}{3q-6}} \; < \; \eta_q$$

for some value $\eta_q > 0$ appropriately small (depending only on q). Together with (4.16), this shows (1.15), Theorem 1.4, for $3 \leq q < \infty$. The proof for $q = \infty$ is given in [3].

We finish this Section with a few last remarks. Using (4.8) and the 3D inequality

$$\| \mathsf{u} \|_{L^{r(q)}} \leq K_q \| \mathcal{D}^2 \mathsf{u} \|_{L^q}, \qquad r(q) = \frac{3q}{3 - 2q} \qquad \left(1 \leq q < \frac{3}{2} \right), \qquad (4.19)$$

where $K_q > 0$ depends only on q, we obtain the lower bound estimate

$$\| \mathcal{D}^2 u(\cdot, t) \|_{L^q} \geq c_q \left(T_f - t \right)^{-\frac{3}{2} \frac{q-1}{q}} \qquad \forall \, 0 \leq t < T_f \qquad (\text{if } T < \infty) \qquad (4.20)$$

for each $1 \leq q < 3/2$, and some constant $c_q > 0$ that depends only on q. The estimate (4.20) has been recently shown in [18] to hold for $q = 2$ as well, but its validity for arbitrary $q \geq 3/2$ seems to be still open. The general fact that the norms $\| \mathcal{D}^n u(\cdot, t) \|_{L^q}$, $1 \leq q \leq \infty$, $n \geq 2$, do all blow up as $t \nearrow T_f$ in case $T_f < \infty$ is a direct consequence of (1.13), (4.8) and the family of 3D Gagliardo inequalities given by

$$\| \mathsf{u} \|_{L^q} \leq K(q,r) \| \mathsf{u} \|_{L^2}^{1-\theta} \| \mathcal{D}^n \mathsf{u} \|_{L^r}^{\theta}, \quad \theta = \frac{1/2 - 1/\mathsf{q}}{1/2 + n/3 - 1/r}, \quad r \geq \max\left\{ 1, \frac{3\,\mathsf{q}}{n\,\mathsf{q} + 3} \right\}$$

for $n \geq 2, 3 \leq q \leq \infty$ arbitrary, provided that $(n, q, r) \neq (2, \infty, 3/2)$, $(n, q, r) \neq (3, \infty, 1)$.

5. Comparison of blow–up functions

Let $3 \leq q < r \leq \infty$ and assume that $T_f < \infty$. Theorem 1.3 yields the lower bounds

$$\| u(\cdot, t) \|_{L^q} \geq c_q \left(T_f - t \right)^{-\kappa(q)}, \qquad \| u(\cdot, t) \|_{L^r} \geq c_r \left(T_f - t \right)^{-\kappa(r)}$$

with positive constants c_q, c_r and

$$0 \leq \kappa(q) = \frac{q - 3}{2q} < \kappa(r) = \frac{r - 3}{2r}.$$

Thus, the lower bound for the L^r norm blows up faster than the lower bound for the L^q norm. This suggests that

$$\frac{\| u(\cdot, t) \|_{L^r}}{\| u(\cdot, t) \|_{L^q}} \to \infty \quad \text{as} \quad t \to T_f \quad \text{if} \quad 3 \leq q < r \leq \infty. \qquad (5.1)$$

A precise result can be obtained by using boundedness of the L^2 norm and interpolation: defining $0 < \lambda < 1$ by

$$\frac{1}{q} = \frac{\lambda}{2} + \frac{1 - \lambda}{r}$$

and recalling the interpolation estimate $\| u(\cdot, t) \|_{L^q} \leq \| u(\cdot, t) \|^{\lambda} \| u(\cdot, t) \|_{L^r}^{1-\lambda}$, one

gets

$$\| u(\cdot, t) \|_{L^r}^{\lambda} \leq \| f \|^{\lambda} \frac{\| u(\cdot, t) \|_{L^r}}{\| u(\cdot, t) \|_{L^q}}, \qquad \lambda = \frac{1/q - 1/r}{1/2 - 1/r}. \qquad (5.2)$$

Using the lower bound on the blow–up of $\| u(\cdot, t) \|_{L^r}$ provided by Theorem 1.3, one then obtains (5.1) with an algebraic lower bound, as described next.

Theorem 5.1. *Let $3 \leq q < r \leq \infty$, and assume that $T_f < \infty$. Then there is a constant $c(f) = c(f; q, r) > 0$, depending on q, r and the initial state f, such that*

$$\frac{\| u(\cdot, t) \|_{L^r}}{\| u(\cdot, t) \|_{L^q}} \geq c(f) \cdot \left(T_f - t \right)^{-\gamma} \qquad \forall \, 0 \leq t < T_f, \qquad (5.3a)$$

where

$$\gamma = \frac{r - 3}{r - 2} \cdot \frac{r - q}{qr} > 0. \qquad (5.3b)$$

In a similar way, using the 3D inequality

$$\| \mathsf{u} \|_{L^q} \leq K(q) \| \mathsf{u} \|^{1-\theta} \| \mathcal{D}\mathsf{u} \|^{\theta}, \qquad \theta = \frac{3}{2} \cdot \frac{q - 2}{q} \qquad (2 \leq q \leq 6) \qquad (5.4)$$

and assuming $T_f < \infty$, one obtains

$$\frac{\| \mathcal{D}u(\cdot, t) \|}{\| u(\cdot, t) \|_{L^q}} \geq \tilde{c}(f) \cdot \left(T_f - t \right)^{-\tilde{\gamma}}, \qquad \tilde{\gamma} = \frac{6 - q}{8q} \geq 0 \qquad (5.5)$$

for all $0 \leq t < T_f$, and every $2 \leq q \leq 6$, where the constant $\tilde{c}(f)$ depends on q and $\| f \|$.

In the remaining part of this section we compare the growth of $\| u(\cdot, t) \|_{L^q}^{q} \| u(\cdot, t) \|_{\infty}^{2}$, for $3 < q < \infty$, with the growth of $\| u(\cdot, t) \|_{L^3} \| u(\cdot, t) \|_{\infty}^{q}$, as $t \nearrow T_f$. Setting $r = \infty$ and $\lambda = 2/q$ in (5.2) above, we have

$$\| u(\cdot, t) \|_{\infty}^{2/q} \leq \| f \|^{2/q} \frac{\| u(\cdot, t) \|_{\infty}}{\| u(\cdot, t) \|_{L^q}} \qquad \forall \, 0 \leq t < T_f. \qquad (5.6)$$

Together with (1.13) and (5.3), one obtains the following theorem.

Theorem 5.2. *Let $3 < q < \infty$, and assume that $T_f < \infty$. Then we have*

$$\lim_{t \nearrow T_f} \frac{\| u(\cdot, t) \|_{\infty}^{q}}{\| u(\cdot, t) \|_{L^q}^{q}} \cdot \frac{\| u(\cdot, t) \|_{L^3}}{\| u(\cdot, t) \|_{\infty}^{2}} = \infty. \qquad (5.7)$$

Remark: Since the proof of (1.13) in [1] is very involved, we give here a direct elementary argument for the weaker statement

$$\sup_{0 \leq t < T_f} \frac{\| u(\cdot, t) \|_{\infty}^{q}}{\| u(\cdot, t) \|_{L^q}^{q}} \cdot \frac{\| u(\cdot, t) \|_{L^3}}{\| u(\cdot, t) \|_{\infty}^{2}} = \infty. \qquad (5.8)$$

Denoting by $\langle u, v \rangle = \sum_i u_i v_i$ the Euclidean inner product in $\mathbb{R}^3$, we have

$$\frac{1}{q} \frac{d}{dt} \| u(\cdot, t) \|_{L^q}^q = \int_{\mathbb{R}^3} | u(x, t) |^{q-2} \langle u(x, t), u_t(x, t) \rangle \, dx$$

$$= - \int_{\mathbb{R}^3} | u |^{q-2} \langle u, \nabla p \rangle \, dx - \int_{\mathbb{R}^3} | u |^{q-2} \langle u, u \cdot \nabla u \rangle \, dx$$

$$+ \int_{\mathbb{R}^3} | u |^{q-2} \langle u, \Delta u \rangle \, dx$$

$$=: \ T_p + T_c + T_v$$

Using integration by parts, one obtains that $T_c = 0$. Also,

$$T_v = \sum_{i, j = 1}^{3} \int_{\mathbb{R}^3} | u |^{q-2} u_i D_j^2 u_i \, dx$$

$$= - \sum_{i, j = 1}^{3} \int_{\mathbb{R}^3} | u |^{q-2} | \mathcal{D} u |^2 \, dx - (q - 2) \sum_{j = 1}^{3} \int_{\mathbb{R}^3} | u |^{q-4} \langle u, D_j u \rangle^2 \, dx$$

$$\leq \ 0.$$

The pressure term is

$$T_p = - \sum_{j = 1}^{3} \int_{\mathbb{R}^3} | u |^{q-2} u_j D_j p \, dx$$

$$= (q - 2) \sum_{j = 1}^{3} \int_{\mathbb{R}^3} | u |^{q-4} \langle u, D_j u \rangle u_j p \, dx$$

$$\leq \ C_q \| u(\cdot, t) \|_{\infty}^{q-2} \| \mathcal{D} u(\cdot, t) \| \| p(\cdot, t) \|.$$

For $\| p(\cdot, t) \|$, we use the bound (from Fourier transform, plus Parseval's relation)

$$\| p(\cdot, t) \| \leq \sum_{i, j = 1}^{3} \| u_i u_j (\cdot, t) \|$$

$$\leq \ C \| u(\cdot, t) \|_{L^4}^2$$

$$\leq \ C \| u(\cdot, t) \|_{L^3} \| \mathcal{D} u(\cdot, t) \|.$$

Thus, we have shown the estimate

$$\frac{d}{dt} \| u(\cdot, t) \|_{L^q}^q \leq C_q \| u(\cdot, t) \|_{\infty}^{q-2} \| u(\cdot, t) \|_{L^3} \| \mathcal{D} u(\cdot, t) \|^2. \tag{5.9}$$

Setting
$$h(t) := \frac{\| u(\cdot, t) \|_{\infty}^q}{\| u(\cdot, t) \|_{L^q}^q} \cdot \frac{\| u(\cdot, t) \|_{L^3}}{\| u(\cdot, t) \|_{\infty}^2}, \tag{5.10}$$

we have that if $h(t)$ were bounded by some quantity h_{max} in the interval $0 \leq t < T_f$, then the estimate (5.9) would give

$$\frac{d}{dt} \, \| \, u(\cdot, t) \, \|_{L^q}^q \; \leq \; C_q \, h_{\mathsf{max}} \, \| \, \mathcal{D}u(\cdot, t) \, \|^2 \, \| \, u(\cdot, t) \, \|_{L^q}^q.$$

Since $\int_0^{T_f} \| \, \mathcal{D}u(\cdot, t) \, \|^2 \, ds$ is finite by Theorem 2.1, this would give (by Gronwall's lemma) boundedness of $\| \, u(\cdot, t) \, \|_{L^q}$ in $0 \leq t < T_f$. This contradiction proves (5.8).

$\diamond$

6. The Beale–Kato–Majda blow–up condition

In this section, we recall a few basic facts for the vorticity $\omega(\cdot, t) := \nabla \times u(\cdot, t)$, which satisfies the related equation

$$\omega_t \, + \, u(\cdot, t) \cdot \nabla \, \omega(\cdot, t) \; = \; \Delta \omega(\cdot, t) \, + \, \omega(\cdot, t) \cdot \nabla u(\cdot, t). \qquad (6.1)$$

From our definition of the L^2-norm $\| \cdot \|$, it is readily seen that $\| \, \omega(\cdot, t) \, \| = \| \, \mathcal{D}u(\cdot, t) \, \|$, and, more generally,

$$\| \, \mathcal{D}^{\ell+1} u(\cdot, t) \, \| \; = \; \| \, \mathcal{D}^\ell \omega(\cdot, t) \, \| \qquad \forall \, \ell \geq 0, \qquad (6.2)$$

so that we have, in case $T_f < \infty$, that $\| \, \mathcal{D}^\ell \omega(\cdot, t) \, \| \to \infty$ as $t \nearrow T_f$ for all $\ell \geq 0$. Similar considerations are obtained from

$$\| \, \mathcal{D}^{\ell+1} u(\cdot, t) \, \|_{L^q} \; \leq \; K(\ell, q) \, \| \, \mathcal{D}^\ell \omega(\cdot, t) \, \|_{L^q} \qquad \forall \, \ell \geq 0, \; 1 < q < \infty, \qquad (6.3)$$

which follows from the Calderon–Zygmund theory of singular operators, see e.g. [32,52]. Another important property of $\omega(\cdot, t)$ is that it stays bounded in L^1, as observed in [54].

Theorem 6.1. (i) *Let* $\omega = \big(\omega_1, \omega_2, \omega_3 \big)$ *be the vorticity. If* $\omega_i(\cdot, 0) \in L^1(\mathbb{R}^3)$ *for some* i*, then* $\omega_i(\cdot, t)$ *remains in* $L^1(\mathbb{R}^3)$ *for* $t > 0$*, with*

$$\| \, \omega_i(\cdot, t) \, \|_{L^1} \; \leq \; \| \, \omega_i(\cdot, 0) \, \|_{L^1} + \frac{1}{2} \, \| \, u(\cdot, 0) \, \|^2 \qquad \forall \, 0 \leq t < T_f. \qquad (6.4)$$

(ii) *If* $\omega(\cdot, 0) \in L^1(\mathbb{R}^3)$*, then*

$$\| \, \omega(\cdot, t) \, \|_{L^1} \; \leq \; \| \, \omega(\cdot, 0) \, \|_{L^1} + \frac{\sqrt{3}}{2} \, \| \, u(\cdot, 0) \, \|^2 \qquad \forall \, 0 \leq t < T_f. \qquad (6.5)$$

Proof: Again, we use regularized sign functions $L'_\delta(\cdot)$ as defined in [35], p. 136, where $\delta > 0$ is arbitrary. Multiplying the i-th component of equation (6.1) above by $L'_\delta(\omega_i(\cdot, t))$ and integrating on $\mathbb{R}^3 \times [0, t]$, we get, letting $\delta \searrow 0$,

$$\| \, \omega_i(\cdot, t) \, \|_{L^1} \; \leq \; \| \, \omega_i(\cdot, 0) \, \|_{L^1} + \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} | \, \omega_j(x, t) \, | \, | \, D_j u_i(x, t) \, | \; dx \, d\tau$$

for all $0 \leq t < T_f$, which gives (6.4) by the Cauchy–Schwarz inequality and (1.7), (6.2). Moreover, summing on $1 \leq i \leq 3$ and applying again the Cauchy–Schwarz inequality to estimate the integral term, one obtains (6.5), using (1.7) and (6.2) once more.   ⬦

We now turn to the L^2 norm of $\omega(\cdot, t)$, which will quickly lead us to the following blow–up result, originally obtained for the Euler's equations in [30].

Theorem 6.2. (*Beale–Kato–Majda*). *If* $T_f < \infty$, *then* $\displaystyle\int_0^{T_f} \| \omega(\cdot, t) \|_\infty \, dt = \infty$.

Proof: Multiplying the i-th component of equation (6.1) by $\omega_i(\cdot, t)$ and integrating on $\mathbb{R}^3 \times [0, t]$, we get, summing on $1 \leq i \leq 3$,

$$\frac{1}{2}\frac{d}{dt} \| \omega(\cdot, t) \|^2 + \| \mathcal{D}\omega(\cdot, t) \|^2 = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \omega_i(x,t)\, \omega_j(x,t)\, D_j\, u_i(x,t)\, dx$$

$$\leq \| \omega(\cdot, t) \|_\infty \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} | \omega_j(x,t)|\, | D_j\, u_i(x,t)|\, dx$$

$$\leq \sqrt{3}\, \| \omega(\cdot, t) \|_\infty\, \| \omega(\cdot, t) \|\, \| \mathcal{D}u(\cdot, t) \|$$

$$= \sqrt{3}\, \| \omega(\cdot, t) \|_\infty\, \| \omega(\cdot, t) \|^2$$

for all $0 \leq t < T_f$, using the Cauchy-Schwarz inequality and that $\| \mathcal{D}u(\cdot, t) \| = \| \omega(\cdot, t) \|$, see (6.2) above. By the standard Gronwall lemma, this gives

$$\| \omega(\cdot, t) \| \leq \| \omega(\cdot, 0) \| \cdot \exp\left\{ \sqrt{3} \int_0^t \| \omega(\cdot, \tau) \|_\infty\, d\tau \right\} \qquad \forall\, 0 \leq t < T_f.$$

Therefore, if we had $\displaystyle\int_0^{T_f} \| \omega(\cdot, t) \|_\infty \, dt < \infty$, we would have $\| \omega(\cdot, t) \|$ bounded in $[\,0, T_f\,[$. That is, $\| \mathcal{D}u(\cdot, t) \|$ would be bounded in $[\,0, T_f\,[$, contradicting Leray's estimate (3.10).   ⬦

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