Bol. Soc. Paran. Mat. (3s.) **v. 35** 1 (2017): 139–145.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v35i1.27933

On arithmetic continuity

Taja Yaying and Bipan Hazarika

ABSTRACT: In this article we introduce the concept of arithmetic continuity and arithmetic compactness and prove some intresting results related to these notions.

Key Words: Continuity; sequences; summability; compactness.

Contents

1. Introduction

A sequence $x = (x_k)$ defined on $\mathbb{N}$ and $n \in \mathbb{N}$ the notation $\sum_{k|n} x_k$ means the finite sum of all the numbers x_k as k ranges over the integers that divide n including 1 and n. In general for integers k and n we write $k|n$ to mean 'k divides n' or 'n is a multiple of k'. We use the symbol $< m, n >$ to denote the greatest common divisor of two integers m and n.

W.H.Ruckle [11], introduced the notions arithmetically summable and arithmetically convergent as follows.

(i) A sequence $x = (x_k)$ defined on $\mathbb{N}$ is called *arithmetically summable* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $\left|\sum_{k|m} x_k - \sum_{k|<m,n>} x_k\right| < \varepsilon$.

(ii) A sequence $y = (y_k)$ is called *arithmetically convergent* if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $\left|y_m - y_{(<m,n>)}\right| < \varepsilon$.

From above two definitions it is clear that a sequence $x = (x_k)$ is arithmetically summable if and only if the sequence $y = (y_k)$ defined by $y_n = \sum_{k|n} x_k$ is arithmetically convergent, but the sequence $y = (y_k)$ is not convergent in the ordinary sense and is in fact periodic.

2000 *Mathematics Subject Classification:* Primary: 40A35 Secondaries: 40A05; 26A15

A sequence $x = (x_k)$ is called periodic if there is a number n such that $x_{k+n} = x_k$ for all $k \in \mathbb{N}$, the smallest such integer n is called the period of the sequence x. Denote by $\mathbf{P}$ the linear space of all periodic sequences and denote by $\mathbf{QP}$ the closure of $\mathbf{P}$ in the space ℓ_∞ of all bounded sequences. For details about the spaces $\mathbf{P}$ and $\mathbf{QP}$ we refer to [2,3,8,9,13]. For details of arithmetically summable and arithmetical functions we refer to [7,12,14].

A subset E of $\mathbb{R}$ is compact if any open covering of E has a finite subcovering, where $\mathbb{R}$ is the set of real numbers. This is equivalent to the statement that any sequence $x = (x_n)$ of points in E has a convergent subsequence whose limit is in E. A real function f is continuous if and only if $(f(x_n))$ is a convergent sequence whenever (x_n) is. Regardless of limit, this is equivalent to the statement that $(f(x_n))$ is Cauchy whenever (x_n) is. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, we introduce the concept of *arithmetic continuity* and *arithmetic compactness* and establish some interesting results related to these notions. For details on continuity of real valued functions we refer to [1,4,5,6,10]

2. Arithmetic Continuity

Definition 2.1. *A function f defined on a subset E of $\mathbb{R}$ is said to be arithmetic continuous if it transforms arithmetic convergent sequences to arithmetic convergent sequences. In other words, the sequence (x_n) is arithmetic convergent implies the sequence $(f(x_n))$ is arithmetic convergent.*

Theorem 2.2. *Sum of two arithmetic continuous functions is again arithmetic continuous.*

Proof: Let f and g be arithmetic continuous functions on a subset E of $\mathbb{R}$. To prove that the function $f+g$ is arithmetic continuous function on E. Let $\varepsilon > 0$ and (x_n) be any arithmetic convergent sequence on E. By the definition of arithmetic continuity the sequences $(f(x_n))$ and $(g(x_n))$ are arithmetic convergent sequences. Since $(f(x_n))$ and $(g(x_n))$ are arithmetic convergent sequences therefore for $\varepsilon > 0$ and a positive integer m

$$|f(x_n) - f(x_{<n,m>})| < \frac{\varepsilon}{2} \text{ for each } n \text{ and } |g(x_n) - g(x_{<n,m>})| < \frac{\varepsilon}{2} \text{ for each } n.$$

Now

$$\begin{aligned} |(f+g)(x_n) - (f+g)(x_{<n,m>})| &= |f(x_n) + g(x_n) - f(x_{<n,m>}) - g(x_{<n,m>})| \\ &\leq |f(x_n) - f(x_{<n,m>})| + |g(x_n) - g(x_{<n,m>})| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for each} \quad n. \end{aligned}$$

This proves the theorem. □

Theorem 2.3. *The difference of two arithmetic continuous functions is again arithmetic continuous.*

Proof: The proof is similar to the above theorem. □

Theorem 2.4. *If f is an arithmetic continuous function then $|f|$ is arithmetic continuous.*

Proof: Let f be an arithmetic continuous function on a subset E of $\mathbb{R}$. Let (x_n) be any arithmetic convergent sequence in E. Then by definition of arithmetic continuity the sequence $(f(x_n))$ is arithmetic convergent.
Therefore for $\varepsilon > 0$ there exists a positive integer m such that

$$|f(x_n) - f(x_{<n,m>})| < \varepsilon \text{ for each } n.$$

Now

$$\begin{aligned} ||f|(x_n) - |f|(x_{<n,m>})| &= ||f(x_n)| - |f(x_{<n,m>})|| \\ &\leq |f(x_n) - f(x_{<n,m>})| < \varepsilon \text{ for each } n. \end{aligned}$$

Hence the result. □

Theorem 2.5. *The composition of two arithmetic continuous functions is again arithmetic continuous.*

Proof: Let f and g be two arithmetic continuous functions on a subset E of $\mathbb{R}$. To prove that the function $f \circ g(x_n) = f(g(x_n))$ is arithmetic continuous function. Let (x_n) be any arithmetic convergent sequence. Since g is arithmetic continuous,so the sequence $(g(x_n))$ is also arithmetic convergent. Furthermoer, it is given that f is arithmetic continuous, hence it transforms arithmetic convergent sequence $(g(x_n))$ to arithmetic convergent sequence $(f(g(x_n)))$.
Hence the result follows. □

Theorem 2.6. *If f is uniformly continuous on a subset E of $\mathbb{R}$ then it is arithmetic continuous.*

Proof: Let f be uniformly continuous and (x_n) be any arithmetic convergent sequence in E. Since f is uniformly continuous in E, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in E$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.
Again the sequence (x_n) is arithmetic convergent, hence for the same $\delta > 0$ there exists a positive integer m such that

$$\begin{aligned} |x_n - x_{<n,m>}| < \delta \text{ for each } n &\Rightarrow |f(x_n) - f(x_{<n,m>})| < \varepsilon \text{ for each } n \\ &\Rightarrow \text{the sequence } (f(x_n)) \text{ is arithmetic convergent.} \\ &\Rightarrow \text{the function} f \text{is arithmetic continuous.} \end{aligned}$$

This completes the proof. □

3. Arithmetic convergence of sequence of functions

Definition 3.1. *A sequence of functions (f_n) defined on a subset E of $\mathbb{R}$ is said to be arithmetic convergent if for any $\varepsilon > 0$ and $\forall\ x \in E$ there exists a positive integer m such that*

$$|f_n(x) - f_{<n,m>}(x)| < \varepsilon\ \forall x \in E \quad \textit{and} \quad n \in \mathbb{N}.$$

Theorem 3.2. *If (f_n) be a sequence of arithmetic functions defined on a subset E of $\mathbb{R}$ and x_0 is a point in E such that*

$$\lim_{x \to x_0} f_n(x) = y_n,\ \ n = 1, 2, 3 \ldots.$$

then (y_n) is arithmetic convergent.

Proof: Since the sequence (f_n) is arithmetic convergent therefore for $\varepsilon > 0$ and a positive integer m

$$|f_n(x) - f_{<n,m>}(x)| < \varepsilon\ \forall x \in E \quad \text{and} \quad n \in \mathbb{N}.$$

Keeping n, m fixed and letting $x \to x_o$,

$$|y_n - y_{<n,m>}| < \varepsilon\ \forall\ n.$$

Hence the sequence (y_n) is arithmetic convergent. □

Theorem 3.3. *If (f_n) is a sequence of arithmetic continuous functions and (f_n) is uniformly convergent to a function f on a subset E of $\mathbb{R}$, then f is arithmetic continuous.*

Proof: Let $\varepsilon > 0$ and (x_n) be any arithmetic convergent sequence on a subset E of $\mathbb{R}$. Since $f_n \to f$ uniformly, we have for a positive integer n_1 such that

$$|f_n(x) - f(x)| < \varepsilon\ \forall n \geq n_1 \text{ and } \forall x \in E. \tag{3.1}$$

In particular for $n = n_1$

$$|f_{n_1}(x) - f(x)| < \varepsilon\ \forall x \in E. \tag{3.2}$$

Furthermore (f_n) is given to be a sequence of arithmetic continuous functions, therefore

$$|f_{n_1}(x_n) - f_{n_1}(x_{<n,m>})| < \varepsilon. \tag{3.3}$$

Also from (3.1) we get

$$|f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| < \varepsilon. \tag{3.4}$$

Therefore using (3.2),(3.3),(3.4) we get

$$\begin{aligned} |f(x_n) - f(x_{<n,m>})| &= |f(x_n) - f_{n_1}(x_n) + f_{n_1}(x_n) - f_{n_1}(x_{<n,m>}) \\ &\quad + f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| \\ &\leq |f(x_n) - f_{n_1}(x_n)| + |f_{n_1}(x_n) - f_{n_1}(x_{<n,m>})| \\ &\quad + |f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$

Hence f is arithmetic continuous, which concludes the proof. □

Theorem 3.4. *The set of all arithmetic continuous functions defined on a subset E of $\mathbb{R}$ is a closed subset of all continuous function on E, i.e. $\overline{acf(E)}$=$acf(E)$, where $acf(E)$ denotes the set of all arithmetic continuous functions defined on E and $\overline{acf(E)}$ denotes the set of all limit points of $acf(E)$.*

Proof: Let f be any element of $\overline{acf(E)}$. Then there exists a sequence of points in $acf(E)$ such that $\lim f_n = f$. Now let (x_n) be any arithmetic convergent sequence in E. Since (f_n) converges to f, there exists a positive integer n_1 such that

$$|f(x) - f_n(x)| < \varepsilon \;\forall n \geq n_1 \text{ and } \forall x \in E. \tag{3.5}$$

Also since f_{n_1} is arithmetic continuous on E, there exists a positive integer m such that

$$|f_{n_1}(x_n) - f_{n_1}(x_{<n,m>})| < \varepsilon. \tag{3.6}$$

Again from(3.5),

$$|f_{n_1}(x_n) - f(x_n)| < \varepsilon \;, n = 1, 2, 3 \ldots. \tag{3.7}$$

Also

$$|f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| < \varepsilon. \tag{3.8}$$

Using (3.6),(3.7),(3.8) we get

$$\begin{aligned} |f(x_n) - f(x_{<n,m>})| &= |f(x_n) - f_{n_1}(x_n) + f_{n_1}(x_n) - f_{n_1}(x_{<n,m>}) \\ &\quad + f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| \\ &\leq |f(x_n) - f_{n_1}(x_n)| + |f_{n_1}(x_n) - f_{n_1}(x_{<n,m>})| \\ &\quad + |f_{n_1}(x_{<n,m>}) - f(x_{<n,m>})| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Hence f is arithmetic continuous on E. This completes the proof of the theorem. □

Corollary 3.5. *The set of all arithmetic continuous functions defined on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions.*

4. Arithmetic Compactness

Definition 4.1. *A subset E of $\mathbb{R}$ is called aritmetic compact if every sequence of points in E has arithmetic convergent subsequence.*

Theorem 4.2. *An arithmetic continuous image of an arithmetic compact subset of $\mathbb{R}$ is arithmetic compact.*

Proof: Let f be an arithmetic continuous function on a subset E of $\mathbb{R}$ and E is arithmetic compact. Let (y_n) be a sequence of points in $f(E)$. Then we can write $y_n = f(x_n)$ where $(x_n) \in E$ for each $n \in \mathbb{N}$. Since E is arithmetic compact, there exists an arithmetic convergent subsequence (x_{n_k}) of (x_n). Again it is given that f is arithmetic continuous, this implies that $f(x_{n_k})$ is arithmetic convergent subsequence of $f(x_n)$. Hence $f(E)$ is arithmetic compact. $\square$

Corollary 4.3. *An arithmetic continuous image of a compact subset of $\mathbb{R}$ is arithmetic compact.*

References

1. J. Antoni and T. Šalát, On the A-continuity of real functions, Acta Math. Univ. Comenian. **39**(1980), 159-164.
2. I. D. Berg and A. Wilansky, Periodic, almost periodic and semi-periodic sequences, Michigan Math. J., 9(1962),263-268.
3. I. D. Berg, The conjugate space of the space of semiperiodic sequences, Michigan Math. J., 13(1962),293-297.
4. I. Çanak, M. Dik, New Types of Continuities, Abstr. Appl. Anal. Hindawi Publ. Corp., New York, ISSN 1085-3375, Volume 2010, Article ID 258980, (2010).
5. J. Connor, K. G. Grosse-Erdmann, Sequential definitions of continuity for real functions, Rocky Mountain J. Math., **33**(1)(2003), 93-121.
6. M. Et, Muhammed Çinar, Murat Karakaş, On λ-statistical convergence of order α of sequences of function, J. Ineq. Appl., 2013, 2013:204.
7. P. Haukkanen, T. Tossavainen, Approximation of arithmetical functions by even functions, JP. J. Algebra Number Theory Appl., 10(1)(2008), 51-63.
8. G. H. Hardy and E. M. Wright, An introduction to the theory of Numbers, Fourth Edition, 1960, Oxford.
9. M. N. Jimenez, Multipliers on the space of semiperiodic sequences, Trans. Amer. Math. Soc., 291(1985),801-811.
10. E. C. Posner, Summability preserving functions, Proc.Amer.Math.Soc. **12**(1961), 73-76.
11. W. H. Ruckle, Arithmetical Summability, J. Math. Anal. Appl. 396(2012),741-748.
12. W. Schwarz, J. Spilker, Arithmetical functions. An Introduction to elementary and analytic properties of arithmetic functions and to some of their almost-periodic properties, in : London Math. Soc. Lecture Note Series, Vol 184, Cambridge University Press, Cambridge, 1994.
13. J. A. Siddiqi, Infinite matrices summing every almost periodic sequence, Pacific J. Math., 39(1971), 235-251.
14. Taja Yaying and Bipan Hazarika, On arithmetical summability and multiplier sequences (preprint).

Taja Yaying
Department of Mathematics, Dera Natung Govt. College,
Itanagar-791 111, Arunachal Pradesh, India
E-mail address: tajayaying20@gmail.com

and

Bipan Hazarika (Corresponding author)
Department of Mathematics, Rajiv Gandhi University,
Rono Hills, Doimukh-791 112, Arunachal Pradesh, India
E-mail address: bh_rgu@yahoo.co.in