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On g_{ij}-closed Bi-Generalized topological spaces

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ABSTRACT: In this paper, generalizations of adherence and convergence of nets and filters on a bi-GTS are introduced and studied. Several properties and interrelations among such adherence and convergence of nets and filters on a bi-GTS are discussed and characterized using graphs of functions. Finally, these results are applied to investigate the behaviour of a generalization of compactness, known as g_{ij}-*closedness* of a bi-GTS.

Key Words: (μ_i, μ_j)-adherence, (μ_i, μ_j)-convergence, μ-IFIP, net of μ-open sets.

Contents

1. Introduction and Preliminaries

In continuation of our work on bi-generalized topological spaces (in short, bi-GTS) [2,1], we introduce and study certain generalizations of adherence and convergence of nets and filters on a bi-GTS. Discussing several properties and interrelations among such adherence and convergence of nets and filters on a bi-GTS, we have characterized them using graphs of functions. Finally, the results obtained in the first part of the paper are applied to investigate the behaviour of a generalization of compactness, called g_{ij}-*closedness* [2] of a bi-GTS.

We list a few known definitions and existing results here, which we require in the following sections.
Let X be a nonempty set and μ be a collection of subsets of X (i.e. $\mu \subseteq \mathcal{P}(X)$). μ is called a *generalized topology* (briefly GT) [3] on X iff $\emptyset \in \mu$ and $G_\lambda \in \mu$ for $\lambda \in \Lambda(\neq \emptyset)$ implies $\cup_{\lambda\in\Lambda} G_\lambda \in \mu$. The pair (X, μ) is called a *generalized topological*

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space (briefly GTS). The elements of μ are called *μ-open sets* and their complements are called *μ-closed sets*. The *generalized closure* of a subset S of X, denoted by $c_\mu S$, is the intersection of all μ-closed sets containing S. The *generalized interior* of a subset S of X, denoted by $i_\mu S$, is the union of all μ-open sets included in S. The set of all μ-open sets containing an element $x \in X$ is denoted by $\mu(x)$. A GT μ is called a *strong GT* if $X \in \mu$.

Let $\psi : X \to exp(expX)$ satisfy $V \in \psi(x)$ for each $x \in V$. Then $\psi(x)$ is called a *generalized neighbourhood* of $x \in X$ and ψ a *generalized neighbourhood system* (briefly GNS) on X. On a GTS (X, μ), ψ_μ defined by $\psi_\mu(x) = \{A \subseteq X : x \in M \subseteq A \ for\ some\ M \in \mu\}$, for each $x \in X$ also forms a GNS on X which is called *GNS generated by the GT μ* (briefly μ-GNS). Each member of $\psi_\mu(x)$ is called a μ-nbd of x. [3]

Let μ_1, μ_2 be two GTs on a nonempty set X. Then (X, μ_1, μ_2) is called a *bi-generalized topological space* (briefly bi-GTS) [6]. On a bi-GTS (X, μ_1, μ_2), $\gamma_{\mu_i,\mu_j} : \mathcal{P}(X) \to \mathcal{P}(X)$, $i, j = 1, 2 (i \neq j)$, is defined by
$\gamma_{\mu_i,\mu_j}(A) = \{x \in X : c_{\mu_j} M \cap A \neq \phi \ for\ all\ M \in \mu_i(x)\}$. [4]
Let (X, μ_1, μ_2) be a bi-GTS. Then $\theta(\mu_i, \mu_j)$ [4] $\subseteq \mathcal{P}(X)(i \neq j)$, defined by
$\theta(\mu_i, \mu_j) = \{A \subseteq X \ : \ for\ each\ x \in X \exists M \in \mu_i(x),\ with\ c_{\mu_j} M \subseteq A\}$ also forms a GT on X. The elements of $\theta(\mu_i, \mu_j)$ are called $\theta(\mu_i, \mu_j)$-open and the complements are called $\theta(\mu_i, \mu_j)$-closed.

Theorem 1.1. [4] *Let (X, μ_1, μ_2) be a bi-GTS and $A \subseteq X$. Then A is $\theta(\mu_i, \mu_j)$-closed iff $A = \gamma_{\mu_i,\mu_j}(A)$.*

Let μ_1, μ_2 be two GTs on a nonempty set X and $A \subseteq X$. A is said to be $r(\mu_i, \mu_j)$-open (resp. $r(\mu_i, \mu_j)$-closed) if $A = i_{\mu_i}(c_{\mu_j}(A))$ (resp. $A = c_{\mu_i}(i_{\mu_j}(A))$) [4]. Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. If $\nu_i (i = 1, 2)$ on the cartesian product $X \times Y$ is given by $\nu_i = \mu_i \times \eta_j$ for $i, j = 1, 2 (i \neq j)$ then $(X \times Y, \nu_1, \nu_2)$ is a bi-GTS. Similarly, for a bi-GTS (X, μ_1, μ_2), $(X \times X, \nu_1, \nu_2)$ is a bi-GTS where $\nu_i = \mu_i \times \mu_j$ for $i, j = 1, 2 (i \neq j)$.
It is well known that a filterbase $\mathcal{F}$ induces a net [7] $P : (\Lambda, \geq) \to X$ defined by $P((x, F)) = x$ where $\Lambda = \{(x, F) : x \in F \in \mathcal{F}\}$ and the binary relation $\geq$ is given by $(x_1, F_1) \geq (x_2, F_2)$ if and only if $F_1 \subseteq F_2$. Similarly, a net (x_α) with the directed set $(\Lambda, \geq)$ induces a filterbase [7] $\{T_\alpha : \alpha \in \Lambda\}$, where each $T_\alpha = \{x_\beta \ : \ \beta \in \Lambda \ and\ \beta \geq \alpha\}$.

2. (μ_i, μ_j)-adherence and (μ_i, μ_j)-convergence of Nets and Filterbases

Definition 2.1. [2] *A filterbase $\mathcal{F}$ on a bi-GTS (X, μ_1, μ_2) is said to*
(i) (μ_i, μ_j)-adhere $(i, j = 1, 2$ and $i \neq j)$ at $x \in X$ if for each $U \in \mu_i(x)$ and each $F \in \mathcal{F}$, $F \cap c_{\mu_j} U \neq \emptyset$.
(ii) (μ_i, μ_j)-converge $(i, j = 1, 2$ and $i \neq j)$ to $x \in X$ if for each $U \in \mu_i(x)$ there exists $F \in \mathcal{F}$, such that $F \subseteq c_{\mu_j} U$.

Definition 2.2. *A net* (x_α) *on a bi-GTS* (X, μ_1, μ_2) *with the directed set* $(\Lambda, \geq)$ *as a domain is said to*
(i) (μ_i, μ_j)*-adhere* $(i, j = 1, 2$ *and* $i \neq j)$ *at* $x \in X$ *if for each* $U \in \mu_i(x)$ *and each* $\alpha \in \Lambda$*, there exists* $\beta \in \Lambda$ *such that* $\beta \geq \alpha$ *and* $x_\beta \in c_{\mu_j}U$.
(ii) (μ_i, μ_j)*-converge to* $x \in X$ $(i, j = 1, 2$ *and* $i \neq j)$ *if for each* $U \in \mu_i(x)$ *there exists* $\alpha_0 \in \Lambda$ *such that* $x_\alpha \in c_{\mu_j}U$ *for all* $\alpha \in \Lambda$ *with* $\alpha \geq \alpha_0$.

Theorem 2.3. *Let* (X, μ_1, μ_2) *be a bi-GTS and* $x_0 \in X$. *Then a filterbase* $\mathcal{F}$ *on* X (μ_i, μ_j)*-converges to* x_0 *iff the net* P *based on* $\mathcal{F}$ (μ_i, μ_j)*-converges to* x_0; $i, j = 1, 2 (i \neq j)$.

Proof: Let a filterbase $\mathcal{F}$ be (μ_i, μ_j)-convergent to x_0 and $P : \Lambda \to X$ be the net based on $\mathcal{F}$. If $U \in \mu_i(x_0)$ then by the convergence of $\mathcal{F}$ there exists $F \in \mathcal{F}$ such that $F \subseteq c_{\mu_j}U$. Choose $p \in F$ so that $(p, F) \in \Lambda$. So if $(x_1, F_1) \geq (p, F)$ then $P[(x_1, F_1)] = x_1 \in F_1$. As $F_1 \subseteq F$, $x_1 \in c_{\mu_j}U$. i.e., P is (μ_i, μ_j)-convergent to x_0.
Conversely, let P be (μ_i, μ_j)-convergent to x_0 and $U \in \mu_i(x_0)$. Then there exists $(x_1, F_1) \in \Lambda$ such that $(y, F) \geq (x_1, F_1)$ implies $P[(y, F)] = y \in c_{\mu_j}U$. Now for each $z \in F_1$ we have $(z, F_1) \geq (x_1, F_1)$, i.e., $z \in c_{\mu_j}U$ and hence $F_1 \subseteq c_{\mu_j}U$. Thus $\mathcal{F}$ is (μ_i, μ_j)-convergent to x_0. □

Theorem 2.4. *Let* (X, μ_1, μ_2) *be a bi-GTS and* $x_0 \in X$. *Then* x_0 *is a* (μ_i, μ_j)*-adherent point of a filterbase* $\mathcal{F}$ *iff the net* P *based on* $\mathcal{F}$ *has* x_0 *as a* (μ_i, μ_j)*-adherent point;* $i, j = 1, 2 (i \neq j)$.

Proof: Let x_0 be a (μ_i, μ_j)-adherent point of a filterbase $\mathcal{F}$ and $P : \Lambda \to X$ be the net based on $\mathcal{F}$. Let $(p, F) \in \Lambda$ and $U \in \mu_i(x_0)$. Then by the adherence of $\mathcal{F}$, $F \cap c_{\mu_j}U \neq \emptyset$. If $x_1 \in F \cap c_{\mu_j}U$ then $(x_1, F) \geq (p, F)$ and $P[(x_1, F)] = x_1 \in c_{\mu_j}U$. Hence x_0 is a (μ_i, μ_j)-adherent point of P.
Conversely, let P have x_0 as a (μ_i, μ_j)-adherent point. Let $U \in \mu_i(x_0)$ and $F \in \mathcal{F}$. Choose $p \in F$ so that $(p, F) \in \Lambda$ and so by the adherence of the net P there exists $(b, K) \in \Lambda$ with $(b, K) \geq (p, F)$, $P[(b, K)] = b \in c_{\mu_j}U$. As $K \subseteq F$, we have $F \cap c_{\mu_j}U \neq \emptyset$, i.e., x_0 is a (μ_i, μ_j)-adherent point of $\mathcal{F}$. □

Theorem 2.5. *Let* (X, μ_1, μ_2) *be a bi-GTS and* $x_0 \in X$. *Then a net* $(x_\alpha)_{\alpha \in \Lambda}$ (μ_i, μ_j)*-converges to* x_0 *iff the filterbase generated by the net is* (μ_i, μ_j)*-convergent to* x_0; $i, j = 1, 2 (i \neq j)$.

Proof: Let a net $(x_\alpha)_{\alpha \in \Lambda}$ (μ_i, μ_j)-converge to x_0 and $U \in \mu_i(x_0)$. Then there exists some $\alpha_0 \in \Lambda$ such that $x_\beta \in c_{\mu_j}U$, $\forall \beta \geq \alpha_0$. The filterbase generated by the net $(x_\alpha)_{\alpha \in \Lambda}$ is $\{T_\alpha : \alpha \in \Lambda\}$ where $T_\alpha = \{x_\beta : \beta \in \Lambda$ *and* $\beta \geq \alpha\}$. It is clear that $T_{\alpha_0} \subseteq c_{\mu_j}U$ and hence $\{T_\alpha : \alpha \in \Lambda\}$ (μ_i, μ_j)-converges to x_0.
Conversely, let $\{T_\alpha : \alpha \in \Lambda\}$ (μ_i, μ_j)-converge to x_0 and $U \in \mu_i(x_0)$. Then there exists some $\alpha \in \Lambda$ such that $T_\alpha \subseteq c_{\mu_j}U$, i.e., $x_\beta \in c_{\mu_j}U$, $\forall \beta \geq \alpha$ and hence $(x_\alpha)_{\alpha \in \Lambda}$ (μ_i, μ_j)-converges to x_0. □

Theorem 2.6. *Let (X, μ_1, μ_2) be a bi-GTS and $x_0 \in X$. Then x_0 is a (μ_i, μ_j)-adherent point of a net $(x_\alpha)_{\alpha \in \Lambda}$ iff x_0 is a (μ_i, μ_j)-adherent point of the filterbase generated by $(x_\alpha)_{\alpha \in \Lambda}$; $i, j = 1, 2 (i \neq j)$.*

Proof: Let x_0 be a (μ_i, μ_j)-adherent point of a net $(x_\alpha)_{\alpha \in \Lambda}$ and $U \in \mu_i(x_0)$. Then for each $\alpha \in \Lambda$ there exists some $\beta \in \Lambda$ with $\beta \geq \alpha$ and $x_\beta \in c_{\mu_j}U$. The filterbase generated by $(x_\alpha)_{\alpha \in \Lambda}$ is $\{T_\alpha : \alpha \in \Lambda\}$ where $T_\alpha = \{x_\beta : \beta \in \Lambda \textit{ and } \beta \geq \alpha\}$. It can be easily shown that $T_\alpha \cap c_{\mu_j}U \neq \emptyset$ for each $\alpha \in \Lambda$ and hence x_0 is a (μ_i, μ_j)-adherent point of $\{T_\alpha : \alpha \in \Lambda\}$.
Conversely, let x_0 be a (μ_i, μ_j)-adherent point of $\{T_\alpha : \alpha \in \Lambda\}$ and $U \in \mu_i(x_0)$. Then for each $\alpha \in \Lambda$, $T_\alpha \cap c_{\mu_j}U \neq \emptyset$, i.e., for each $\alpha \in \Lambda$ there exists some $\beta \in \Lambda$ with $\beta \geq \alpha$ and $x_\beta \in c_{\mu_j}U$ and hence x_0 is a (μ_i, μ_j)-adherent point of $(x_\alpha)_{\alpha \in \Lambda}$. □

Theorem 2.7. *Let (X, μ_1, μ_2) be a bi-GTS, where μ_i is a strong GT and $x_0 \in X$. Then x_0 is a (μ_i, μ_j)-adherent point of a net $(x_\alpha)_{\alpha \in \Lambda}$ in X iff there exists a subnet (x_{α_λ}) of $(x_\alpha)_{\alpha \in \Lambda}$, which (μ_i, μ_j)-converge to x_0; $i, j = 1, 2 (i \neq j)$.*

Proof: Let x_0 be a (μ_i, μ_j)-adherent point of a net $(x_\alpha)_{\alpha \in \Lambda}$. Let $M = \{(\alpha, c_{\mu_j}U) : x_0 \in U \in \mu_i \textit{ and } x_\alpha \in c_{\mu_j}U\}$. Define $(\alpha_1, c_{\mu_j}U_1) \geq (\alpha_2, c_{\mu_j}U_2)$ iff $\alpha_1 \geq \alpha_2$ and $c_{\mu_j}U_1 \subseteq c_{\mu_j}U_2$. Let $\phi : M \to \Lambda$ be defined by $\phi[(\alpha, c_{\mu_j}U)] = \alpha$. So ϕ defines a subnet of $(x_\alpha)_{\alpha \in \Lambda}$. Now if $U \in \mu_i(x_0)$ then for some $\alpha \in \Lambda$, $x_\alpha \in c_{\mu_j}U$, and so $(\beta, c_{\mu_j}V) \geq (\alpha, c_{\mu_j}U)$ implies $x_\beta \in c_{\mu_j}V \subseteq c_{\mu_j}U$. Hence the subnet (μ_i, μ_j)-converges to x_0.
Conversely, let $(x_\alpha)_{\alpha \in \Lambda}$ be a net with the directed set $(\Lambda, \geq)$ as a domain. Let (x_{α_λ}) a subnet of $(x_\alpha)_{\alpha \in \Lambda}$ with the domain M, which (μ_i, μ_j)-converges to x_0. Let $U \in \mu_i(x_0)$ and $\alpha_0 \in \Lambda$. Then there exists $\lambda_0 \in M$ such that for each $\lambda \geq \lambda_0$, $x_{\alpha_\lambda} \in c_{\mu_j}U$. Take $\lambda_1 \in M$ such that $\alpha_{\lambda_1} \geq \alpha_0$. Let λ_2 be such that $\lambda_2 \geq \lambda_0$ and $\lambda_2 \geq \lambda_1$, then $\alpha_{\lambda_2} \geq \alpha_{\lambda_1} \geq \alpha_0$ and $x_{\alpha_{\lambda_2}} \in c_{\mu_j}U$. Hence x_0 is a (μ_i, μ_j)-adherent point of $(x_\alpha)_{\alpha \in \Lambda}$. □

3. Adherence and Convergence of nets and filters in terms of graph of a function

Definition 3.1. [2] *Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. Then $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is said to be $(\mu_i\mu_j, \eta_k)$-continuous at $x \in X$ if for each $V \in \eta_k(f(x))$, there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j}U) \subseteq V$; $i, j, k = 1, 2 (i \neq j)$. If f is $(\mu_i\mu_j, \eta_k)$-continuous at each $x \in X$ then f is called $(\mu_i\mu_j, \eta_k)$-continuous on X or simply $(\mu_i\mu_j, \eta_k)$-continuous.*

Definition 3.2. *Let μ be a GT on a nonempty set X. Then a filterbase $\mathcal{F}$ on X is said to μ-converge to $x \in X$ if for each $U \in \mu(x)$ there exists $F \in \mathcal{F}$, such that $F \subseteq U$.*

Theorem 3.3. *Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. If $f : X \to Y$ is a $(\mu_i\mu_j, \eta_k)$-continuous function then for every filterbase $\mathcal{F}$ on X, $\mathcal{F}$ (μ_i, μ_j)-converges to some $x \in X$ implies $f(\mathcal{F})$ η_k-converges to $f(x)$, where $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is a filterbase on Y; $i, j, k = 1, 2$ $(i \neq j)$.*

Proof: Let V be any η_k-open set containing $f(x)$. Then there exists a μ_i-open set U containing x such that $f(c_{\mu_j}U) \subseteq V$. Again since $\mathcal{F}$ (μ_i, μ_j)-converges to x, there is $F \in \mathcal{F}$ such that $F \subseteq c_{\mu_j}U$, i.e., $f(F) \subseteq f(c_{\mu_j}U) \subseteq V$. Hence, $f(\mathcal{F})$ η_k-converges to $f(x)$. $\square$

The converse of Theorem 3.3 is also true if we take μ_i as a topology on X.

Theorem 3.4. *Let $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ be a function between two bi-GTS. If for every filterbase $\mathcal{F}$ on X, $\mathcal{F}$ (μ_i, μ_j)-converges to x whenever $f(\mathcal{F})$ η_k-converges to $f(x)$, where μ_i is a topology on X and $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ then $f : X \to Y$ is $(\mu_i\mu_j, \eta_k)$-continuous; $i, j, k = 1, 2$ $(i \neq j)$.*

Proof: Suppose $f : X \to Y$ is not $(\mu_i\mu_j, \eta_k)$-continuous at some point $x \in X$. Then there exists some $V \in \eta_k(f(x))$ such that $f(c_{\mu_j}U) \nsubseteq V$ for every $U \in \mu_i(x)$. Now $\mathcal{F} = \{c_{\mu_j}U : U \in \mu_i(x)\}$ is a filterbase on X such that $\mathcal{F}$ (μ_i, μ_j)-converges to x but $f(\mathcal{F})$ does not η_k-converge to $f(x)$. $\square$

In what follows, by $G(f)$ we denote the graph of a function $f : X \to Y$; i.e., $G(f) = \{(x, y) \in X \times Y : y \in f(x)\}$. Clearly, for any $f : X \to Y$, if $A \subseteq X$ and $B \subseteq Y$, $f(A) \cap B = \{y \in Y : (x, y) \in ((A \times B) \cap G(f)),\ for\ some\ x \in X\}$.

Theorem 3.5. *Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. If $f : X \to Y$ has a $\theta(\nu_i, \nu_j)$-closed graph then for every filterbase $\mathcal{F}$ on X, $\mathcal{F}$ (μ_i, μ_j) converges to some $x \in X$ implies (η_j, η_i)-ad $(f(\mathcal{F})) \cup \{f(x)\} = \{f(x)\}$, where (η_i, η_j)-adΩ denotes the collection of all (η_i, η_j)-adherent points of a filterbase Ω; $i, j = 1, 2 (i \neq j)$.*

Proof: Let $y \in (\eta_j, \eta_i)$-*ad* $f(\mathcal{F})$ such that $y \neq f(x)$. Then $(x, y) \in X \times Y \backslash G(f)$. Since $y \in (\eta_j, \eta_i)$-*ad* $f(\mathcal{F})$, for any $W \in \eta_j(y)$ and for any $f(F) \in f(\mathcal{F})$ we have, $f(F) \cap c_{\eta_i}W \neq \emptyset$. Again since, $\mathcal{F}$ (μ_j, μ_i)-converges to x, for any $V \in \mu_i(x)$ there exists $F \in \mathcal{F}$ such that $F \subseteq c_{\mu_j}V$. Then $f(F) \subseteq f(c_{\mu_j}V)$ and hence $f(c_{\mu_j}V) \cap c_{\eta_i}W \neq \emptyset$. It then follows that for every $V \in \mu_i(x)$ and $W \in \eta_j(y)$, $(c_{\mu_j}V \times c_{\eta_i}W) \cap G(f) \neq \emptyset$, i.e., $c_{\nu_j}(V \times W) \cap G(f) \neq \emptyset$. i.e., $(x, y) \in \gamma_{\nu_i,\nu_j}G(f)$. So by Theorem 1.1 f can not have $\theta(\nu_i, \nu_j)$-closed graph. $\square$

The converse of Theorem 3.5 is also true if we take μ_i as a topology on X.

Theorem 3.6. *Let $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ be a function between two bi-GTS. If for a filterbase $\mathcal{F}$ on X, (η_j, η_i)-ad $(f(\mathcal{F})) \cup \{f(x)\} = \{f(x)\}$ for some $x \in X$ implies $\mathcal{F}$ (μ_i, μ_j)-converges to x, where μ_i is a topology, then $f : X \to Y$ has a $\theta(\nu_i, \nu_j)$-closed graph. Here (η_i, η_j)-adΩ denotes the collection of all (η_i, η_j)-adherent points of a filterbase Ω; $i, j = 1, 2$ $(i \neq j)$.*

Proof: Suppose graph of f is not $\theta(\nu_i, \nu_j)$-closed. Then by Theorem 1.1 there exists $(x, y) \in X \times Y \backslash G(f)$ such that $(x, y) \in \gamma_{\nu_i,\nu_j}G(f)$. i.e., $y \neq f(x)$ and for each $V \in \mu_i(x)$, $W \in \eta_j(y)$, $c_{\nu_j}(V \times W) \cap G(f) \neq \emptyset$. So $(c_{\mu_j}V \times c_{\eta_i}W) \cap G(f) \neq \emptyset$ and hence $f(c_{\mu_j}V) \cap c_{\eta_i}W \neq \emptyset$. Now $\mathcal{F} = \{c_{\mu_j}V : V \in \mu_i(x)\}$ is a filterbase on X such that $\mathcal{F}$ (μ_i, μ_j)-converges to x but $f(\mathcal{F})$ is a filterbase on Y such that there exists some $y \in (\eta_j, \eta_i)$-ad $(f(\mathcal{F}))$ other than $f(x)$, a contradiction to the hypothesis. Hence f has $\theta(\nu_i, \nu_j)$-closed graph. □

Theorem 3.7. *Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. Suppose $f:(X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ has a $\theta(\nu_i, \nu_j)$-closed graph $G(f)$. If $\mathcal{F}$ is a filterbase on X such that $\mathcal{F}$ (μ_i, μ_j)-converges to some point p and $f(\mathcal{F})$ (η_j, η_i)-converges to some q in Y, then $f(p) = q$; $i, j = 1, 2 (i \neq j)$.*

Proof: If possible let $f(p) \neq q$, then $(p, q) \notin G(f)$. Since $G(f)$ is $\theta(\nu_i, \nu_j)$-closed, by Theorem 1.1 $(p, q) \notin \gamma_{\nu_i, \nu_j} G(f)$. Thus there exist $U \in \mu_i(p)$ and $V \in \eta_j(q)$ such that $c_{\nu_j}(U \times V) \cap G(f) = \emptyset$, i.e., $(c_{\mu_j} U \times c_{\eta_i} V) \cap G(f) = \emptyset$. Since $\mathcal{F}$ (μ_i, μ_j)-converges to p and $f(\mathcal{F})$ (η_j, η_i)-converges to q, there exists $A_\alpha \in \mathcal{F}$ such that $A_\alpha \subseteq c_{\mu_j} U$ and $f(A_\alpha) \subseteq c_{\eta_i} V$. Consequently $(c_{\mu_j} U \times c_{\eta_i} V) \cap G(f) \neq \emptyset$, a contradiction. □

4. g_{ij}-closed spaces

Definition 4.1. [2] *A bi-GTS (X, μ_1, μ_2) is called g_{ij}-closed if for every μ_i-open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $X = \cup_{U \in \mathcal{U}_0} c_{\mu_j} U$; $i, j = 1, 2 (i \neq j)$.*

Theorem 4.2. *Let (X, μ_1, μ_2) be a bi-GTS. Then for $i, j = 1, 2 (i \neq j)$ the following are equivalent:*

1. *X is g_{ij}-closed;*
2. *any filterbase (μ_i, μ_j)-adheres in X;*
3. *any net (μ_i, μ_j)-adheres in X.*

Proof: (1) $\Rightarrow$ (2) Suppose $\mathcal{F}$ is a filterbase on (X, μ_1, μ_2) such that it has no (μ_i, μ_j)-adherent point. So for each $x \in X$, there exists a $U_x \in \mu_i(x)$ and $F_x \in \mathcal{F}$ such that $F_x \cap c_{\mu_j} U_x = \emptyset$. Let us consider the μ_i-open cover $\{U_x : x \in X\}$ of X. Then there exist $x_1, x_2, \cdots, x_n$ such that $X = \cup_{k=1}^n c_{\mu_j} U_{x_k}$. Now $F_{x_k} \cap c_{\mu_j} U_{x_k} = \emptyset$ for $k = 1, 2, \cdots, n$, i.e., $(\cap_{k=1}^n F_{x_k}) \cap (\cup_{k=1}^n c_{\mu_j} U_{x_k}) = \emptyset$. Since $\mathcal{F}$ is a filterbase, there is some $F \in \mathcal{F}$ such that $F \subseteq \cap_{k=1}^n F_{x_k} \subseteq X - \cup_{k=1}^n c_{\mu_j} U_{x_k} = \emptyset$, a contradiction.
(2) $\Rightarrow$ (1) Let $\{G_\lambda : \lambda \in \Lambda\}$ be a μ_i-open cover of X such that for any finite subset Λ_0 of Λ, $\cup_{\lambda \in \Lambda_0} c_{\mu_j} G_\lambda \neq X$. Consider $\mathcal{F} = \{X - \cup_{\lambda \in \Lambda_0} c_{\mu_j} G_\lambda : \Lambda_0\ is\ a\ finite\ subset\ of \Lambda\}$. Then clearly $\mathcal{F}$ is a filterbase on X. So by (2) there exists some $x \in X$ such that $\mathcal{F}$ (μ_i, μ_j)-adheres at x. Since $\{G_\lambda : \lambda \in \Lambda\}$ is a μ_i-open cover of X there is some $\lambda_0 \in \Lambda$ such that $x \in G_{\lambda_0}$. Now $X - c_{\mu_j} G_{\lambda_0} = F \in \mathcal{F}$ such that $F \cap c_{\mu_j} G_{\lambda_0} = \emptyset$, a contradiction to the fact that $\mathcal{F}$ is (μ_i, μ_j)-adheres at x.
(2) $\Rightarrow$ (3) Follows from Theorem 2.6.
(3) $\Rightarrow$ (2) Follows from Theorem 2.4. □

Definition 4.3. *Let μ be a GT on a nonempty set X. Then A family $\mathcal{F}$ of subsets of X is said to have μ-interiorly finite intersection property (in short μ-IFIP) if for every finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$ there exists a non-void μ-open set U such that $U \subseteq \cap\mathcal{F}_0$.*

Theorem 4.4. *Let (X, μ_1, μ_2) be a g_{ij}-closed bi-GTS. Then every family of μ_i-closed set in X with μ_j-IFIP has a non-void intersection; $i, j = 1, 2 (i \neq j)$.*

Proof: Let $\mathcal{F}$ be a family of μ_i-closed sets in X with μ_j-IFIP such that $\cap\mathcal{F} = \emptyset$. Then $\mathcal{U} = \{X - F : F \in \mathcal{F}\}$ is a μ_i-open cover of X. Now for any subcollection $\{X - F_1, \cdots, X - F_n\}$, there exists a non-null μ_j-open set U such that $U \subseteq \cap_{r=1}^{n} F_r$. Then $\cup_{r=1}^{n} c_{\mu_j}(X - F_r) \subseteq c_{\mu_j}(\cup_{r=1}^{n}(X - F_r)) \subseteq c_{\mu_j}(X - \cap_{r=1}^{n} F_r) \subseteq c_{\mu_j}(X - U) = X - U \neq X$. Thus X is not g_{ij}-closed, a contradiction to the hypothesis. □

Definition 4.5. *Let μ be a GT on a nonempty set X. A filterbase $\mathcal{F}$ on X is said to be a μ-open filterbase if $\mathcal{F} \subseteq \mu$.*

Definition 4.6. *Let μ be a GT on a nonempty set X. A point x of X is said to be a μ-adherent point of a filterbase $\mathcal{F}$ on X if for each $U \in \mu(x)$ and each $F \in \mathcal{F}$, $F \cap U \neq \emptyset$.*

Theorem 4.7. *Let (X, μ_1, μ_2) be a g_{ij}-closed bi-GTS. Then every μ_j-open filterbase has a μ_i-adherent point; $i, j = 1, 2 (i \neq j)$.*

Proof: Let $\mathcal{F}$ be a μ_j-open filterbase such that it has no μ_i-adherent point. Then for each $x \in X$ there exist $G_x \in \mu_i(x)$ and $F_x \in \mathcal{F}$ such that $G_x \cap F_x = \emptyset$. Consider the μ_i-open cover $\{G_x : x \in X\}$ of X. Then by g_{ij}-closedness of X we have $x_1, x_2, \cdots, x_n \in X$ such that $X = \cup_{k=1}^{n} c_{\mu_j} G_{x_k}$. Again, $G_{x_k} \cap F_{x_k} = \emptyset \Rightarrow c_{\mu_j} G_{x_k} \cap F_{x_k} = \emptyset$ (since, F_{x_k} is μ_j-open)$\Rightarrow F_{x_k} \subseteq X \backslash c_{\mu_j} G_{x_k} \Rightarrow \cap_{k=1}^{n} F_{x_k} \subseteq \cap_{k=1}^{n}(X \backslash c_{\mu_j} G_{x_k}) = X \backslash \cup_{k=1}^{n} c_{\mu_j} G_{x_k} = \emptyset$, a contradiction to the fact that $\mathcal{F}$ is a filterbase. □

The converse of Theorem 3.6 is also true if we take μ_j as a topology on X.

Theorem 4.8. *Let (X, μ_1, μ_2) be a bi-GTS. If every μ_j-open filterbase has a μ_i-adherent point, where μ_j is a topology, then X is g_{ij}-closed, $i, j = 1, 2 (i \neq j)$.*

Proof: If possible let X be not g_{ij}-closed. Then there exists a μ_i-open cover $\{G_\alpha : \alpha \in \Lambda\}$ such that for every finite subset Λ_0 of Λ, $X \neq \cup_{\alpha \in \Lambda_0} c_{\mu_j} G_\alpha \Rightarrow X \backslash \cup_{\alpha \in \Lambda_0} c_{\mu_j} G_\alpha \neq \emptyset \Rightarrow \cap_{\alpha \in \Lambda_0}(X \backslash c_{\mu_j} G_\alpha) \neq \emptyset$. Now $\mathcal{F} = \{\cap_{\alpha \in \Lambda_0}(X \backslash c_{\mu_j} G_\alpha) : \Lambda_0 \; is \; a \; finite \; subset \; of \Lambda\}$ forms a μ_j-open filterbase on X. So by the hypothesis $\mathcal{F}$ μ_i-adheres to some $x \in X$. Now, $x \in G_{\alpha_0} \in \mu_i$ for some $\alpha_0 \in \Lambda$. Again $X \backslash c_{\mu_j} G_{\alpha_0} \in \mathcal{F}$, contradicts that $\mathcal{F}$ μ_i-adheres at x. □

Definition 4.9. *Let μ be a GT on a nonempty set X and $(\Lambda, \geq)$ be a directed set. Then $\{O_\alpha \in \mu : \alpha \in \Lambda\}$ is said to be a net of μ-open sets.*

Definition 4.10. *Let (X, μ_1, μ_2) be a bi-GTS and $\{O_\alpha : \alpha \in \Lambda\}$ be a net of μ_j-open sets. Then a point x of X is called μ_i-adherent point of the net $\{O_\alpha : \alpha \in \Lambda\}$ of μ_j-open sets iff for each $V \in \mu_i(x)$ and each $\alpha \in \Lambda$, there exists $\beta \in \Lambda$ such that $\beta \geq \alpha$ and $V \cap O_\beta \neq \emptyset$; $i, j = 1, 2 (i \neq j)$.*

Theorem 4.11. *Let (X, μ_1, μ_2) be a g_{ij}-closed bi-GTS. Then every net of μ_j-open sets in X has μ_i-adherent point; $i, j = 1, 2 (i \neq j)$.*

Proof: Let $\{O_\alpha : \alpha \in \Lambda\}$ be a net of μ_j-open sets. Consider $F_\alpha = c_{\mu_i}[\cup\{O_\beta : \beta \in \Lambda \text{ and } \beta \geq \alpha\}]$ for each α. Then clearly $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ is a family of μ_i-closed sets with μ_j-IFIP. By Theorem 4.4 there exists $x \in \cap \mathcal{F}$. Let $\alpha \in \Lambda$ and $V \in \mu_i(x)$. Then $V \cap (\cup_{\beta \geq \alpha} O_\beta) \neq \emptyset$, i.e., there exists $\beta \in \Lambda$ with $\beta \geq \alpha$ such that $V \cap O_\beta \neq \emptyset$, proving that x is a μ_i-adherent point of the given net. □

Theorem 4.12. *A bi-GTS (X, μ_1, μ_2) is g_{ij}-closed iff every family $\mathcal{U}$ of $r(\mu_j, \mu_i)$-closed sets having the property that for each $x \in X$, there is $U \in \mathcal{U}$ such that U is a μ_i-nbd of x, has a finite subcover; $i, j = 1, 2 (i \neq j)$.*

Proof: Let X be a g_{ij}-closed space and $\mathcal{U}$ a family satisfying the given condition. So for each $x \in X$, we can find some $U_x \in \mathcal{U}$ and μ_i-open set V_x such that $x \in V_x \subseteq U_x$. It then follows that $\{V_x : x \in X\}$ is a μ_i-open cover of X. Then for a finite subset $\{x_1, x_2, \cdots, x_n\}$ of X, $X = \cup_{k=1}^n c_{\mu_j} V_{x_k} \subseteq \cup_{k=1}^n c_{\mu_j} U_{x_k} = \cup_{k=1}^n U_{x_k}$. Conversely, for any μ_i-open cover $\mathcal{U}$ of X, $\{c_{\mu_j} U : U \in \mathcal{U}\}$ is a family which satisfies the hypothesis of the theorem and the rest is clear. □

Theorem 4.13. *Let (X, μ_1, μ_2) be a bi-GTS. Then X is g_{ij}-closed iff each filterbase $\mathcal{F}$ on X with at most one (μ_i, μ_j)-adherent point, (μ_i, μ_j)-converges; $i, j = 1, 2 (i \neq j)$.*

Proof: Let $\mathcal{F}$ be a filterbase on g_{ij}-closed bi-GTS (X, μ_1, μ_2) with at most one (μ_i, μ_j)-adherent point. So by Theorem 4.2 there exists a point $x_0 \in X$ such that (μ_i, μ_j)-ad$\mathcal{F} = \{x_0\}$. If $\mathcal{F}$ does not (μ_i, μ_j)-converge to x_0 then there exists a $V \in \mu_i(x_0)$ such that for each $F \in \mathcal{F}$, $F \nsubseteq c_{\mu_j} V$. i.e., $F \cap (X \backslash c_{\mu_j} V) \neq \emptyset$. Now $\mathcal{G} = \{F \cap (X \backslash c_{\mu_j} V) : F \in \mathcal{F}\}$ is a filterbase on X. Since X is g_{ij}-closed, $\mathcal{G}$ has non-void (μ_i, μ_j)-adherence by Theorem 4.2. Consequently $\cap_{G \in \mathcal{G}} \gamma_{\mu_i, \mu_j} G \neq \emptyset$. Again $\cap_{G \in \mathcal{G}} \gamma_{\mu_i, \mu_j} G = \cap_{F \in \mathcal{F}} \gamma_{\mu_i, \mu_j} (F \cap (X \backslash c_{\mu_j} V)) \subseteq (\mu_i, \mu_j)$-ad$\mathcal{F} \cap \gamma_{\mu_i, \mu_j} (X \backslash c_{\mu_j} V) = \{x_0\} \cap \gamma_{\mu_i, \mu_j} (X \backslash c_{\mu_j} V)$, i.e., $x_0 \in \gamma_{\mu_i, \mu_j} (X \backslash c_{\mu_j} V)$, which is a contradiction. Hence $\mathcal{F}$ (μ_i, μ_j) converges to x_0.

Conversely, If possible let X be not g_{ij}-closed. Then there exists a filterbase $\mathcal{F}$ on X which has no adherent point in X. So by the hypothesis $\mathcal{F}$ is (μ_i, μ_j)-converges to some point $x \in X$. Since x is not a (μ_i, μ_j)-adherent point of $\mathcal{F}$, there exist $U \in \mu_i(x)$ and $F_1 \in \mathcal{F}$ such that $F_1 \cap c_{\mu_j} U = \emptyset$. Again since $\mathcal{F}$ (μ_i, μ_j)-converges to x, we have some $F_2 \in \mathcal{F}$ such that $F_2 \subseteq c_{\mu_j} U$. But $\mathcal{F}$ is a filterbase on X and so there exist $F \in \mathcal{F}$ such that $F \subseteq F_1 \cap F_2$, which contradicts $F_1 \cap c_{\mu_j} U = \emptyset$ and $F_2 \subseteq c_{\mu_j} U$ to hold simultaneously. □

Definition 4.14. [5] *Let (X, μ_1, μ_2) be a bi-GTS. Then X is said to be (μ_i, μ_j)-regular if for any $x \in X$ and any μ_i-closed set F not containing x, there exist $U \in \mu_i$ and $V \in \mu_j$ with $x \in U, F \subseteq V$ such that $U \cap V = \emptyset$; $i, j = 1, 2 (i \neq j)$.*

Theorem 4.15. [5] *Let (X, μ_1, μ_2) be a (μ_i, μ_j)-regular bi-GTS. Then $\mu_i \subseteq \theta(\mu_i, \mu_j)$; $i, j = 1, 2 (i \neq j)$.*

Theorem 4.16. *A (μ_i, μ_j)-regular bi-GTS (X, μ_1, μ_2) is g_{ij}-closed iff every cover of X by $\theta(\mu_i, \mu_j)$-open sets of X has a finite subcover; $i, j = 1, 2 (i \neq j)$.*

Proof: Let X be g_{ij}-closed space and $\mathfrak{U}$ a cover of X by $\theta(\mu_i, \mu_j)$-open sets. Then for each $x \in X$, there is $U_x \in \mathfrak{U}$ such that $x \in U_x$, and then $x \in V_x \subseteq c_{\mu_j} V_x \subseteq U_x$ for a μ_i-open set V_x. Now $\{V_x : x \in X\}$ is a μ_i-open cover of X and hence by g_{ij}-closedness of X, $X = \cup_{k=1}^{n} c_{\mu_j} V_{x_k}$, for a finite subset $\{x_1, x_2, ..., x_n\}$ of X. Then $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ is a finite subcover of $\mathfrak{U}$.
Conversely, let X be (μ_i, μ_j)-regular and $\mathfrak{U}$ be a μ_i-open cover of X. Then by Theorem 4.15 $\mathfrak{U}$ is also a $\theta(\mu_i, \mu_j)$-open cover of X and so there exists a finite subcollection $\mathfrak{U}_0$ of $\mathfrak{U}$ such that $X = \cup_{U \in \mathfrak{U}_0} U$, i.e., $X = \cup_{U \in \mathfrak{U}_0} c_{\mu_j} U$. □

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