



Efficient Galerkin solution of stochastic fractional differential equations using second kind Chebyshev wavelets

Fakhroddin Mohammadi

ABSTRACT: Stochastic fractional differential equations (SFDEs) have been used for modeling many physical problems in the fields of turbulence, heterogeneous, flows and materials, viscoelasticity and electromagnetic theory. In this paper, an efficient wavelet Galerkin method based on the second kind Chebyshev wavelets are proposed for approximate solution of SFDEs. In this approach, operational matrices of the second kind Chebyshev wavelets are used for reducing SFDEs to a linear system of algebraic equations that can be solved easily. Convergence and error analysis of the proposed method is considered. Some numerical examples are performed to confirm the applicability and efficiency of the proposed method.

Key Words: Fractional calculus; Stochastic calculus; Stochastic fractional differential equations; Second kind Chebyshev wavelets; Operational matrix; Wavelet Galerkin method.

Contents

1 Introduction	195
2 Preliminary definitions	197
2.1 Stochastic calculus	197
2.2 Fractional calculus	198
2.3 Block pulse functions	199
3 Second kind Chebyshev wavelets	200
3.1 Second kind Chebyshev wavelets and BPFs	202
4 Operational matrices for second kind Chebyshev wavelets	203
5 Description of the numerical method	205
6 Convergence analysis	206
7 Numerical results	210
8 Conclusion	212

1. Introduction

Recently, stochastic analysis has been an interesting research area in mathematics, fluid mechanics, geophysics, biology, chemistry, epidemiology, microelectronics, physics, economics, and finance [1,2,3]. The behavior of dynamical systems in

these areas are often dependent on a noise source and a Gaussian white noise, governed by certain probability laws, so that modeling such phenomena naturally requires use of various stochastic differential equations or, in more complicated cases, stochastic integral equations and stochastic integro-differential equations. Since analytic solutions of stochastic integral and differential equations are not available in many cases, numerical approximation becomes a practical way to face this difficulty. Therefore, Many researchers have considered numerical solution of stochastic integral and differential equations. For example, Runge-Kutta method [4,5,6,7], Galerkin finite element method [8,9,10], operational method and orthogonal functions [11,12] and spectral methods [13] have been used for solving stochastic differential and integral equations.

Fractional integrals and derivatives have been applied for modeling many physical phenomena in fields of nonlinear oscillation of earthquake, fluid-dynamic traffic, continuum and statistical mechanics, signal processing, control theory, and dynamics of interfaces between nanoparticles and substrates [14,15,16,17,18]. Consequently, considerable attentions have been given for deriving approximate solution of fractional differential and integral equations. Recently, several numerical methods such as Fourier transforms method [19], Laplace transforms method [20], fractional differential transform method [21], finite difference method [22], orthogonal functions [23,24,25,26], wavelets method [27,28], Adomian decomposition method [29], variational iteration method [30], and homotopy analysis method [31] have been used for producing approximate solution of fractional differential and integral equations.

Recently, different orthogonal basis functions, such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, were used to estimate solutions of functional equations. As a powerful tool, wavelets have been extensively used in computational mathematics, signal processing, image processing and time-frequency analysis and many other areas [27,28,32,33,34,35,36,37].

In this paper a second kind Chebyshev wavelet Galerkin method is proposed for numerical solution of the following SFDE

$$D_*^\alpha u(t) = f(t) + \int_0^t u(s)k_1(s,t)ds + \int_0^t u(s)k_2(s,t)dB(s), \quad t \in [0, 1], \quad (1.1)$$

with these initial conditions

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \dots, n-1, \quad n-1 < \alpha \leq n, \quad (1.2)$$

where $u(t)$, $f(t)$ and $k_i(s,t)$, $i = 1, 2$ are the stochastic processes defined on the same probability space (Ω, F, P) , and $u(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_2(s,t)u(s)dB(s)$ is the Itô integral. Many phenomena in science that have been modeled by fractional differential equations have some uncertainty, so for deriving a more accurate solution, we need the solution of SFDEs [13,40,41,42]. For deriving an approximate solution of SFDEs (1.1) we first derive some operational matrices for the second kind Chebyshev wavelets. Then, these operational matrices along with second kind Chebyshev wavelet are used to obtain approximate solution.

The reminder of the paper is organized as follows: In section 2 some preliminary definitions of stochastic calculus, fractional calculus and Block Pulse Functions (BPFs) are reviewed. Section 3 is devoted to the basic definitions of the second kind Chebyshev wavelets and their properties. In section 4 general procedures for forming operational matrices of the second kind Chebyshev wavelets are explained. In section 5 a wavelet Galerkin method based on the second kind Chebyshev wavelets and their operational matrices are proposed for solving SFDEs. Numerical examples are included in section 6. Finally, a conclusion is given in section 7.

2. Preliminary definitions

In this section we review some necessary definitions and mathematical preliminaries about stochastic calculus, fractional calculus and BPFs which are required for establishing our results in the next sections [1,2].

2.1. Stochastic calculus

Definition 2.1. (*Brownian motion*) A real-valued stochastic process $B(t), t \in [0, T]$ is called Brownian motion, if it satisfies the following properties:

- (i) The process has independent increments for $0 \leq t_0 \leq t_1 \leq \dots \leq t_n \leq T$,
- (ii) For all $t \geq 0$, $B(t+h) - B(t)$ is a normal distribution with mean 0 and variance h ,
- (iii) The function $t \rightarrow B(t)$ is a continuous function of t .

Definition 2.2. Let $\{\mathcal{N}_t\}_{t \geq 0}$ be an increasing family of σ -algebras of subsets of Ω . A process $g : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ is called \mathcal{N}_t -adapted if for each $t \geq 0$ the function $\omega \rightarrow g(t, \omega)$ is \mathcal{N}_t -measurable.

Definition 2.3. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions $f : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ such that

- (i) The function $(t, \omega) \rightarrow f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ -measurable, where \mathcal{B} denotes the Borel algebra on $[0, \infty)$ and \mathcal{F} is the σ -algebra on Ω .
- (ii) f is adapted to \mathcal{F}_t , where \mathcal{F}_t is the σ -algebra generated by the random variables $B(s), s \leq t$.

$$(iii) E \left(\int_S^T f^2(t, \omega) dt \right) < \infty.$$

Definition 2.4. (*The Itô integral*) Let $f \in \mathcal{V}(S, T)$, then the Itô integral of f is defined by

$$\int_S^T f(t, \omega) dB_t(\omega) = \lim_{n \rightarrow \infty} \int_S^T \varphi_n(t, \omega) dB_t(\omega), \quad (\text{lim in } L^2(\Omega, H)),$$

where, φ_n is a sequence of elementary functions such that

$$E \left(\int_s^T (f(t, \omega) - \varphi_n(t, \omega))^2 dt \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

For more details about stochastic calculus and integration please see [1,2,43].

2.2. Fractional calculus

Fractional order calculus is a branch of calculus which deal with integration and differentiation operators of non-integer order. Among the several formulations of the generalized derivative, the Riemann-Liouville and Caputo definition are most commonly used. Here we give some necessary definitions and mathematical preliminaries of the fractional calculus which are required for establishing our results [20].

Definition 2.5. A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ and a function $f_1(t) \in C[0, \infty)$ such that $f(t) = t^p f_1(t)$, and it is said to be in the space $C_\mu^n, n \in \mathbb{N}$ if $f^{(n)} \in C_\mu$.

Definition 2.6. The Riemann-Liouville fractional integration of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$(I^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \quad (2.1)$$

The Riemann-Liouville fractional operator J^α has the following properties:

- (a) $J^\alpha (J^\beta f(t)) = J^\beta (J^\alpha f(t))$,
- (b) $J^\alpha (J^\beta f(t)) = J^{\alpha+\beta} f(t)$,
- (c) $J^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} t^{\nu+\alpha}, \alpha, \beta \geq 0, \nu > -1$.

Definition 2.7. Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D^\alpha f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \quad n \in \mathbb{N}, \quad n-1 < \alpha \leq n. \quad (2.2)$$

The Riemann-Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, a modified fractional differential operator D_*^α was proposed by Caputo [20].

Definition 2.8. The fractional derivative of order $\alpha > 0$ in the Caputo sense is defined as

$$D_*^\alpha f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & 0 \leq n-1 < \alpha < n. \end{cases} \quad (2.3)$$

where n is an integer, $t > 0$, and $f \in C_1^n$.

Some useful relation between the Riemann-Liouville and Caputo fractional operators is given by the following expression:

- (a) $J^\alpha D_*^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad n-1 < \alpha \leq n, \quad t > 0.$
- (b) $D_*^\alpha J^\alpha f(t) = f(t).$
- (c) $J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}.$
- (d) $D_*^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & \beta \geq \alpha, \\ 0 & \beta < \alpha. \end{cases}$

For more details about fractional calculus please see [20].

2.3. Block pulse functions

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [11,27].

The m -set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

in which $t \in [0, T)$, $i = 1, 2, \dots, m$ and $h = \frac{T}{m}$. The set of BPFs are disjoint with each other in the interval $[0, T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), \quad i, j = 1, 2, \dots, m, \quad (2.5)$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0, T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, \dots, m. \quad (2.6)$$

If $m \rightarrow \infty$ the set of BPFs is a complete basis for $L^2[0, T)$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0, T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^m f_i b_i(t), \quad (2.7)$$

where

$$f_i = \frac{1}{h} \int_0^T b_i(t)f(t)dt, \quad i = 1, 2, \dots, m. \quad (2.8)$$

Rewriting Eq. (2.7) in the vector form we have

$$f(t) \simeq \sum_{i=1}^m f_i b_i(t) = F^T \Phi(t) = \Phi^T(t)F, \quad (2.9)$$

in which

$$\Phi(t) = [b_1(t), b_2(t), \dots, b_m(t)]^T, \quad (2.10)$$

$$F = [f_1, f_2, \dots, f_m]^T. \quad (2.11)$$

Moreover, a two dimensional function $k(s, t) \in L^2([0, T_1] \times [0, T_2])$ can be expanded with respect to BPFs such as

$$k(s, t) = \Phi^T(t) K \Phi(t), \quad (2.12)$$

where $\Phi(t)$ is the m -dimensional BPFs vectors and K is the $m \times m$ BPFs coefficient matrix with (i, j) -th element

$$k_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} k(s, t) b_i(t) b_j(s) dt ds, \quad i, j = 1, 2, \dots, m, \quad (2.13)$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t) \Phi(t) = 1, \quad (2.14)$$

and

$$\Phi(t) \Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m \times m}. \quad (2.15)$$

For an m -vector F we have

$$\Phi(t) \Phi^T(t) F = \tilde{F} \Phi(t), \quad (2.16)$$

where \tilde{F} is an $m \times m$ matrix, and $\tilde{F} = \text{diag}(F)$. Also, it is easy to show that for an $m \times m$ matrix A

$$\Phi^T(t) A \Phi(t) = \hat{A}^T \Phi(t), \quad (2.17)$$

where $\hat{A} = (a_{11}, a_{22}, \dots, a_{mm})$ is an m -vector.

3. Second kind Chebyshev wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function ψ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [10,32,35,36]

$$\psi_{a,b}(t) = a^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0. \quad (3.1)$$

The second kind Chebyshev wavelets $\psi_{nm}(x) = \psi(k, n, m, x)$ are defined on the interval $[0, 1)$ by

$$\psi_{nm}(t) = \begin{cases} \sqrt{\frac{2}{\pi}} 2^{\frac{k+1}{2}} U_m(2^{k+1}t - 2n - 1), & \frac{n}{2^k} \leq x \leq \frac{n+1}{2^k} \\ 0, & \text{otherwise} \end{cases}, \quad (3.2)$$

where $U_m(t)$ is the second kind Chebyshev polynomials of degree m , given by [44]

$$U_m(t) = \frac{\sin((m+1)\theta)}{\sin(\theta)}, \quad t = \cos(\theta) \quad (3.3)$$

The second kind Chebyshev wavelets $\{\psi_{nm}(t) | n = 0, 1, \dots, 2^k - 1, m = 0, 1, 2, \dots, M-1\}$ forms an orthonormal basis for $L^2_{w_{nk}}[0, 1]$ with respect to the weight function $w_{nk}(t) = w(2^{k+1}t - 2n - 1)$, in which $w(t) = \sqrt{1-t^2}$.

By using the orthonormality of the second kind Chebyshev wavelets, any square integrable function $f(t)$ defined over $[0, 1)$ can be expanded in terms of the second kind Chebyshev wavelets as

$$f(t) \simeq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \quad (3.4)$$

where $c_{mn} = (f(t), \psi_{mn}(t))_{w_{nk}}$ and $(\cdot, \cdot)_{w_{nk}}$ denotes the inner product on $L^2_{w_{nk}}[0, 1]$. If the infinite series in (3.4) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{mn} \psi_{mn}(x) = C^T \Psi(t), \quad (3.5)$$

where C and $\Psi(t)$ are $\hat{m} = 2^k M$ column vectors given by

$$C = [c_{00}, \dots, c_{0(M-1)} | c_{10}, \dots, c_{1(M-1)} |, \dots, | c_{(2^k-1)0}, \dots, c_{(2^k-1)(M-1)}]^T, \quad (3.6)$$

$$\Psi(x) = \begin{bmatrix} \psi_{00}(t), \dots, \psi_{0(M-1)}(t) | \psi_{10}(t), \dots, \psi_{1(M-1)}(t) |, \dots, | \psi_{(2^k-1)0}(t), \\ \dots, \psi_{(2^k-1)(M-1)}(t) \end{bmatrix}^T.$$

By changing indices in the vectors $\Psi(t)$ and C the series (3.5) can be rewritten as

$$f(t) \simeq \sum_{i=1}^{\hat{m}} c_i \psi_i(t) = C^T \Psi(t), \quad (3.7)$$

where

$$C = [c_1, c_2, \dots, c_{\hat{m}}], \quad \Psi(x) = [\psi_1(x), \psi_2(x), \dots, \psi_{\hat{m}}(x)], \quad (3.8)$$

and

$$c_i = c_{nm}, \quad \psi_i(t) = \psi_{nm}(t), \quad i = (n-1)M + m + 1. \quad (3.9)$$

Similarly, a two dimensional function $k(s, t) \in L^2([0, 1] \times [0, 1])$ can be expanded into second kind Chebyshev wavelets basis as

$$k(s, t) \approx \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} k_{ij} \psi_i(s) \psi_j(t) = \Psi^T(s) K \Psi(t), \quad (3.10)$$

where $K = [k_{ij}]$ and $k_{ij} = \left(\psi_i(s), (\psi_j(t))_{w_{nk}} \right)_{w_{nk}}$.

3.1. Second kind Chebyshev wavelets and BPFs

In this section we will review the relation between the second kind Chebyshev wavelets and BPFs. It is worth mention that here we set $T = 1$ in definition of BPFs.

Theorem 3.1. *Let $\Psi(t)$ and $\Phi(t)$ be the \hat{m} -dimensional second kind Chebyshev wavelets and BPFs vector respectively, the vector $\Psi(t)$ can be expanded by BPFs vector $\Phi(t)$ as*

$$\Psi(t) \simeq Q\Phi(t), \quad (3.11)$$

where Q is an $\hat{m} \times \hat{m}$ block matrix and

$$Q_{ij} = \psi_i \left(\frac{2j-1}{2\hat{m}} \right), \quad i, j = 1, 2, \dots, \hat{m} \quad (3.12)$$

Proof: Let $\psi_i(t), i = 1, 2, \dots, \hat{m}$ be the i -th element of second kind Chebyshev wavelets vector. Expanding $\psi_i(t)$ into an \hat{m} -term vector of BPFs, we have

$$\psi_i(t) \simeq \sum_{k=1}^{\hat{m}} Q_{ik} b_k(t), \quad i = 1, 2, \dots, \hat{m}, \quad (3.13)$$

taking the collocation points $\eta_j = \frac{2j-1}{2\hat{m}}$ and evaluating relation (3.13) we get

$$\psi_i(\eta_j) \simeq \sum_{k=1}^{\hat{m}} Q_{ik} b_k(\eta_j) = Q_{ij}, \quad i, j = 1, 2, \dots, \hat{m}, \quad (3.14)$$

and this prove the desired result. \square

The following remarks are consequence of relations (2.16), (2.17) and Theorem 3.1.

Remark 3.2. *For an \hat{m} -vector F we have*

$$\Psi(t)\Psi^T(t)F = \tilde{F}\Psi(t), \quad (3.15)$$

in which \tilde{F} is an $\hat{m} \times \hat{m}$ matrix as

$$\tilde{F} = Q\bar{F}Q^{-1}, \quad (3.16)$$

where $\bar{F} = \text{diag}(Q^T F)$.

Remark 3.3. *Let A be an arbitrary $\hat{m} \times \hat{m}$ matrix, then for the second kind Chebyshev wavelets vector $\Psi(t)$ we have*

$$\Psi^T(t)A\Psi(t) = \hat{A}^T\Psi(t), \quad (3.17)$$

where $\hat{A}^T = UQ^{-1}$ and U is an \hat{m} -vector that its elements are diagonal entries of matrix $Q^T A Q$.

4. Operational matrices for second kind Chebyshev wavelets

In this section some operational matrices for the second kind Chebyshev wavelets vector $\Psi(t)$ are derived. Next theorems provide general procedures for forming these matrices. First, we remind some useful results for BPFs [11].

Lemma 4.1. [11] *Let $\Phi(t)$ be the \hat{m} -dimensional BPFs vector defined in (2.10), then integration of this vector can be derived as*

$$\int_0^t \Phi(s) ds \simeq P\Phi(t), \quad (4.1)$$

where P is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2} \begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{\hat{m} \times \hat{m}}. \quad (4.2)$$

Lemma 4.2. [45] *Let $\Phi(t)$ be the \hat{m} -dimensional BPFs vector defined in (2.10), then integration of this vector can be derived as*

$$J^\alpha \Phi(t) = P^\alpha \Phi(t) \quad (4.3)$$

where P^α is called the operational matrix of integration for BPFs and is given by

$$P^\alpha = \frac{h^\alpha}{\Gamma(\alpha + 2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \dots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4.4)$$

Lemma 4.3. [11] *Let $\Phi(t)$ be the \hat{m} -dimensional BPFs vector defined in (2.10), the Itô integral of this vector can be derived as*

$$\int_0^t \Phi(s) dB(s) \simeq P_s \Phi(t), \quad (4.5)$$

where P_s is called the stochastic operational matrix of BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \dots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B\left(\frac{(2\hat{m}-1)h}{2}\right) - B((\hat{m}-1)h) \end{bmatrix}_{\hat{m} \times \hat{m}} \quad (4.6)$$

Now we are ready to derive operational matrices of stochastic and fractional integration for the second kind Chebyshev wavelets.

Theorem 4.4. Suppose $\Psi(t)$ be the \hat{m} -dimensional second kind Chebyshev wavelets vector defined in (3.8), the integral of this vector can be derived as

$$\int_0^t \Psi(s)ds \simeq QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \quad (4.7)$$

where Q is introduced in (3.11) and P is the operational matrix of integration for BPFs derived in (4.2).

Proof: Let $\Psi(t)$ be the second kind Chebyshev wavelets vector, by using Theorem 3.1 and Lemma 4.2 we have

$$\int_0^t \Psi(s)ds \simeq \int_0^t Q\Phi(s)ds = Q \int_0^t \Phi(s)ds = QP\Phi(t), \quad (4.8)$$

now Theorem 3.1 gives

$$\int_0^t \Psi(s)ds \simeq QP\Phi(t) = QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \quad (4.9)$$

by using this identity we obtain the desired result. \square

Theorem 4.5. Let $\Psi(t)$ be the \hat{m} -dimensional second kind Chebyshev wavelets vector defined in (3.8), the operational matrix of the fractional order integration for $\Psi(t)$ can be derived as

$$J^\alpha \Psi(t) = QP^\alpha Q^{-1}\Psi(t) = \Lambda^\alpha \Psi(t), \quad (4.10)$$

where Λ^α is called the operational matrix of second kind Chebyshev wavelets, Q is the matrix introduced in (3.11) and F^α is the operational matrix of fractional integration for BPFs derived in (4.4).

Proof: By using Theorem 3.1 we have

$$J^\alpha \Psi(t) = J^\alpha Q\Phi(t) = QF^\alpha \Phi(t) = QF^\alpha Q^{-1}\Psi(t) = P^\alpha \Psi(t), \quad (4.11)$$

so, the second kind Chebyshev wavelet operational matrix of the fractional order integration P^α is given by

$$P^\alpha = QF^\alpha Q^{-1}. \quad (4.12)$$

and this completes the proof. \square

Theorem 4.6. Suppose $\Psi(t)$ be the \hat{m} -dimensional second kind Chebyshev wavelets vector defined in (3.8), the Itô integral of this vector can be derived as

$$\int_0^t \Psi(s)dB(s) \simeq QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \quad (4.13)$$

where Λ_s is called stochastic operational matrix for second kind Chebyshev wavelets, Q is introduced in (3.11) and P_s is the stochastic operational matrix of integration for BPFs derived in (4.6).

Proof: Let $\Psi(t)$ be the second kind Chebyshev wavelets vector, by using Theorem 3.1 and Lemma 4.3 we have

$$\int_0^t \Psi(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q \int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \quad (4.14)$$

now Theorem 3.1 result

$$\int_0^t \Psi(s)dB(s) = QP_s\Phi(t) = QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \quad (4.15)$$

and this complete the proof. \square

5. Description of the numerical method

Here we present a wavelet Galerkin method based on the second kind Chebyshev wavelets and their operational matrices for solving SFDEs (1.1). For this purpose, and by using the relation of the fractional derivative and integral, the solution $u(t)$ can be derived as

$$\begin{aligned} u(t) &= \sum_{k=0}^{n-1} u^{(k)}(0^+) + J^\alpha f(t) + J^\alpha \left(\int_0^t u(s)k_1(s,t)ds \right) \\ &\quad + J^\alpha \left(\int_0^t u(s)k_2(s,t)dB(s) \right), \end{aligned} \quad (5.1)$$

now functions $u(t)$, $f(t)$ and $k_i(s,t)$, $i = 1, 2$, can be expanded in term of the second kind Chebyshev wavelets as

$$f(t) \simeq F^T\Psi(t) = \Psi^T(t)F, \quad (5.2)$$

$$u(t) \simeq C^T\Psi(t) = \Psi^T(t)C, \quad (5.3)$$

$$k_i(s,t) \simeq \Psi^T(t)K_i\Psi(s) = \Psi^T(s)K_i^T\Psi(t), \quad i = 1, 2, \quad (5.4)$$

where C and F are second kind Chebyshev wavelets coefficients vectors, and K_i , $i = 1, 2$, are second kind Chebyshev wavelets coefficient matrices defined in Eqs. (3.8) and (3.10). Substituting above approximations in Eq. (5.1), we get

$$C^T\Psi(t) = F_0^T\Psi(t) + J^\alpha F^T\Psi(t) + J^\alpha \left(\Psi^T(t)K_1 \int_0^t \Psi(s)\Psi^T(s)Cds \right)$$

$$+J^\alpha \left(\Psi^T(t) K_1 \int_0^t \Psi(s) \Psi^T(s) C dB(s) \right),$$

now remarks 3.2 results

$$\begin{aligned} C^T \Psi(t) &= F_0^T \Psi(t) + F^T \Lambda^\alpha \Psi(t) + J^\alpha \left(H^T(t) K_1 \int_0^t \tilde{C} \Psi(s) ds \right) \\ &\quad + J^\alpha \left(\Psi^T(t) K_2 \int_0^t \tilde{C} \Psi(s) dB(s) \right) \\ &= F_0^T \Psi(t) + F^T \Lambda^\alpha \Psi(t) + J^\alpha \left(\Psi^T(t) K_1 \tilde{C} \Lambda \Psi(t) \right) + J^\alpha \left(\Psi^T(t) K_2 \tilde{C} \Lambda_s \Psi(t) \right) \\ &= F_0^T \Psi(t) + F^T \Lambda^\alpha \Psi(t) + J^\alpha (C_1^T \Psi(t)) + J^\alpha (C_2^T \Psi(t)) \\ &= F_0^T \Psi(t) + F^T \Lambda^\alpha \Psi(t) + C_1^T \Lambda^\alpha \Psi(t) + C_2^T \Lambda^\alpha \Psi(t), \end{aligned}$$

where $\tilde{C} = \text{diag}(C)$ is a $\hat{m} \times \hat{m}$ matrix, $C_1 = \text{diag}(K_1 \tilde{C} \Lambda)$ and $C_2 = \text{diag}(K_2 \tilde{C} \Lambda_s)$ are \hat{m} -vectors. As this equation is hold for all $t \in [0, 1)$, the standard Galerkin method results

$$C^T = F_0^T + F^T \Lambda^\alpha + C_1^T \Lambda^\alpha + C_2^T \Lambda^\alpha. \quad (5.5)$$

The vectors C_1 and C_2 are linear functions of vector C , so Eq. (5.5) is a linear system of algebraic equations for unknown vector C . Solving this linear system we obtain vector C , which can be used to approximate solution of SFDE (1.1) by substituting in Eq. (5.3).

6. Convergence analysis

The aim of this section is to analyze the proposed the second kind Chebyshev wavelets numerical scheme for solving SFDEs.

Theorem 6.1. Suppose $f(x) \in L_{w_n}^2[0, 1]$ with bounded second derivative, say $|f''(x)| \leq L$, and let $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x)$ be its infinite second kind Chebyshev wavelets expansion, then

$$|c_{mn}| \leq \frac{L\sqrt{\pi}}{2^3 (n+1)^{\frac{5}{2}} (m^2 + 2m - 3)}, \quad (6.1)$$

this means the second kind Chebyshev wavelets series converges uniformly to $f(x)$ and

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \quad (6.2)$$

Proof: From definition of coefficient c_{mn} we have

$$c_{mn} = \int_0^1 f(t) \psi_{mn}(t) w_{nk}(t) dt \quad (6.3)$$

$$= \frac{2^{\frac{k}{2}+1}}{\sqrt{\pi}} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t) U_m(2^{k+1}t - 2n - 1) w(2^{k+1}t - 2n - 1) dt, \quad (6.4)$$

by substituting $2^{k+1}t - 2n - 1 = \cos(\theta)$ in (6.4) we get

$$c_{mn} = \frac{2^{-\frac{k}{2}}}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos(\theta) + 2n + 1}{2^{k+1}}\right) \sin((m+1)\theta) \sin(\theta) d\theta \quad (6.5)$$

$$= \frac{2^{-\frac{k}{2}-1}}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos(\theta) + 2n + 1}{2^{k+1}}\right) [\cos(m\theta) - \cos((m+2)\theta)] d\theta, \quad (6.6)$$

using integration by part two times we obtain

$$c_{mn} = \frac{1}{2^{\frac{5k}{2}+3}\sqrt{\pi}} \int_0^\pi f''\left(\frac{\cos(\theta) + 2n + 1}{2^{k+1}}\right) h_m(\theta) d\theta, \quad (6.7)$$

where

$$\begin{aligned} h_m(\theta) &= \frac{\sin(\theta)}{m} \left(\frac{\sin((m-1)\theta)}{m-1} - \frac{\sin((m+1)\theta)}{m+1} \right) \\ &\quad - \frac{\sin(\theta)}{m+2} \left(\frac{\sin((m+1)\theta)}{m+1} - \frac{\sin((m+3)\theta)}{m+3} \right), \end{aligned}$$

so, we have

$$|c_{mn}| \leq \frac{L}{2^{\frac{5k}{2}+3}\sqrt{\pi}} \int_0^\pi |h_m(\theta)| d\theta \leq \frac{L\sqrt{\pi}}{2^{\frac{5k}{2}+3}(m^2 + 2m - 3)}, \quad (6.8)$$

since $n \leq 2^k - 1$, we obtain

$$|c_{mn}| \leq \frac{L\sqrt{\pi}}{2^{\frac{5k}{2}+3}(m^2 + 2m - 3)} \leq \frac{L\sqrt{\pi}}{2^3(n+1)^{\frac{5}{2}}(m^2 + 2m - 3)}. \quad (6.9)$$

□

Theorem 6.2. Let $f(x)$ be a continuous function defined on $[0, 1)$, with second derivatives $f''(x)$ bounded by L , then we have the following accuracy estimation

$$\begin{aligned} \sigma_{M,k} &\leq \left(\frac{\pi L^2}{2^6} \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} \frac{1}{(n+1)^5 (m^2 + 2m - 3)^2} \right. \\ &\quad \left. + \frac{\pi L^2}{2^6} \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} \frac{1}{(n+1)^5 (m^2 + 2m - 3)^2} \right)^{\frac{1}{2}}, \end{aligned}$$

where

$$\sigma_{M,k} = \left(\int_0^1 \left(f(x) - \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \right)^2 dx \right)^{\frac{1}{2}}.$$

Proof: We have

$$\begin{aligned} \sigma_{M,k}^2 &= \int_0^1 \left(f(x) - \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \right)^2 dx \\ &= \int_0^1 \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x) - \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \right)^2 dx \\ &= \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 \int_0^1 \psi_{nm}^2(x) dx + \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} c_{nm}^2 \int_0^1 \psi_{nm}^2(x) dx \\ &= \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} c_{nm}^2, \end{aligned}$$

now by considering the relation (6.1) we achieve the desired result. \square

Now we state the main result of this section which investigate the convergency of the proposed method for the approximate solution of SFDE (1.1). Hereafter $e_n(t)$ is error function of the second kind Chebyshev wavelets approximate solution $u_n(t)$ and $\|\cdot\|$ denotes L^2 norm in $[0, 1]$ defined by

$$\|u(t)\| = \left(\int_0^1 |u(t)|^2 dt \right)^{\frac{1}{2}}.$$

Theorem 6.3. Suppose $u(t)$ is the exact solution of (1.1) and $u_n(t), k_{1n}(s, t), k_{2n}(s, t)$ are the second kind Chebyshev wavelets approximate solution for $u(t), k_1(s, t), k_2(s, t)$ respectively. Also assume that

$$a) \quad \|u(t)\| \leq \rho, \quad t \in [0, 1], \quad (6.10)$$

$$b) \quad \|k_i(s, t)\| \leq M_i, \quad (s, t) \in [0, 1] \times [0, 1], \quad i = 1, 2, \quad (6.11)$$

$$b) \quad M_B = \sup |B(t)|, \quad t \in [0, 1], \quad (6.12)$$

then $\lim_{n \rightarrow \infty} \|e_n(t)\| = 0$.

Proof: Let $e_n(t) = u(t) - u_n(t)$, from (5.1) we get

$$\begin{aligned} e_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{f(\tau) - f_n(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \frac{k_1(s, \tau) u(s) - k_{1n}(s, \tau) u_n(s)}{(t-\tau)^{1-\alpha}} ds d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \frac{k_2(s, \tau) u(s) - k_{2n}(s, \tau) u_n(s)}{(t-\tau)^{1-\alpha}} dB(s) d\tau, \end{aligned} \quad (6.13)$$

consequently, we can write

$$\begin{aligned} \|e_n(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau \left\| \frac{f(\tau) - f_n(\tau)}{(t-\tau)^{1-\alpha}} \right\| d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{k_1(s, \tau) u(s) - k_{1n}(s, \tau) u_n(s)}{(t-\tau)^{1-\alpha}} \right\| ds d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{k_2(s, \tau) u(s) - k_{2n}(s, \tau) u_n(s)}{(t-\tau)^{1-\alpha}} \right\| ds d\tau. \end{aligned} \quad (6.14)$$

So we get

$$\begin{aligned} \|e_n(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau \left\| \frac{f(\tau) - f_n(\tau)}{(t-\tau)^{1-\alpha}} \right\| d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{k_1(s, \tau) (u(s) - u_n(s))}{(t-\tau)^{1-\alpha}} \right\| ds d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{(k_1(s, \tau) - k_{1n}(s, \tau)) (u(s) - u_n(s))}{(t-\tau)^{1-\alpha}} \right\| ds d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{(k_1(s, \tau) - k_{1n}(s, \tau)) u(s)}{(t-\tau)^{1-\alpha}} \right\| ds d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{k_2(s, \tau) (u(s) - u_n(s))}{(t-\tau)^{1-\alpha}} \right\| dB(s) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{(k_2(s, \tau) - k_{2n}(s, \tau)) (u(s) - u_n(s))}{(t-\tau)^{1-\alpha}} \right\| dB(s) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\tau \left\| \frac{(k_2(s, \tau) - k_{2n}(s, \tau)) u(s)}{(t-\tau)^{1-\alpha}} \right\| dB(s) d\tau. \end{aligned} \quad (6.15)$$

Theorem 6.1 shows that second kind Chebyshev wavelets expansion of any square integrable function converges uniformly. So, for any $\varepsilon > 0$ there exist n such that

$$\begin{aligned} \|f_n(t) - f(t)\| &\leq \varepsilon, \\ \|u_n(t) - u(t)\| &\leq \varepsilon, \\ \|k_{1n}(s, t) - k_1(s, t)\| &\leq \varepsilon, \\ \|k_{2n}(s, t) - k_2(s, t)\| &\leq \varepsilon, \end{aligned} \quad (6.16)$$

therefore from (6.15) we get

$$\begin{aligned}
\|e_n(t)\| &\leq \frac{\varepsilon}{\Gamma(\alpha+1)} + \frac{M_1 \|e_n(t)\|}{\Gamma(\alpha+3)} + \frac{\varepsilon \|e_n(t)\|}{\Gamma(\alpha+3)} + \frac{\varepsilon \rho}{\Gamma(\alpha+3)} \\
&\quad + \frac{M_B M_2 \|e_n(t)\|}{\Gamma(\alpha+2)} + \frac{M_B \varepsilon \|e_n(t)\|}{\Gamma(\alpha+2)} + \frac{M_B \varepsilon \rho}{\Gamma(\alpha+2)} \\
&= \frac{\Gamma(\alpha+2)(M_1 + \varepsilon) + \Gamma(\alpha+3)M_B(M_2 + \varepsilon)}{\Gamma(\alpha+3)\Gamma(\alpha+2)} \|e_n(t)\| \\
&\quad + \frac{\varepsilon \Gamma(\alpha+2)\Gamma(\alpha+3) + \varepsilon \rho \Gamma(\alpha+1)\Gamma(\alpha+3) + \varepsilon \rho M_B \Gamma(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha+3)},
\end{aligned}$$

or

$$\|e_n(t)\| \leq \frac{\varepsilon \Gamma(\alpha+2)\Gamma(\alpha+3) + \varepsilon \rho \Gamma(\alpha+1)\Gamma(\alpha+3) + \varepsilon \rho M_B \Gamma(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha+1)[\Gamma(\alpha+3)\Gamma(\alpha+2) - \Gamma(\alpha+2)(M_1 + \varepsilon) + \Gamma(\alpha+3)M_B(M_2 + \varepsilon)]},$$

and the proof is complete. \square

7. Numerical results

In this section, we implement the proposed algorithm in section 5 for solving SFDEs. In all examples the algorithms are performed by Maple 17 with 20 digits precision.

Example 7.1. Consider the following SFDE

$$D^\alpha u(t) = \frac{\Gamma(2)t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{t^3}{3} + \int_0^t su(s)ds + \int_0^t u(s)dB(s), \quad s, t \in [0, 1],$$

subject to the initial condition $u(0) = 0$. The exact solution of this SFDE is unknown. Here we use the wavelet Galerkin method proposed in section 5 to solve it. Table 1 lists the approximate solution for different values of t and α with $\hat{m} = 128$. Moreover, Fig. 1 shows the approximate solutions obtained for different values of α .

Table 1: Numerical results for different values of t , α and $\hat{m} = 128$.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.1	0.0927744211	0.0922521030	0.0917906320
0.3	0.2721378110	0.2863044024	0.2928005524
0.5	0.7682012950	0.8361875094	0.8953568153
0.7	0.6550951414	0.6509386054	0.6556671400
0.9	0.9781496416	0.9214944630	0.8970816620

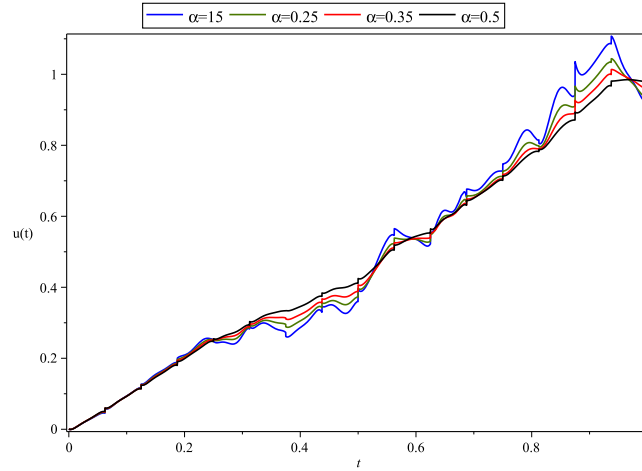


Figure 1: The approximate solution for $\alpha = 0.25, \alpha = 0.5$ and $\alpha = 0.75$.

Example 7.2. As the second example consider the following SFDE

$$D^\alpha u(t) = \frac{7}{12}t^4 - \frac{5}{6}t^3 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^t (s+t)u(s)ds + \int_0^t su(s)dB(s),$$

$s, t \in [0, 1],$

subject to the initial condition $u(0) = 0$. The exact solution of this SFDE is unknown. The approximate solution obtained by the proposed method for various values of t and α are listed in Table 2. Fig. 2 plots the approximate solution for different values of α with $\hat{m} = 128$.

Table 2: Numerical results for different values of t and α .

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.1	0.0848476129	0.0845854008	0.0836981075
0.3	0.2030817352	0.2061982209	0.2070198626
0.5	0.4178064937	0.4486021937	0.4686001068
0.7	0.1671430740	0.1804842703	0.1890702090
0.9	0.0289527458	0.0480352224	0.0592956315

Example 7.3. Consider the following SFDE

$$D^\alpha u(t) = \frac{\Gamma(3)t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^4 \sin(t)}{4} + \int_0^t \sin(t)s^2u(s)ds + \int_0^t se^t u(s)dB(s), \quad s, t \in [0, 1],$$

subject to the initial condition $u(0) = 0$. The exact solution of this SFDE is unknown. Here the proposed wavelet Galerkin method is used for deriving numerical

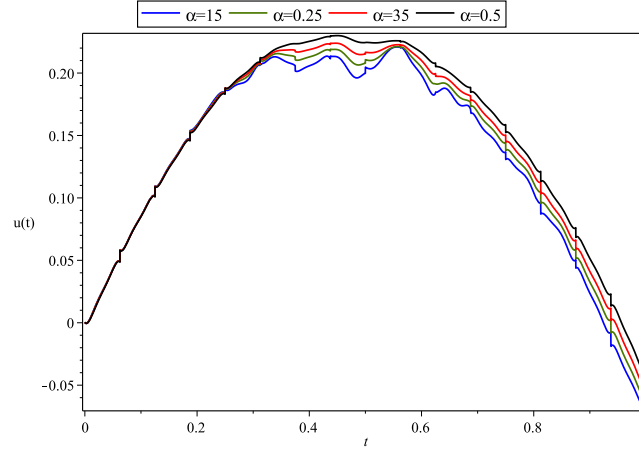


Figure 2: The approximate solution for $\alpha = 0.25, \alpha = 0.5$ and $\alpha = 0.75$.

solution of it. The approximate solution for different values of α and t with $\hat{m} = 128$ is listed in Table 3. Moreover, Fig. 3 shows the approximate solutions for different values of α and $\hat{m} = 128$.

Table 3: Numerical results for different values of t and α .

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.1	0.0088179999	0.0088039620	0.0087980641
0.3	0.0851554626	0.0867376737	0.0875707189
0.5	0.4390176500	0.4566864654	0.4713103382
0.7	0.4950304749	0.4876362989	0.4832995027
0.9	0.9761532307	0.8973556536	0.8564481929

8. Conclusion

Many phenomena in science that have been modeled by fractional differential equations have some uncertainty, so for deriving a more accurate solution, we have to solve a SFDEs. In this paper, we proposed a Galerkin scheme based on the second kind Chebyshev wavelets for solving SFDEs. In this scheme, we used the operational matrices of fractional and stochastic integration for the second kind Chebyshev wavelets. The main advantage of this method is to reduce the SFDEs into a problem consisting of a system of algebraic equations. The reduction is based on the operational matrices and the Galerkin method. The efficiency and applicability of the suggested scheme is confirmed on some examples.

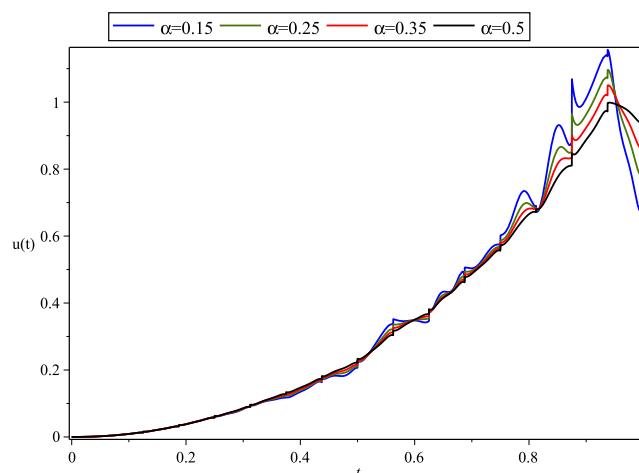


Figure 3: The approximate solution for $\alpha = 0.25, \alpha = 0.5$ and $\alpha = 0.75$.

References

1. P. E. Kloeden, E. Platen, Numerical solution of stochastic differential equations, Springer, 1992.
2. B. Oksendal, Stochastic Differential Equations: An Introduction with Applications, fifth ed., Springer-Verlag, New York, 1998.
3. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM review. 43 (3) (2001) 525-546.
4. M. Khodabin, K. Maleknejad, M. Rostami and Mostafa Nouri. Numerical solution of stochastic differential equations by second order Runge-Kutta methods. Mathematical and Computer Modelling 53 (9) (2011) 1910-1920.
5. A. Foroush Bastani, S. M. Hosseini. A new adaptive Runge-Kutta method for stochastic differential equations. Journal of computational and applied mathematics, 206 (2) (2007) 631-644.
6. A. Tocino, R. Ardanuy. Runge-Kutta methods for numerical solution of stochastic differential equations. Journal of Computational and Applied Mathematics 138 (2) (2002) 219-241.
7. K. Burrage, P. M. Burrage, High strong order explicit Runge-Kutta methods for stochastic ordinary differential equations. Applied Numerical Mathematics 22 (1) (1996) 81-101.
8. I. Babuska, R. Tempone, G. E. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. SIAM Journal on Numerical Analysis 42 (2) (2004) 800-825.
9. M. Deb, K. Manas, I. M. Babuska, J. T. Oden. Solution of stochastic partial differential equations using Galerkin finite element techniques. Computer Methods in Applied Mechanics and Engineering, 190 (48) (2001) 6359-6372.
10. F. Mohammadi, A wavelet-based computational method for solving stochastic Itô-Volterra integral equations, J. Comp. Phys. 298 (2015) 254-265.
11. K. Maleknejad, M. Khodabin, M. Rostami, Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions, Math. Comput. Model. (55) (2012) 791-800.

12. M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek, C. Cattani, A computational method for solving stochastic Itô-Volterra integral equations based on stochastic operational matrix for generalized hat basis functions, *Journal of Computational Physics*, 270 (2014) 402-415.
13. M. Kamrani, Numerical solution of stochastic fractional differential equations. *Numerical Algorithms* (2014) 1-13.
14. J. He, Nonlinear oscillation with fractional derivative and its applications, *International conference on vibrating engineering*. 98 (1998) 288-291.
15. J. He, Some applications of nonlinear fractional differential equations and their approximations, *Bull. Sci. Technol.*, 15 (2) (1999) 86-90.
16. F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics A. Carpinteri, F. Mainardi (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien (1997), 291-348.
17. R. Panda and M. Dash, Fractional generalized splines and signal processing, *Signal Process*, 86 (2006) 2340-2350.
18. T. Chow, Fractional dynamics of interfaces between soft-nanoparticles and rough substrates, *Phys. Lett. A*, 342 (2005) 148-155.
19. L. Gaul, P. Klein, S. Kemple, Damping description involving fractional operators, *Mech. Syst. Signal. Process.* 5 (1991) 81-88.
20. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, New York, 1998.
21. A. Arikoglu, I. Ozkol, Solution of fractional integro-differential equations by using fractional differential transform method, *Chaos Solitons Fractals* 40 (2009) 521-529.
22. M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.* 56 (2006) 80-90.
23. M. P. Tripathi, VK Baranwal, RK Pandey, O. P. Singh, A new numerical algorithm to solve fractional differential equations based on operational matrix of generalized hat functions. *Commun. Nonlinear Sci. Numer. Simul.* 18 (6) (2013) 1327-1340.
24. A. Saadatmandi M. Dehghan. A new operational matrix for solving fractional-order differential equations. *Computers and mathematics with applications*. 59 (3) (2010) 1326-1336.
25. A. H. Bhrawy, A. S. Alofi. The operational matrix of fractional integration for shifted Chebyshev polynomials. *Applied Mathematics Letters* 26 (1) (2013) 25-31.
26. Y. Li, N. Sun. Numerical solution of fractional differential equations using the generalized block pulse operational matrix. *Computers and Mathematics with Applications*. 62 (3) (2011) 1046-1054.
27. Y. Li, Solving a nonlinear fractional differential equation using Chebyshev wavelets, *Commun. Nonlinear Sci. Numer. Simul.* 15 (2010) 2284-2292.
28. L. Zhu, Q. Fan. Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. *Commun. Nonlinear Sci. Numer. Simul.* 17 (6) (2012) 2333-2341.
29. S. Momani, K. Al-Khaled, Numerical solutions for systems of fractional differential equations by the decomposition method, *Applied Mathematics and Computation*, 162 (3), (2005) 1351-1365.
30. Z. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, *Int. J. Nonlinear Sci. Numer. Simul.* 7 (2006) 27-34.
31. I. Hashim, O. Abdulaziz, S. Momani, Homotopy analysis method for fractional IVPs, *Commun. Nonlinear Sci. Numer. Simul.* 14 (3) (2009) 674-684.
32. S. Mallat, *A wavelet tour of signal processing*, 2nd ed., Academic Press, 1999.

33. Y. Wang, Q. Fan, The second kind Chebyshev wavelet method for solving fractional differential equations. *Applied Mathematics and Computation* 218 (17) (2012) 8592-8601.
34. M. Razzaghi, S. Yousefi, The Legendre wavelets operational matrix of integration, *International Journal of Systems Science* 32.4 (2001): 495-502.
35. F. Mohammadi, M. M. Hosseini, and Syed Tauseef Mohyud-Din. Legendre wavelet Galerkin method for solving ordinary differential equations with non-analytic solution, *International Journal of Systems Science* 42 (4) (2011) 579-585.
36. F. Mohammadi, Haar Wavelets Approach For Solving Multidimensional Stochastic Itô-Volterra Integral Equations, *Applied Mathematics E-Notes* 15 (2015) 80-96.
37. M. H. Heydari, M. R. Hooshmandasl, G. Barid Loghmani, C. Cattani, Wavelets Galerkin method for solving stochastic heat equation, *International Journal of Computer Mathematics* (2015) 1-18.
38. A. Abdulle, G.A. Pavliotis, Numerical methods for stochastic partial differential equations with multiple scales, *J. Comp. Phys.* 231 (2012) 2482-2497.
39. E. Weinan, D. Liu and E. Vanden-Eijnden, Analysis of multiscale methods for stochastic differential equations, *Commun. Pure Appl. Math.* 58 (11) (2005) 1544-1585.
40. M. Enelund, B. L. Josefson, Time-domain finite element analysis of viscoelastic structures with fractional derivatives constitutive relations. *AIAA J.* 35 (10) (1997) 1630-1637.
41. C. Friedrich, Linear viscoelastic behavior of branched polybutadiens: a fractional calculus approach. *Acta Polym.* (1995) 385-390.
42. T. E. Govindan, M. C. Joshi, Stability and optimal control of stochastic functional-differential equations with memory. *Numer. Funct. Anal. Optim.* 13 (3-4) (1992) 249-265.
43. D. Keck, M. A. McKibben. Functional integro-differential stochastic evolution equations in Hilbert space. *International Journal of Stochastic Analysis*, 16 (2) (2003) 141-161.
44. J. C. Mason, D. C. Handscomb, Chebyshev polynomials, CRC Press, 2010.
45. A. Kilicman and Z. A. Zhour, Kronecker operational matrices for fractional calculus and some applications, *Appl. Math. Comput.*, 187(1) (2007), 250-65.

Fakhroddin Mohammadi
Department of Mathematics, Hormozgan University,
P. O. Box 3995, Bandarabbas, Iran
E-mail address: f.mohammadi62@hotmail.com