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Moment Asymptotic Expansions of the Wavelet Transforms*

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ABSTRACT: Using distribution theory we present the moment asymptotic expansion of continuous wavelet transform in different distribution spaces for large and small values of dilation parameter a. We also obtain asymptotic expansions for certain wavelet transform.

Key Words: Asymptotic expansion, Wavelet transform,Distribution.

Contents

1. Introduction

In past few decades there were many mathematicians who have done great work in the field of asymptotic expansion like Wong [6] .Using Mellin transform technique of Wong and asymptotic expansions of the Fourier transform of the function and the wavelet Pathak [3] and Pathak & Pathak [3,4] have obtained the asymptotic expansions of the continuous wavelet transform for large and small values of dilation and translation parameters. Estrada & Kanwal [5] have obtained the asymptotic expansions of generalized functions on different spaces of test functions. In present paper we exploit the distributional theory of asymptotic expansions due to Estrada & Kanwal and obtain asymptotic expansions of wavelet transform when asymptotic

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behaviour of the function and the wavelet are known.

The continuous wavelet transform of f with respect to wavelet ψ is defined by

$$(W_\psi f)(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx, \;\; b \in \mathbb{R}, a > 0, \tag{1.1}$$

provided the integral exists [3]. We can also write

$$\begin{aligned}(W_\psi f)(a,b) &= \sqrt{a} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(x - \frac{b}{a}\right)} dx \\ &= \sqrt{a} \left\langle f(ax), \psi\left(x - \frac{b}{a}\right) \right\rangle. \end{aligned} \tag{1.2}$$

This paper is arranged in following manner. In section second, third , fourth and fifth we drive the asymptotic expansion in the distributional spaces $\mathscr{E}'(\mathbb{R})$, $\mathscr{P}'(\mathbb{R})$,$\mathcal{O}'_\gamma(\mathbb{R})$,$\mathcal{O}'_c(\mathbb{R})$ and $\mathcal{O}'_M(\mathbb{R})$ respectively, studied in [5].

2. The moment asymptotic expansion of $(W_\psi f)(a,b)$ as $a \to \infty$ in the space $\mathscr{E}'(\mathbb{R})$ for given b

The space $\mathscr{E}(\mathbb{R})$ is the space of all smooth functions on $\mathbb{R}$ and it's dual space $\mathscr{E}'(\mathbb{R})$, is the space of distributions with compact support. If $\psi \in \mathscr{E}(\mathbb{R})$, then clearly $\psi\left(x - \frac{b}{a}\right) \in \mathscr{E}(\mathbb{R})$. We consider the seminorms in the following two cases.

Case 1 For $b \geq 0$:

$$\left\| \psi\left(x - \frac{b}{a}\right) \right\|_{\alpha,M} = Max \left\{ \left| D^\alpha \psi\left(x - \frac{b}{a}\right) \right| : \frac{b}{a} - M < x < b + M \right\}, \tag{2.1}$$

Case 2 For $b < 0$:

$$\left\| \psi\left(x - \frac{b}{a}\right) \right\|_{\alpha,M} = Max \left\{ \left| D^\alpha \psi\left(x - \frac{b}{a}\right) \right| : b - M < x < \frac{b}{a} + M \right\}, \tag{2.2}$$

for $\alpha \in \mathbb{N}$ and $M > 0$. These seminorm generate the topology of $\mathscr{E}(\mathbb{R})$. If $q = 0, 1, 2, 3, ...,$ we set

$$X_q = \{\psi \in \mathscr{E}(\mathbb{R}) : D^\alpha \psi(0) = 0 \; for \; \alpha < q\}, \tag{2.3}$$

Lemma 2.1. *Let $\psi \in X_q$, then for every $\alpha \in \mathbb{N}$ and $M > 0$,*

$$\left\| \psi\left(\frac{x-b}{a}\right) \right\|_{\alpha,M} = O\left(\frac{1}{a^q}\right) \quad as \;\; a \to \infty \tag{2.4}$$

Proof: *Assume that $b \geq 0$. For $\psi \in X_q$ we can find a constant K such that*

$$\left| \psi\left(x - \frac{b}{a}\right) \right| \leq K \left| x - \frac{b}{a} \right|^q, \frac{b}{a} - 1 < x < b + 1. \tag{2.5}$$

Therefore, if $a > M$ we obtain

$$\begin{aligned}\left\|\psi\left(\frac{x-b}{a}\right)\right\|_{0,M} &= Max\left\{\left|\psi\left(\frac{x-b}{a}\right)\right| : \frac{b}{a}-M < x < b+M\right\} \\ &\leq O\left(\frac{M}{a^q}\right). \end{aligned} \tag{2.6}$$

If $\alpha \leq q$ and $\psi \in X_q$ then $D^\alpha\psi \in X_{q-\alpha}$ and thus

$$\begin{aligned}\left\|\psi\left(\frac{x-b}{a}\right)\right\|_{\alpha,M} &= \left\|a^\alpha D^\alpha\psi\left(\frac{x-b}{a}\right)\right\|_{0,M} \\ &= \frac{1}{a^\alpha}O\left(\frac{1}{a^{q-\alpha}}\right) \\ &= O\left(\frac{1}{a^q}\right).\end{aligned}$$

Similarly by using (2.2) we can prove that

$$\left\|\psi\left(\frac{x-b}{a}\right)\right\|_{\alpha,M} = O\left(\frac{1}{a^q}\right) \quad for \ \ b<0.$$

□

Now, by using Lemma 2.1 we obtain the following theorem.

Theorem 2.1. *Let wavelet $\psi \in \mathscr{E}(\mathbb{R}), f \in \mathscr{E}'(\mathbb{R})$ and $\mu_\alpha = \langle f, x^\alpha\rangle$ be its moment sequence . Then for a fixed b the moment asymptotic expansion of the wavelet transform is*

$$\sqrt{a}\left\langle f(ax), \psi\left(x-\frac{b}{a}\right)\right\rangle = \sum_{\alpha=0}^{N}\frac{\mu_\alpha D^\alpha\psi(-b/a)}{\alpha!\ a^{\alpha+1/2}} + O\left(\frac{1}{a^{N+1/2}}\right) \quad as \ \ a\to\infty. \tag{2.7}$$

Proof: Let $P_N(x,b/a) = \sum_{\alpha=0}^{N}\frac{D^\alpha\psi(-b/a)}{\alpha!}x^\alpha$ be the polynomial of order N of the function $\psi\left(x-\frac{b}{a}\right)$. Then we have

$$\begin{aligned}\left\langle f(ax), \psi\left(x-\frac{b}{a}\right)\right\rangle &= \left\langle f(ax), P_N(x,b/a)\right\rangle + \left\langle f(ax), \psi\left(x-\frac{b}{a}\right) - P_N(x,b/a)\right\rangle \\ &= \sum_{\alpha=0}^{N}\frac{\mu_\alpha D^\alpha\psi(-b/a)}{\alpha! a^{\alpha+1}} + R_N(a),\end{aligned}$$

where the remainder $R_N(a)$ is given by $R_N(a) = \left\langle f(ax), \psi\left(x - \frac{b}{a}\right) - P_N(x, b/a)\right\rangle$.
Since $\psi\left(x - \frac{b}{a}\right) - P_N(x, b/a)$ is in X_{N+1} we obtain

$$\begin{aligned} |R_N(a)| &= \left|\left\langle f(ax), \psi\left(x - \frac{b}{a}\right) - P_N(x, b/a)\right|\right. \\ &= \frac{L}{a}\sum_{\alpha=0}^{q} \|\psi_N\left(\frac{x-b}{a}\right)\|_{\alpha,M} \\ &= O\left(\frac{1}{a^{N+1}}\right), \end{aligned}$$

where the existence of L, q and M is guaranteed by the continuity of f. Hence we get the required asymptotic expansion (2.7). □

Example 2.1. *In this example we choose ψ to be Mexican-Hat wavelet and derive the asymptotic expansion of Mexican-Hat wavelet transform by using Theorem2.1. The Mexican-Hat wavelet is given by [3]*

$$\psi(x) = (1 - x^2)e^{-\frac{x^2}{2}} \in \mathscr{E}(\mathbb{R}). \tag{2.8}$$

Then

$$P_2(x, b/a) = \frac{e^{-\frac{b^2}{2a^2}}}{a^2}\left((a^2 - b^2) + \frac{b(3a^2 - b^2)}{a}x + \frac{(6a^2b^2 - 3a^4 - b^4)}{2a^2}x^2\right).$$

Now, using Theorem2.1 we get the asymptotic expansion of Mexican-Hat wavelet transform

$$\begin{aligned} \sqrt{a}\left\langle f(ax), \psi\left(x - \frac{b}{a}\right)\right\rangle &= \frac{e^{-\frac{b^2}{2a^2}}}{a^2}\left(\frac{(a^2 - b^2)}{\sqrt{a}}\mu_0 + \frac{b(3a^2 - b^2)}{a^{3/2}}\mu_1\right. \\ &+ \left.\frac{(6a^2b^2 - 3a^4 - b^4)}{2a^{5/2}}\mu_2\right) + O\left(\frac{1}{a^{9/2}}\right) \quad as \ a \to \infty, \end{aligned}$$

where $\mu_i = \langle f, x^i\rangle, i = 0, 1, 2.$

3. The moment asymptotic expansion of $(W_\psi f)(a, b)$ for large and small values of a in the space $\mathscr{P}'(\mathbb{R})$ for a given b

Case 1. Let $\psi \in \mathscr{P}(\mathbb{R})$.
We now consider the moment asymptotic expansion in the space $\mathscr{P}'(\mathbb{R})$ of distributions of "less than exponential growth". The space $\mathscr{P}(\mathbb{R})$ consists of those smooth functions $\phi(x)$ that satisfy

$$lim_{x\to\infty}e^{-\gamma|x|}D^\beta\phi(x) = 0 \ \ for \ \ \gamma > 0 \ and \ each \ \beta \in \mathbb{N},$$

with seminorms

$$\|\phi(x)\|_{\gamma,\beta} = sup \left\{ | \ e^{-\gamma|x|} \ D^{\beta}\phi(x) \ | : x \in \mathbb{R} \right\}.$$

Let wavelet $\psi(x) \in \mathscr{P}(\mathbb{R})$. Then

$$\begin{aligned} \|\psi(x)\|_{\ \gamma,\ \beta,\ \frac{b}{a}} &:= sup\left\{\left|e^{-\gamma|x|}D^{\beta}\psi\left(x-\frac{b}{a}\right)\right| : x \in \mathbb{R}\right\} \\ &= sup\left\{\left|e^{-\gamma|x-\frac{b}{a}|}D^{\beta}\psi\left(x-\frac{b}{a}\right)\frac{e^{-\gamma|x|}}{e^{-\gamma|x-\frac{b}{a}|}}\right| : x \in \mathbb{R}\right\} \\ &= \left\|\psi\left(x-\frac{b}{a}\right)\right\|_{\gamma,\ \beta} A\left(x, b/a\right), \end{aligned}$$

where

$$\begin{aligned} A\left(x, b/a\right) &= sup\left\{\left|\frac{e^{-\gamma|x|}}{e^{-\gamma|x-\frac{b}{a}|}}\right| : x \in \mathbb{R}\right\} \\ &\leq e^{\gamma\left|\frac{b}{a}\right|} < \infty, \end{aligned}$$

for a given $\gamma > 0$ and $b \in \mathbb{R}$.
So $\|\psi\left(x\right)\|_{\ \gamma,\ \beta,\ \frac{b}{a}}$ is also seminorm on $\mathscr{P}(\mathbb{R})$ for $\gamma > 0$, $\beta \in \mathbb{N}$ and for a given $b \in \mathbb{R}$. Therefore these seminorms generate the topology of the space $\mathscr{P}(\mathbb{R})$. If

$$X_q = \{\psi \in \mathscr{P}(\mathbb{R}) : D^{\alpha}\psi(0) = 0,\ for\ \ \alpha < q\},$$

then for any $\gamma > 0$ we can find a constant C such that

$$\left|\psi\left(x-\frac{b}{a}\right)\right| \leq C\left|x-\frac{b}{a}\right|^{q} e^{\frac{\gamma|x|}{2}} e^{\gamma\left|\frac{b}{a}\right|}.$$

If $a > 1$, then

$$e^{-\gamma|x|}\left|\psi\left(\frac{x-b}{a}\right)\right| \leq C\left|\frac{x-b}{a}\right|^{q} e^{-\frac{\gamma|x|}{2}} e^{\gamma\left|\frac{b}{a}\right|} \leq \frac{C_1}{a^q},$$

and thus

$$\left\|\psi\left(\frac{x}{a}\right)\right\|_{\gamma,\ 0,\ \frac{b}{a}} = O\left(\frac{1}{a^q}\right)\ as\ a \to \infty,\ \ \psi \in X_q. \tag{3.1}$$

Hence using above equation we get

$$\left\|\psi\left(\frac{x}{a}\right)\right\|_{\gamma,\ \beta,\ \frac{b}{a}} = O\left(\frac{1}{a^q}\right)\ as\ a \to \infty. \tag{3.2}$$

Using (3.2) we obtain the following theorem

Theorem 3.1. *Let $\psi \in \mathscr{P}(\mathbb{R}), f \in \mathscr{P}'(\mathbb{R})$ and $\mu_\alpha = \langle f, x^\alpha \rangle$ be its moment sequence. Then for a fixed b the asymptotic expansion of wavelet transform is*

$$\sqrt{a}\left\langle f(ax), \psi\left(x - \frac{b}{a}\right)\right\rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha D^\alpha \psi(-b/a)}{\alpha!\ a^{\alpha+1/2}} \quad as \quad a \to \infty. \tag{3.3}$$

The Proof is similar to that of Theorem 2.1.

Example 3.1. *Let $\psi(x) = (1 - x^2)e^{-\frac{x^2}{2}} \in \mathscr{P}(\mathbb{R})$ is Mexican-Hat wavelet and $f(x) \in \mathscr{P}'(\mathbb{R})$. Therefore by Theorem 3.3 moment asymptotic expansion of continuous Mexican-Hat wavelet transform for large a in $\mathscr{P}'(\mathbb{R})$ is given by*

$$\sqrt{a}\left\langle f(ax), \psi\left(x - \frac{b}{a}\right)\right\rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha D^\alpha[(1-x^2)e^{-\frac{x^2}{2}}]_{x=-\frac{b}{a}}}{\alpha!\ a^{\alpha+1/2}} \quad as \quad a \to \infty.$$

Case 2. In this case we consider wavelet $\psi(x) \in \mathscr{P}'(\mathbb{R})$ and $f(x) \in \mathscr{P}(\mathbb{R})$. Then the wavelet transform (1.1) can we rewrite as

$$(W_\psi f)(a,b) = \frac{1}{\sqrt{a}}\left\langle \psi\left(\frac{x}{a}\right), f(x+b)\right\rangle.$$

Similarly as Theorem 3.1 we can also obtain the following theorem

Theorem 3.2. *Let $\psi \in \mathscr{P}'(\mathbb{R}), f \in \mathscr{P}(\mathbb{R})$ and $\mu_\alpha = \langle \psi, x^\alpha \rangle$ be its moment sequence. Then for a fixed b the asymptotic expansion of wavelet transform is*

$$\frac{1}{\sqrt{a}}\left\langle \psi\left(\frac{x}{a}\right), f(x+b)\right\rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha D^\alpha f(b) a^{\alpha+1/2}}{\alpha!} \quad as \quad a \to 0. \tag{3.4}$$

Example 3.2. *In this example again we consider the Mexican-Hat wavelet which is less then exponential growth , so by applying Theorem 3.2 and using formula [30,pp.320, 1], we get the asymptotic expansion of wavelet transform for small values of a*

$$\frac{1}{\sqrt{a}}\left\langle \psi\left(\frac{x}{a}\right), f(x+b)\right\rangle \sim \sum_{\alpha=0}^{\infty} -2^{\frac{1}{2}(2\alpha-1)}\Gamma\left(\frac{2\alpha+1}{2}\right) \frac{D^{2\alpha} f(b) a^{2\alpha+1/2}}{(2\alpha)!} \quad as \quad a \to 0.$$

4. The moment asymptotic expansion of $(W_\psi f)(a,b)$ as $a \to \infty$ in the space $\mathcal{O}'_\gamma(\mathbb{R})$ for given b

A function ψ belongs to test $\mathcal{O}_\gamma(\mathbb{R})$, if it is smooth and $D^\alpha \psi(x) = O(|x|^\gamma)$ as $x \to \infty$ for every $\alpha \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. The family of seminorms

$$\|\psi(x)\|_{\alpha,\gamma} = sup\{\rho_\gamma(|x|)|D^\alpha \psi(x)| : x \in \mathbb{R}\},$$

where

$$\rho_\gamma(|x|) = \left\{ \begin{array}{cc} 1, & 0 \le |x| \le 1 \\ |x|^{-\gamma}, & |x| > 1 \end{array} \right\}, \tag{4.1}$$

generates a topology for $\mathcal{O}_\gamma(\mathbb{R})$. The space $\mathcal{O}_c(\mathbb{R})$ is the inductive limit $lim\mathcal{O}_\gamma(\mathbb{R})$ as $\gamma \to \infty$. The elements of $\mathcal{O}'_c(\mathbb{R})$ are generalized functions that decay fast at infinity in the distributional senses. We have $\Im[\mathcal{O}_c] = \mathcal{O}'_M$ and $\Im[\mathcal{O}'_M] = \mathcal{O}_c$ [7,p.402].
Now with the help of the translation version of $\psi(x)$, we can define the seminorms on $\mathcal{O}_\gamma(\mathbb{R})$ as

$$\begin{aligned} \|\psi(x)\|_{\alpha,\gamma,b/a} &= sup\left\{\rho_\gamma(|x|)\left|D^\alpha\psi\left(x-\frac{b}{a}\right)\right| : x \in \mathbb{R}\right\} \\ &= sup\left\{\rho_\gamma\left(\left|x-\frac{b}{a}\right|\right)\left|D^\alpha\psi\left(x-\frac{b}{a}\right)\right| : x \in \mathbb{R}\right\}\nabla(x,b/a) \\ &= \left\|\psi\left(x-\frac{b}{a}\right)\right\|_{\alpha,\gamma}\nabla(x,b/a), \end{aligned}$$

where $\nabla(x,b/a) = sup\left\{\frac{\rho_\gamma(|x|)}{\rho_\gamma(|x-\frac{b}{a}|)} : x \in \mathbb{R}\right\}$.
Notice that if $\gamma > 0$, then

$$\nabla(x,b/a) \le \left\{\begin{array}{cc} 1, & for\ 0 \le |x| \le 1\ \ and\ 0 \le |x-b/a| \le 1 \\ \left(1+\frac{|b/a|}{1-|b/a|}\right)^\gamma, & for\ |x| > 1\ \ and\ |x-b/a| > 1 \\ (1+|\frac{b}{a}|)^\gamma, & for\ 0 \le |x| \le 1\ \ and\ |x-b/a| > 1 \\ 1, & for\ |x| > 1\ \ and\ |x-b/a| \le 1 \end{array}\right\}.$$

Also, if $\gamma < 0$, we have

$$\nabla(x,b/a) \le \left\{\begin{array}{cc} 1, & for\ 0 \le |x| \le 1\ \ and\ 0 \le |x-b/a| \le 1 \\ \left(1+|b/a|\right)^{-\gamma}, & otherwise \end{array}\right\}.$$

Thus $sup\left\{\frac{\rho_\gamma(|x|)}{\rho_\gamma(|x-\frac{b}{a}|)} : x \in \mathbb{R}\right\} \le \left(1+|b/a|\right)^{|\gamma|} = K < \infty, \quad \forall\ \gamma\ \in \mathbb{R}.$
Therefore $\|\psi(x)\|_{\alpha,\ \gamma,\ b/a}$ are also seminorms on $\mathcal{O}_\gamma(\mathbb{R})$.These seminorm generate the topology of the space $\psi(x) \in \mathcal{O}_\gamma(\mathbb{R})$.
Now, set

$$X_q = \{\psi \in \mathcal{O}_\gamma(\mathbb{R}) : D^\alpha\psi(0) = 0,\ for\ \ \alpha < q\}.$$

Then for any $\gamma \in \mathbb{R}$ we can find a constant C such that

$$\rho_\gamma(|x|)\left|\psi\left(x-\frac{b}{a}\right)\right| \le C\rho_\gamma(|x|)\left|x-\frac{b}{a}\right|^q \nabla(x,b/a).$$

If $a > 1$, then

$$\rho(|x|)\left|\psi\left(x-\frac{b}{a}\right)\right| \le \frac{M}{a^q}.$$

Hence using above equation we get

$$\left\| \psi\left(\frac{x}{a}\right) \right\|_{\alpha,\ \gamma\ ,b/a} = O\left(\frac{1}{a^q}\right) \ as\ a \to \infty. \tag{4.2}$$

Similarly as Theorem 3.1 we can obtain the following theorem.

Theorem 4.1. *Let* $\psi \in \mathcal{O}_\gamma(\mathbb{R}), f \in \mathcal{O}'_\gamma(\mathbb{R}),\ N = [[\gamma]] - 1$ *and* $\mu_\alpha = \langle f, x^\alpha \rangle$ *be its moment sequence. Then for a fixed* $b \in \mathbb{R}$ *the asymptotic expansion of wavelet transform is*

$$\sqrt{a}\Big\langle f(ax), \psi\left(x - \frac{b}{a}\right)\Big\rangle = \sum_{\alpha=0}^{N} \frac{\mu_\alpha D^\alpha \psi(-b/a)}{\alpha! a^{\alpha+1/2}} + O\Big(\frac{1}{a^{N+1/2}}\Big) \ as\ a \to \infty. \tag{4.3}$$

Since $\mathcal{O}'_c(\mathbb{R}) = \bigcap \mathcal{O}'_\gamma(\mathbb{R})$, we obtain the asymptotic expansion of wavelet transform in the space $\mathcal{O}'_c(\mathbb{R})$.

Theorem 4.2. *Let* $\psi \in \mathcal{O}_c(\mathbb{R}), f \in \mathcal{O}'_c(\mathbb{R})$ *and* $\mu_\alpha = \langle f, x^\alpha \rangle$ *be its moment sequence. Then for a fixed* $b \in \mathbb{R}$ *the asymptotic expansion of wavelet transform is*

$$\sqrt{a}\Big\langle f(ax), \psi\left(x - \frac{b}{a}\right)\Big\rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha D^\alpha \psi(-b/a)}{\alpha! a^{\alpha+1/2}} + O\Big(\frac{1}{a^{N+1/2}}\Big) \ as\ a \to \infty. \tag{4.4}$$

5. The moment asymptotic expansion of $(W_\psi f)(a,b)$ as $a \to \infty$ in the space $\mathcal{O}'_M(\mathbb{R})$ for given b

The space $\mathcal{O}_M(\mathbb{R})$ consist of all c^∞-function whose derivatives are bounded by polynomials (of probably different degrees). Let $\psi \in \mathcal{O}_M(\mathbb{R})$ then its translation version is also in $\mathcal{O}_M(\mathbb{R})$. Then by using Theorem 9 [5] we can also derive the asymptotic expansion of wavelet transform in $\mathcal{O}'_M(\mathbb{R})$.

Theorem 5.1. *Let* $\psi \in \mathcal{O}_M(\mathbb{R}), f \in \mathcal{O}'_M(\mathbb{R})$ *and* $\mu_\alpha = \langle f, x^\alpha \rangle$ *be its moment sequence. Then for a fixed* $b \in \mathbb{R}$ *the asymptotic expansion of wavelet transform is*

$$\langle f(x), \psi_{a,b}(x) \rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha(f) D^\alpha \psi(-\frac{b}{a})}{\alpha!\ a^{\alpha+1/2}} \ as\ a \to \infty. \tag{5.1}$$

Proof: By using (1.7.3) given in [3] we can be write the wavelet transform

$$\sqrt{a}\Big\langle f(ax), \psi\left(x - \frac{b}{a}\right)\Big\rangle = \frac{\sqrt{a}}{2\pi}\langle e^{ib\omega} \hat{f}(\omega), \hat{\psi}(a\omega)\rangle, \tag{5.2}$$

where $\psi(x) \in \mathcal{O}_M(\mathbb{R})$ and $f(x) \in \mathcal{O}'_M(\mathbb{R})$ then its Fourier transforms $\hat{\psi}(\omega) \in \mathcal{O}'_c(\mathbb{R})$ and $\hat{f}(\omega) \in \mathcal{O}_c(\mathbb{R})$ respectively.
Now by using Theorem 4.2 we get

$$\langle f(x), \psi_{a,b}(x) \rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha(e^{-i\frac{b}{a}\omega}\hat{\psi}(\omega))D^\alpha(\hat{f}(0)}{2\pi\alpha!\ a^{\alpha+1/2}} \ as\ a \to \infty. \tag{5.3}$$

But by the properties of Fourier transform we have

$$\begin{aligned}\mu_\alpha(e^{-iba\omega}\hat{\psi}(\omega)) &= \langle e^{-i\frac{b}{a}\omega}\hat{\psi}(\omega), \omega^\alpha\rangle \\ &= 2\pi i^\alpha D^\alpha \psi\left(-\frac{b}{a}\right), \;\; D^\alpha(\hat{f}(\omega))_{\omega=0} = (-i)^\alpha \mu_\alpha(f(x)),\end{aligned}$$

and hence

$$\langle f(x), \psi_{a,b}(x)\rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha(f) D^\alpha \psi(-\frac{b}{a})}{\alpha! \;\; a^{\alpha+1/2}} \;\; as \;\; a \to \infty. \tag{5.4}$$

□

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