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Solution of Optimal Control Problems with Payoff Term and Fixed State Endpoint by Using Bezier Polynomials

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ABSTRACT: In this paper, a new numerical method for solving the optimal control problems with payoff term or fixed state endpoint by quadratic performance index is presented. The method is based on Bezier polynomial. The properties of Bezier polynomials in any intervel as $[a, b]$ are presented. The operational matrices of integration and derivative are utilized to reduce the solution of the optimal control problems to a nonlinear programming one to which existing well-developed algorithms may be applied. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Key Words: Optimal control, Bezier and Bernstein polynomial family, Best approximating, Operational matrices of derivative and integration, payoff term, Fixed state endpiont.

Contents

1. Introduction

One of the widely used methods to solve optimal control problems is the direct method. There is a large number of research papers that employ this method to solve optimal control problems (see for example [2,3,5,7,8,9,10,11,12,13,14,17] and the references therein). Razzaghi, et. al. used direct method for variational problems by using hybrid of block-pulse and Bernoulli polynomials [14]. Optimal control of switched systems based on Bezier control points presented in [6]. Edrisi-Tabriz et al. used B-spline functions to solve constrained quadratic optimal control problems [9]. A new approach using linear combination property of intervals and discretization is proposed to solve a class of nonlinear optimal control problems, containing a nonlinear system and linear functional [16,18]. Bernstein polynomials have been utilized for solving different equations by using various approximate methods [13]. An accurate method is proposed to solve problems such as identification, analysis and optimal control using the Bernstein orthonormal polynomials operational matrix of integration [17].

In this paper, we present a computational method to solve optimal control problems with payoff term and fixed state endpoint by using Bezier polynomial. The method is based on approximating the state variables and the control variables with Bezier polynomials [5,13,14]. Our method consists of reducing the optimal control problem to a NLP one by first expanding the state rate $\dot{x}(t)$ the control $u(t)$ as a Bezier polynomial with unknown coefficients. These linear cardinal Bezier polynomials are introduced. In order to approximate the integral and differential parts of the problem and the performance index, the operational matrix of integration P_Φ and differentiation D_Φ are given.

The paper is organized as follows: In Section 2 we describe the basic formulation of the Bezier functions required for our subsequent development. Section 3 is devoted to the formulation of optimal control problems. Section 4 summarizes the application of this method to the optimal control problems, and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed method.

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2. Some Properties Of Bernstein And Bezier Polynomials On [a,b]

The Bernstein basis polynomial of degree n on [a,b] are defined as [17]

$$B_{i,n}(t) = \binom{n}{i}(t-a)^i(b-t)^{n-i}, \ \ i \in [0,n] \tag{2.1}$$

where i is integer nummber and the binomial coefficients are given by

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!}, & i \in [0,n], \\ 0, & elsewhere. \end{cases}$$

Some properties of these polynomials are
(i) $B_{i,n}(a) = \delta_{i,0}(b-a)^n$ and $B_{i,n}(b) = \delta_{i,n}(b-a)^n$, where δ is the Kronecker delta function.
(ii) $B_{i,n}(t)$ has two roots, each of multiplicity i and $n-i$, at $t = a$ and $t = b$ respectively.
(iii) $B_{i,n}(t) \geq 0$ for $t \in$[a,b] and $B_{i,n}(b-t) = B_{n-i,n}(t-a)$.
(iv) The Bernstein polynomials form a partition of unity i.e. $\sum_{i=0}^{n} B_{i,n}(t) = (b-a)^n$.
(v) It has a degree dowing property in the sense that any of the upper-degree polynomials (degree $> n-1$) can be expressed as a linear combinations of polynomials of degree $n-1$. We have,
$B_{i,n}(t) = (b-t)B_{i,n-1}(t) + (t-a)B_{i-1,n-1}(t)$.

Bernstein Polynomials on [a,b] satisfy in the following relations:
(i) Derivative: $\frac{d}{dt}B_{i,n}(t) = \frac{1}{b-a}[(n+1-i)B_{i-1,n}(t)+(2i-n)B_{i,n}(t)-(i+1)B_{i+1,n}]$.
(ii) Integral: $\int_a^t B_{i,n}(x)dx = \frac{1}{n+1}\sum_{j=i+1}^{n+1} B_{j,n+1}(t)$ and $\int_a^b B_{i,n}(x)dx = \frac{(b-a)^{n+1}}{n+1}$.
(iii) Product: $B_{i,m}(t)B_{j,n}(t) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}}B_{i+j,m+n}(t)$ and
$\int_a^b B_{i,m}(t)B_{j,n}(t)dt = \frac{(b-a)^{n+m+1}\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}(m+n+1)}$.

2.1. Definition Of Bezier Polynomials On [a,b]

We will express Bezier (polynomials) curves in terms of Bernstein polynomials, defined explicitly by

$$p_n(t) = \sum_{i=0}^{n} c_i B_{i,n}(t), \ \ t \in [a,b], \tag{2.2}$$

where the $c_i, i = 1, 2, \ldots, n$ are given by $c_i = c[a^{<n-i>}, b^{<i>}]$ and they are control points or Bezier pionts and $a^{<n-i>}$ means that a appears $n-i$ times. For example,

$c[a^{<3>}, b^{<0>}] = c[a, a, a]$. Some property of Bezier polynomials on [a,b] are :
(i) Symmetry: $\sum_{i=0}^{n} c_i B_{i,n}(t-a) = \sum_{i=0}^{n} c_{n-i} B_{i,n}(b-t)$.
(ii) Linear precision: $\frac{1}{(b-a)^{n-1}} \sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t) = t - a$.

2.2. The Operational Matrices Of Derivative And Integration For The Bezier Polynomials

Suppose $\Phi_n(t)$ on [a,b] is given by

$$\Phi_n(t) = [B_{0,n}(t), B_{1,n}(t), ..., B_{n,n}(t)]^T, \tag{2.3}$$

where T denotes transposition.

The differentiation of vector $\Phi_n(t)$ can be expressed as

$$\Phi_n'(t) = D_\Phi \Phi_n(t), \tag{2.4}$$

where D_Φ is the $(n+1)(n+1)$ operational matrix of derivative for the Bezier polynomials given as follows:

$$D_\Phi = \frac{1}{b-a} \begin{bmatrix} -n & -1 & 0 & \dots & 0 & 0 \\ n & -(n-2) & -2 & 0 & \vdots & 0 \\ 0 & (n-1) & -(n-4) & 0 & \vdots & 0 \\ 0 & 0 & (n-k+1) & -(n-2k) & 0 & \vdots \\ \vdots & \dots & \dots & \dots & \ddots & \\ 0 & & \dots & & 1 & -(n-2n) \end{bmatrix}. \tag{2.5}$$

The integral of the vector $\Phi_n(t)$ defined in Eq. (2.3) is given as

$$\int_a^t \Phi_n(x)dx \simeq P_\Phi \Phi_n(t). \tag{2.6}$$

where P_Φ is the $(n+1) \times (n+1)$ operational matrix of integration for the Bezier polynomials given as:

$$P_\Phi = (b-a)^{n+1} P_\phi, P_\phi = WV, \tag{2.7}$$

where P_ϕ is the $(n+1) \times (n+1)$ operational matrix of integration for the Bezier polynomials on [0,1] and $W = (w_{i,j})$ and $V = (v_{j,k})$ are $(n+1) \times (n+1)$ matrices as:

$$w_{i,j} = \begin{cases} a_{i,0} - \frac{1}{3}a_{i,1}, & j = 0, \\ \frac{1}{2j-1}a_{i,j-1} - \frac{1}{2j+3}a_{i,j+1}, & j = 1, 2, \ldots, n-1, \\ \frac{1}{2n+1}a_{i,n-1}, & j = n, \end{cases} \tag{2.8}$$

where

$$a_{i,j} = \frac{2j+1}{n+j+1}\binom{n}{i}\sum_{k=0}^{j}(-1)^{k+j}\frac{\binom{j}{k}\binom{j}{k}}{\binom{n+j}{k+i}}, \qquad i, j = 0, 1, \ldots, n, \tag{2.9}$$

and

$$v_{j,k} = \frac{1}{\binom{n}{k}}\sum_{i=r}^{min\{j,k\}}(-1)^{j+i}\binom{j}{i}\binom{j}{i}\binom{n-j}{k-i}, \; r = max\{0, j+k-n\}, \; j, k = 0, 1, \ldots, n. \tag{2.10}$$

2.3. Function Approximation

Any function $f \in L^2[a, b]$ can be approximated using Bezier polynomials as

$$f(t) \simeq S_0 = \sum_{i=0}^{n} c_i B_{i,n}(t) = C^T \Phi_n(t), \tag{2.11}$$

where $C = [c_0 \ldots c_n]^T$ can be obtained as

$$C =< f, \Phi_n >= \int_a^b f(t)\Phi_n(t)dt = [< f, B_{0,n} >, \ldots, < f, B_{n,n} >]^T. \tag{2.12}$$

Let $R =< \Phi_n, \Phi_n >$ which is a $(n + 1) \times (n + 1)$ matrix and is called the dual matrix of $\Phi_n(t)$, and it can obtain as:

$$R_{i+1,j+1} =<B_{i,n}, B_{j,n}>=\int_a^b B_{i,n}(t)B_{j,n}(t)dt = (b-a)^{2n+1}\frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}, \; i, j = 0 \ldots n. \tag{2.13}$$

Lemma 2.1. *Suppose that the function $f : [a, b] \to R$ is $n + 1$ times continuously differentiable (i.e. $f \in C^{n+1}[a, b]$), and S_n=span$\{\Phi_n(t)\}$. If $C^T B$ is the best approximation of f out of S_n, then*

$$\|f - C^T B\|_{L^2[a,b]} \leq \frac{\hat{K}}{(n+1)!}\sqrt{\frac{b^{2m+3} - a^{2n+3}}{2n+3}} \tag{2.14}$$

where $\hat{K}$=max$|f^{(n+1)}(t)|$, $t \in [a, b]$.

Proof: We know that Set$\{1, x, x^2, ..., x^n\}$ is a basis for polynomials space of degree n . Therefore we define $y_1(x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + ... + \frac{x^n}{n!}f^{(n)}(a)$. Using Taylor expansion we have:

$$|f(x) - y_1(x)| = |f^{(n+1)}(\xi_x)\frac{x^{n+1}}{(n+1)!}| \tag{2.15}$$

where $\xi_x \in (a, b)$. Since $C^T B$ is the best approximation of f out of S_n and $y_1 \in S_n$ using (2.15) we obtain

$\|f - C^T B\|^2_{L^2[a,b]} \leq \|f - y_1\|^2_{L^2[a,b]} = \int_a^b |f(x) - y_1(x)|^2 dx$

$= \int_a^b |f^{(n+1)}(\xi_x)|^2 (\frac{x^{n+1}}{(n+1)!})^2 dx \leq (\frac{\hat{K}}{(n+1)!})^2 \int_a^b x^{2n+2} dx = (\frac{\hat{K}}{(n+1)!})^2 (\frac{b^{2n+3}-a^{2n+3}}{2n+3})$

$= (\frac{\hat{K}}{(n+1)!})^2 (\frac{b^{2n+3}(1-(\frac{a}{b})^{2n+3})}{2n+3}) \cong (\frac{\hat{K}b^n}{(n+1)!})^2 (\frac{b^3}{2n+3})$.

□

We can rewrite Eq. (2.14) as:

$$|f - C^T B_{L^2[a,b]}| \leq \frac{\hat{K}}{(n+1)!}\sqrt{\frac{b^{2m+3} - a^{2n+3}}{2n+3}} \cong \frac{\hat{K}}{(n+1)!}\sqrt{(\frac{b^{2n+3}(1 - (\frac{a}{b})^{2n+3})}{2n+3})}$$
$$\cong \frac{\hat{K}b^n}{(n+1)!}\sqrt{\frac{b^3}{2n+3}}, \tag{2.16}$$

which shows that the error vanishes as $n \to \infty$

3. Problem Statement

Consider the following class of nonlinear systems with inequality constraints,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3.1}$$
$$x(a) = x^0, \ x(b) = x^1 \tag{3.2}$$

where $A(t) = (a_{i,j}(t))_{n\times n}$ and $B(t) = (b_{i,j}(t))_{m\times m}$ are matrices functions and $x(t)$ and $u(t)$ are $n \times 1$ and $m \times 1$ state and control vectors respectively. The problem is finding the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, $a \leq t \leq b$ satisfying Eqs. (3.1) and (3.2) while minimize (or maximize) the quadratic performance index

$$Z = \frac{1}{2}x^T(b)Gx(b) + \frac{1}{2}\int_a^b (x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt, \tag{3.3}$$

where $G(t) = (g_{i,j}(t))_{n\times n}$, $Q(t) = (q_{i,j}(t))_{n\times n}$ are symmetric positive semi-definite matrices and and $R(t) = (r_{i,j}(t))_{m\times m}$ is a symmetric positive definite matrix.

3.1. Variational Problems

Consider the following variational problem:

$$Z(x(t)) = \int_a^b F(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t))dt, \tag{3.4}$$

with the boundary conditions

$$x(a) = a_0,\ \dot{x}(a) = a_1,\ \ldots,\ x^{(n-1)}(a) = a_{n-1}, \tag{3.5}$$

$$x(b) = b_0,\ \dot{x}(b) = b_1,\ \ldots,\ x^{(n-1)}(b) = b_{n-1}, \tag{3.6}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. The problem is to find the extremum of Eq. (3.4), subject to boundary conditions (3.5) and (3.6). The method consists of reducing the variational problem into a set of algebraic equations by first expanding $x(t)$ in terms of Bezier polynomials with unknown coefficients [14].

4. The Proposed Method

Let

$$x_i(t) \simeq \Phi_n^T(t)X^i, \tag{4.1}$$

$$u_j(t) \simeq \Phi_n^T(t)U^j, \tag{4.2}$$

where $X^i, i = 1, \ldots, n$, and $U^j, j = 1, \ldots, m$ are $(n+1)\times 1$ state and control coefficient vectors respectively. Then using (2.4) we get

$$\dot{x}_i(t) \simeq [D_\Phi \Phi_n(t)]^T X^i, \tag{4.3}$$

$$\dot{u}_j(t) \simeq [D_\Phi \Phi_n(t)]^T U^j. \tag{4.4}$$

Using Eqs. (4.1) and (4.2) we have

$$x(t) \simeq [\Phi_n^T(t)X]^T = [\sum_{j=0}^{n} B_{j,n}(t)X_j^1, \dots, \sum_{j=0}^{n} B_{j,n}(t)X_j^n]^T, \tag{4.5}$$

$$u(t) \simeq [\Phi_n^T(t)U]^T = [\sum_{j=0}^{n} B_{j,n}(t)U_j^1, \dots, \sum_{j=0}^{n} B_{j,n}(t)U_j^m]^T, \tag{4.6}$$

where $X = (X_i^k)_{(n+1)\times n}$ and $U = (U_j^r)_{(n+1)\times m}$ are state and control coefficient matrices respectively. The boundary conditions in Eq. (3.2) can be rewritten as

$$x(a) = x^0 = d^0 \otimes E\Phi_n(t), \tag{4.7}$$

$$x(b) = x^1 = d^1 \otimes E\Phi_n(t). \tag{4.8}$$

where d^0 and d^1 are $n\times 1$ constant vectors, $E = [1, \dots, 1]$ is $1\times(n+1)$ constant vector, and the symbol $\otimes$ denotes Kronecker product [19]. If $x(a)$ or $x(b)$ is unknown in Eq. (3.2), then we put

$$x(a) \simeq [\Phi_n^T(a)X]^T = [\sum_{j=0}^{n} B_{j,n}(a)X_j^1, \dots, \sum_{j=0}^{n} B_{j,n}(a)X_j^n]^T, \tag{4.9}$$

$$x(b) \simeq [\Phi_n^T(b)X]^T = [\sum_{j=0}^{n} B_{j,n}(b)X_j^1, \dots, \sum_{j=0}^{n} B_{j,n}(b)X_j^m]^T. \tag{4.10}$$

4.1. Performance Index Approximation

By substituting Eqs. (4.5), (4.6) and (4.8) in Eq. (3.3) we get

$$\begin{aligned} min(max)Z = \frac{1}{2}{x^1}^T G(b)x^1 + \frac{1}{2}X^T[\int_a^b \Phi_n^T(t)Q(t)\Phi_n(t)dt]X \\ + \frac{1}{2}U^T[\int_a^b \Phi_n^T(t)R(t)\Phi_n(t)dt]U. \end{aligned} \tag{4.11}$$

For problems with time-varying performance index, $Q(t)$ and $R(t)$ are functions of time. Let

$$P_x = \int_a^b \Phi_n^T(t)Q(t)\Phi_n(t)dt, \quad \text{and} \quad P_u = \int_a^b \Phi_n^T(t)R(t)\Phi_n(t)dt. \tag{4.12}$$

Eq. (4.12) can be evaluated by numerical integration techniques. By substituting Eqs. (4.10) and (4.12) in Eq. (4.11) we get

$$Z[X,U] = \frac{1}{2}X^T(\hat{P}+P_x)X + \frac{1}{2}U^T P_u U, \tag{4.13}$$

where

$$\hat{P} = \Phi_n^T(b)G(b)\Phi_n(b).$$

The boundary conditions in Eq. (3.2) can be expressed as

$$q_k^0 = x_k(a) - x_k^0 \quad , \quad k = 1,\dots,n, \tag{4.14}$$

$$q_k^1 = x_k(b) - x_k^1 \quad , \quad k = 1,\dots,n. \tag{4.15}$$

We now find the extremum of Eq. (4.13) subject to Eqs. (4.14) and (4.15) using the Lagrange multiplier technique. Let

$$Z[X,U,\lambda^0,\lambda^1] = Z[X,U] + \lambda^0 Q^0 + \lambda^1 Q^1. \tag{4.16}$$

where $Q^0 = (q_k^0), k = 1,\dots,n$ and $Q^1 = (q_k^1), k = 1,\dots,n$ are $(n\times 1)$ constant vectors. The necessary condition for the extremum of (4.16) is

$$\nabla Z[X,U,\lambda^0,\lambda^1] = 0. \tag{4.17}$$

4.2. Performance Index Approximation For The Variational Problem

By expanding $x(t)$ using the Bezier polynomials we have

$$x(t) = X^T\Phi_n(t), \tag{4.18}$$

where$X$ is a vector of order $(n+1)\times 1$. By derivating Eq. (4.18) with respect to t we get

$$x'(t) = X^T D_\Phi \Phi_n(t), \tag{4.19}$$

where D_Φ is operational matrix of derivative given in Eq. (2.5). By n times derivating of Eq. (4.18) with respect to t we have

$$x^{(n)}(t) = X^T D_\Phi^n \Phi_n(t). \tag{4.20}$$

We expand $(t-a)^i, i = 0, 1, \ldots, n-1$ in terms of Bezier polynomial as

$$(t-a)^i = d_i \Phi_n(t), \quad i = 0, 1, \ldots, n-1, \tag{4.21}$$

where $d_i, i = 0, 1, \ldots, n-1$, are constant vectors of order $1 \times (n+1)$ and are given as

$$d_i = \frac{1}{\binom{n}{i}(b-a)^{n-i}}[0, \ldots, \binom{i}{i}, \binom{i+1}{i}, \ldots, \binom{n}{i}], \quad i = 0, 1, \ldots, n-1. \tag{4.22}$$

So Eq. (3.4) can be rewritten as

$$Z[x(t)] = Z[X]. \tag{4.23}$$

The boundary conditions in Eqs. (3.5) and (3.6) can be expressed as

$$r_k^0 = x^{(k)}(a) - a_k = 0 \ , \ \ k = 0, \ldots, n-1, \tag{4.24}$$

$$r_k^1 = x^{(k)}(b) - b_k = 0 \ , \ \ k = 0, \ldots, n-1. \tag{4.25}$$

We now find the extremum of Eq. (4.23) subject to Eq. (4.25) using the Lagrange multiplier technique. Let

$$Z[x, \lambda] = Z[x, \lambda^0, \lambda^1] + \lambda^0 R^0 + \lambda^1 R^1, \tag{4.26}$$

where $R^0 = (q_k^0), k = 1, \ldots, n$ and $R^1 = (q_k^1), k = 1, \ldots, n$ are $(n \times 1)$ constant vectors. The necessary conditions for the extremum of (4.26) are

$$\nabla Z[X, \lambda^0, \lambda^1] = 0. \tag{4.27}$$

5. Illustrative Examples

This section is devoted to numerical examples. We implemented the proposed method in last section with MALAB (2012) in personal computer. To illustrate our technique, we present four numerical examples, and make a comparison with some of the results in the literatures.

Example 5.1. This example is adapted from [15]

$$\min\ Z = \int_0^4 u^2(t) + x(t)dt, \tag{5.1}$$

subject to

$$\dot{x}(t) = u(t), \tag{5.2}$$

with the boundery conditions

$$x(0) = 0,\ x(4) = 1. \tag{5.3}$$

Here we solve this problem using Bezier polynomials by choosing $n = 3$. Let

$$x(t) = \Phi_3^T(t)X, \tag{5.4}$$

$$u(t) = \Phi_3^T(t)U, \tag{5.5}$$

where

$$X = [X_0, X_1, X_2, X_3]^T, \tag{5.6}$$

$$U = [U_0, U_1, U_2, U_3]^T. \tag{5.7}$$

Using Eqs. (2.4) and (5.4) we get

$$\dot{x}(t) = [D_\Phi \Phi_3(t)]^T X, \tag{5.8}$$

where D_Φ is the operational matrix of derivative given in Eq. (2.5). By substituting Eqs. (5.6) and (5.7) in Eq. (5.2) we obtain

$$[D_\Phi^T X - U]^T \Phi_3(t) = 0. \tag{5.9}$$

Let

$$Z_{\chi[0,4]} = \int_0^4 u^2(t) + x(t)dt. \tag{5.10}$$

Using Eqs. (5.3) and (5.4) in (5.9) we have

$$\begin{aligned} Z_{\chi[0,4]} &= \int_0^4 (\Phi_3^T(t)U)^T(\Phi_3^T(t)U) + \Phi_3^T(t)X\,dt \\ &= \int_0^4 (U^T\Phi_3(t))(\Phi_3^T(t)U) + \Phi_3^T(t)X\,dt \\ &= U^T\left(\int_0^4 \Phi_3(t)\Phi_3^T(t)dt\right)U + \int_0^4 \Phi_3^T(t)dt \\ &= U^TV_{\chi[0,4]}U + v_{\chi[0,4]}X, \end{aligned} \tag{5.11}$$

where $V_{\chi[0,4]} = \int_0^4 \Phi_3(t)\Phi_3^T(t)dt$ and $v_{\chi[0,4]} = \int_0^4 \Phi_3^T(t)dt$ are constant matrix and vector are of order (4×4) respectively. Using the Lagrange multiplier technique to find the extremum of (5.1) subject to the conditions (5.3) we have

$$Z[X, U, \lambda^0, \lambda^1] = Z[X, U] + \lambda^0 R^0 + \lambda^1 R^1. \tag{5.12}$$

where $R^0 = \Phi_3^T(0)X - 0$, $R^1 = \Phi_3^T(4)X - 1$.
The necessary conditions are

$$\nabla Z[X, U, \lambda^0, \lambda^1] = 0. \tag{5.13}$$

The exact solutions of the problem are:

$$x^*(t) = \frac{t^2 - 3t}{4},\ u^*(t) = \frac{2t-3}{4},\ Z^* = \frac{11}{12}. \tag{5.14}$$

we obtain:

$$\begin{aligned} X &= [0, -\tfrac{1}{64}, -\tfrac{1}{96}, \tfrac{1}{64}], \\ U &= [-\tfrac{3}{256}, -\tfrac{1}{768}, -\tfrac{7}{768}, -\tfrac{5}{256}], \end{aligned}$$

and the approximate solutions of state and control functions are as follows:

$$\begin{aligned} x_p(t) &= -\tfrac{1}{64}B_{1,3}(t) - \tfrac{1}{96}B_{2,3}(t) + \tfrac{1}{64}B_{3,3}(t) \\ &= \tfrac{t^2-3t}{4}, \\ u_p(t) &= -\tfrac{3}{256}B_{0,3}(t) - \tfrac{1}{768}B_{1,3}(t) - \tfrac{7}{768}B_{2,3}(t) - \tfrac{5}{256}B_{3,3}(t) \\ &= \tfrac{2t-3}{4}, \\ Z_p &= \tfrac{11}{12}. \end{aligned}$$

which are the exact solutions.

Example 5.2. This example is adapted from [4]

$$\text{extermum } Z = \int_0^1 (x^2(t) + u^2(t))dt, \tag{5.15}$$

subject to

$$\dot{x}(t) = u(t),$$
$$x(0) = 1, x(1) \text{ is indefinite.}$$

Let $n = 3$ then we have

$$x(t) = \Phi_3^T(t)X, \quad u(t) = \Phi_3^T(t)U. \tag{5.16}$$

With payoff term the other condition is obtained as follow:

$$\dot{x}(1) = 0. \tag{5.17}$$

Using the presented method in the previous section, we obtain

$$X = [1, \tfrac{4147}{5537}, \tfrac{3592}{5537}, \tfrac{3592}{5537}],$$

$$U = [-\tfrac{4170}{5537}, -\tfrac{2500}{5537}, -\tfrac{1110}{5537}, 0],$$

and the approximate solutions of state and control functions are as follows

$$x(t) = B_{0,3}(t) + \tfrac{4147}{5537}B_{1,3}(t) + \tfrac{3592}{5537}B_{2,3}(t) + \tfrac{3592}{5537}B_{3,3}(t),$$

$$u(t) = -\tfrac{4170}{5537}B_{0,3}(t) - \tfrac{2500}{5537}B_{1,3}(t) - \tfrac{1110}{5537}B_{2,3}(t).$$

The analytical solutions are [4]

$$\begin{aligned} x^*(t) &= -\frac{sinh(1-t)}{cosh1}, \\ u^*(t) &= \frac{cosh(1-t)}{cosh1}, \\ Z^* &= 0.761594155955765. \end{aligned} \tag{5.18}$$

Table 1, shows the approximate value of Z toghether with absolute value of errors for $n = 3$ and 5. Figures 1 and 2, show plot of errors for $x(t)$ and $u(t)$ for $n = 3$ and 5 respectively.

Table 1: The approximate values of Z, and absolute value of error for Example 5.2.

n	Approximate values of Z	Absoulte error
3	0.761603756546867	$9.6e-06$
5	0.761594156033200	$7.74e-11$

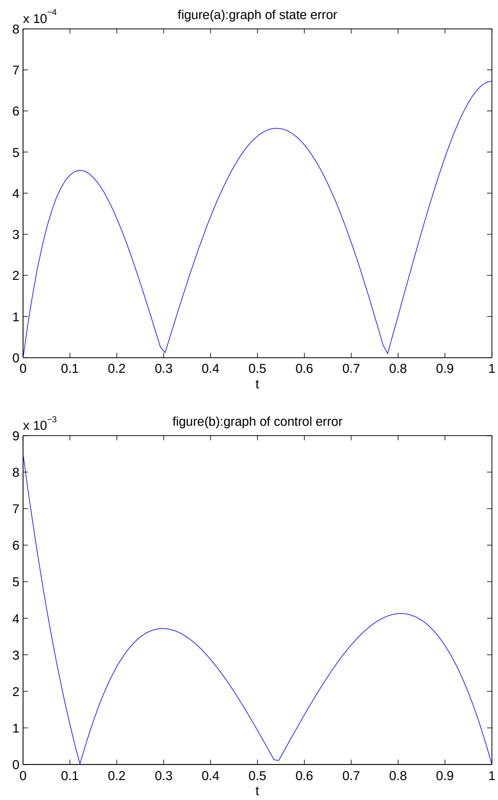

Figure 1: Plots of errors for state (left) and control (right) functions for n=3

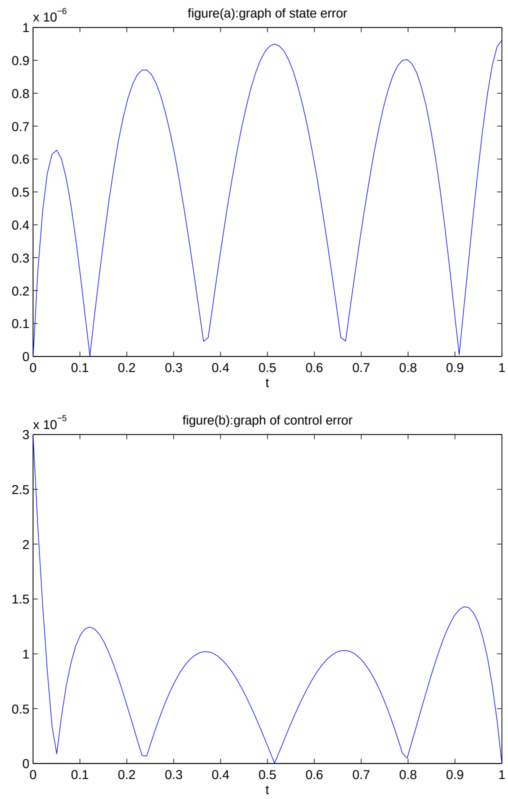

Figure 2: Plots of errors for state (left) and control (right) functions for n=5

Example 5.3. Find the extermum of the functional [4]

$$Z = \int_0^T (x^2(t) + u^2(t))dt \tag{5.19}$$

subject to

$$\dot{x}(t) + x(t) = u(t),$$
$$x(0) = x_0,\ x(T) = 0, \text{T is indefinite.}$$

The exact solutions are:

$$x^*(t) = \frac{x_0 sinh(\sqrt{2}(T-t))}{sinh(\sqrt{2}T)}, \tag{5.20}$$

$$u^*(t) = \frac{x_0(sinh(\sqrt{2}(T-t)) - \sqrt{2}cosh\sqrt{2}(T-t))}{sinh(\sqrt{2}T)}, \tag{5.21}$$

$$z^* = x_0^2(5.918916999941614e-01)T. \tag{5.22}$$

For $n = 3$, we obtin

$$X = \frac{x_0}{T^3}[1, \frac{47}{99}, \frac{49}{198}, 0],$$
$$U = \frac{x_0}{T^3}[-\frac{19}{33}, -\frac{50}{99}, -\frac{5}{11}, -\frac{49}{66}].$$

then we get the approximation solutions as following:

$$x_p(t) = \frac{x_0}{T^3}[B_{0,3}(t) + \frac{47}{99}B_{1,3}(t) + \frac{49}{198}B_{2,3}(t)], \tag{5.23}$$

$$u_p(t) = \frac{x_0}{T^3}[-\frac{19}{33}B_{0,3}(t) - \frac{50}{99}B_{1,3}(t) - \frac{5}{11}B_{2,3}(t) - \frac{49}{66}B_{3,3}(t)], \tag{5.24}$$

$$z_p = \frac{293}{495}Tx_0^2. \tag{5.25}$$

Table 2: The approximate values of Z for n=3, for Example 5.3

T	Exact	Presented Method	error
1	$5.918916555204872e-01$	$5.919191919191920e-01$	$2.75e-05$
5	$2.959458277602435e+00$	$2.959595959595960e+00$	$1.37e-04$
10	$5.918916555204870e+00$	$5.919191919191919e+00$	$2.75e-04$
15	$8.878374832807307e+00$	$8.878787878787879e+00$	$4.13e-04$
1000	$5.918916555204871e+02$	$5.919191919191919e+02$	$2.75e-02$

Table 3: The approximate values of Z for n =5 for Example 5.3

T	Exact	Presented Method	error
1	$5.918916555204872e-01$	$5.918916565792995e-01$	$1.059e-09$
5	$2.959458277602435e+00$	$2.959458282896498e+00$	$5.29e-09$
10	$5.918916555204870e+00$	$5.918916565792996e+00$	$1.058e-08$
15	$8.878374832807307e+00$	$8.878374848689493e+00$	$1.58e-08$
1000	$5.918916555204871e+02$	$5.918916565792996e+02$	$1.058e-06$

Tables 2 and 3 show the approximate values of Z toghether with absolute values of errors for different values of T and $n = 3, 5$.

Example 5.4. Find the extremum of the functional [14]

$$Z(x(t)) = \int_0^{\pi/4} (x^2(t) - \dot{x}^2(t))dt, \tag{5.26}$$

with the conditions

$$x(0) = 1, \;\; \dot{x}(\pi/4) = 0. \tag{5.27}$$

The exact solution is $x(t) = sin(t) + cos(t)$.

Table 4: The approximate values of $x(t)$, for Example 5.4

t	Exact	Presented Method $n = 3$	Presented Method $n = 5$	Method [14]
0	1	1	1	0.999999
0.1	1.09483758	1.09513135	1.09483761	1.094838
0.3	1.25085669	1.25065471	1.25085665	1.250857
0.5	1.35700810	1.35667301	1.35700798	1.357008
0.7	1.40905987	1.40936310	1.40905988	1.409059

Table 5: The approximate values of $Z(x(t))$, for $n = 3, 5$, for Example 5.4

n	Exact	Presented Method	error
3	1	0.9999945002929741	$5.49e-06$
5	1	0.9999999999829694	$1.70e-11$

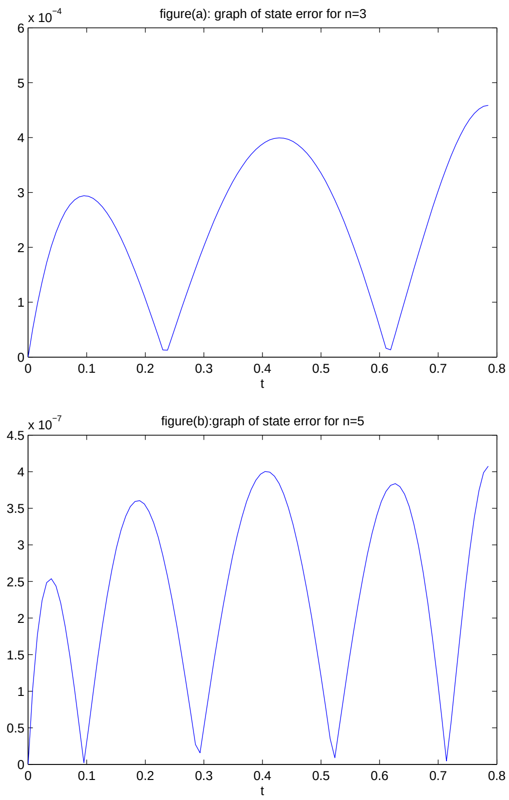

Figure 3: Plots of errors for $x(t)$ for $n = 3$ (left) and for $n = 5$ (right)

Table 4, shows the approximate values of $x(t)$ toghether for different values of t and n. Table 5, show the results for Z. Figure 3 shows the plots of errors for $x(t)$ for $n = 3, 5$.

6. Conclusion

In this paper we presented a numerical scheme for solving linear constrained quadratic optimal control problems. The Bezier polynomials was employed. Several test problems were used to show the applicability and efficiency of the presented method. The obtained results show that the new approach can solve the problem effectively.

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