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A Non-Smooth Three Critical Points Theorem for General Hemivariational Inequality on Bounded Domains

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ABSTRACT: In this paper we are concerned with the study of a hemivariational inequality with nonhomogeneous Neumann boundary condition. We establish the existence of at least three solutions of the problem by using the nonsmooth three critical points theorem and the principle of symmetric criticality for Motreanu-Panagiotopoulos type functionals.

Key Words: Nonsmooth critical point theory, Hemivariational inequality, $p(x)-$Laplacian.

Contents

1. Introduction

In this paper, we study the following nonlinear elliptic differential inclusion with $p(x)-$Laplacian

$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = -\mu g(x,u) & \textit{in } \Omega \\ -|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} \in -\lambda \partial F(x,u) & \textit{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outward unit normal derivative on $\partial\Omega$, $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function satisfying

$$1 < p^- = \inf_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ = \sup_{x\in\bar{\Omega}} p(x) < +\infty,$$

and $\lambda, \mu \in [0, \infty)$. $F : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a function in which $F(\cdot, u)$ is measurable for every $u \in \mathbb{R}$ and $F(x, \cdot)$ is locally Lipschitz for a.e. $x \in \partial\Omega$. $\partial F(x, u)$ denotes the generalized Clarke gradient of $F(x, u)$ at $u \in \mathbb{R}$. Moreover, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $G(x, u) = \int_0^u g(x, t)dt$.
In this paper, a class of problem for hemivariational inequality is studied which is defined on domains of the type $\mathcal{B}$ which are nonempty, closed, convex cone subsets of $W_0^{1,p(x)}(\Omega)$.

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Our purpose is to study the following hemivariational inequality problem:
Find $u \in \mathcal{B}$ (it is called a weak solution of problem (1.1)) if for all $v \in \mathcal{B}$,

$$\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla(v-u)dx + \int_{\Omega} a(x)|u|^{p(x)-2}u(v-u)dx$$

$$+\lambda \int_{\partial\Omega} F^0(x,u;u-v)d\sigma + \mu \int_{\Omega} g(x,u)(v-u)dx \geq 0. \tag{1.2}$$

To indicate the existence for solutions of (1.2), we consider the functional $\mathcal{I}(u)$: $W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ by $\mathcal{I}(u) = \phi(u) - \lambda\mathcal{F}(u) + \mu\mathcal{G}(u) + \chi(u)$ associated to (1.2), such that

$$\phi(u) = \int_{\Omega} \frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}]dx, \qquad \forall u \in W_0^{1,p(x)}(\Omega),$$

$$\mathcal{F}(u) = \int_{\partial\Omega} F(x,u)d\sigma, \qquad \forall u \in W_0^{1,p(x)}(\Omega),$$

$$\mathcal{G}(u) = \int_{\Omega} G(x,u)dx, \qquad \forall u \in W_0^{1,p(x)}(\Omega),$$

where $\chi(u)$ is the indicator function of the set $\mathcal{B}$.

The $p(x)-$Laplace operator $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is a natural generalization of the $p-$Laplacian operator $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. The main difference between them is that $p-$Laplacian operator is $(p-1)-$homogenous, but the $p(x)-$Laplacian operator, when $p(x)$ is not constant is not homogeneous. For $p(x)-$Laplacian operator, we refer the readers to (cf. [13], [14], [15], [18], [23]) and references therein.
In recent years, differential equations and variational problems have been studied in many papers, we refer to some interesting works (cf. [27], [28]). For a thorough treatment of the hemivariational inequality problems we refer to the monographs Naniewicz and Panagiotopoulos (cf. [26]) (based on pseudomonotonicity), Motreanu and Panagiotopoulos (cf. [24]), Motreanu and Rádulescu (cf. [25]) (based on compactness arguments). In these works (and in references therein) there are studied the elliptic problems on bounded domains.
It is well known that many problems in mathematics and physics that comes from the real world by some authors have investigated (see cf. [1], [2], [29], [30], [31]). The applications to nonsmooth variational problems have been seen in (cf. [3]), Bonanno and Candito studied a class of variational-hemivariational inequalities; in (cf. [32]), Zhang and Liu studied an elliptic equation with discontinuous nonlinearities in $\mathbb{R}^N$.
In recent years, the study of the three-critical-points theorem nonsmooth variational problems was investigated. The goal of this article is to apply a version for locally Lipschitz functionals (was established by Kristály, Marzantowicz and Varga

in (cf. [21])).
In the present article, we use a class of perturbed Motreanu-Panagiotopoulos functionals. We prove the existence of at least three solutions for a hemivariational inequality depending on two parameters.
The paper is organized as follows. We prepare the basic definitions and properties in the framework of the generalized Lebesgue and Sobolev spaces. For this introductory part we refer to (cf. [6], [8], [9], [11], [12]). Moreover, some important properties of the $p(x)-$biharmonic operator, some basic notions about generalized directional derivative and hypotheses on F, the basic definitions and facts about the non-smooth three-critical-points theorem are given. Finally, we will give the proofs of our main results.

2. Preliminaries

We recall some basic facts about the variable exponent Lebesgue-Sobolev.
The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$\{u : \Omega \longrightarrow \mathbb{R} : \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

It is endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \{ \lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx\} \leq 1\}.$$

For $p \equiv const.$, the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ coincides with the standard norm $\|\cdot\|_p$ of the Lebesgue space $L^p(\Omega)$. Then $(L^{p(x)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space (cf. [22]).

Let p' be the function obtained by conjugating the exponent p pointwise, that is $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \bar{\Omega}$, then p' belongs to $C_+(\bar{\Omega})$.

Proposition 2.1. *(cf. [22]) $L^{p(\cdot)}(\Omega)$ is a separable, reflexive Banach space and $L^{\acute{p}(\cdot)}(\Omega)$ is its dual space.*

Proposition 2.2. *(cf. [11]) (i) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the following Hölder type inequality valid*

$$\int_\Omega |u(x)v(x)| dx \leq (\frac{1}{p^-} + \frac{1}{p'^-}) \|u\|_{p(x)} \|v\|_{p'(x)}.$$

(ii) If $p, q \in C(\bar{\Omega})$ and $1 \leq p \leq q$ in Ω, then the embedding $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ is continuous.

Proposition 2.3. *(cf. [11]) Let p be a function in $C_+(\overline{\Omega})$. Set $\varphi_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx$. If $u, (u_n)_n$ are in $L^{p(\cdot)}(\Omega)$, with $1 \leq p^- \leq p^+ \leq \infty$, then the following relations hold:*

(i) $\|u\|_{p(\cdot)} \geq 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \leq \varphi_{p(\cdot)} \leq \|u\|_{p(\cdot)}^{p^+}$,

(ii) $\|u\|_{p(\cdot)} \leq 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leq \varphi_{p(\cdot)} \leq \|u\|_{p(\cdot)}^{p^-}$.

The generalized Lebesgue-Sobolev space $W^{L,p(x)}(\Omega)$ for $L = 1, 2, ...$ is defined as

$$W^{L,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$. The space $W^{L,p(x)}(\Omega)$ with the norm

$$\|u\|_{W^{L,p(\cdot)}}(\Omega) = \sum_{|\alpha|\leq L} \|D^\alpha u\|_{p(\cdot)},$$

is a separable and reflexive Banach space.

The space $W_0^{L,p(x)}(\Omega)$ denotes the closure in $W^{L,p(\cdot)}(\Omega)$ of the set of all $W^{L,p(\cdot)}(\Omega)-$functions with compact support.

Proposition 2.4. *(cf. [7]) $W_0^{L,p(\cdot)}(\Omega)$ is a separable, uniformly convex and reflexive Banach space.*

For every $u \in W_0^{L,p(\cdot)}(\Omega)$ the Poincaré inequality holds, i.e., there exists a positive constant C_p such that

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C_p \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

(see (cf. [13])).
Hence, an equivalent norm for the space $W_0^{L,p(\cdot)}(\Omega)$ is given by

$$\|u\|_{W_0^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|=L} \|D^\alpha u\|_{p(\cdot)}.$$

Let p_L^* denote the critical variable exponent related to p, defined for all $x \in \bar{\Omega}$ by the pointwise relation

$$p_L^*(x) = \begin{cases} \frac{Np(x)}{N-Lp(x)} & Lp(x) < N, \\ +\infty & Lp(x) \geq N, \end{cases} \tag{2.1}$$

is the critical exponent related to p.

Proposition 2.5. *(cf. [11], [22]) For $p, q \in C_+(\overline{\Omega})$ such that $q(x) \leq p_L^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Remark 2.6. (i) *By the Proposition 2.5 there is a continuous and compact embedding of $W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}$ where $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.*
(ii) *Define $\|u\| = \inf\{\lambda > 0 : \int_\Omega [|\frac{\nabla u}{\lambda}|^{p(x)} + a(x)|\frac{u}{\lambda}|^{p(x)}]dx \leq 1\}$, for all $u \in W_0^{1,p(x)}(\Omega)$, then $\|u\|$ is a norm on $W_0^{1,p(x)}(\Omega)$.*

In this paper, we denote by $X = W_0^{1,p(x)}(\Omega)$ and $X^\star$ the dual space.

Proposition 2.7. *Set* $\Phi(u) = \int_\Omega [|\nabla u|^{p(x)} + a(x)|u(x)|^{p(x)}dx]$. *For* $u, u_n \in X$ *we have*

(i) $\|u\| < (=; >)1 \Leftrightarrow \Phi(u) < (=; >)1$,

(ii) $\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq \Phi(u) \leq \|u\|^{p^-}$,

(iii) $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \Phi(u) \leq \|u\|^{p^+}$,

(iv) $\|u_n\| \to 0 \Leftrightarrow \Phi(u_n) \to 0$,

(v) $\|u_n\| \to \infty \Leftrightarrow \Phi(u_n) \to \infty$.

The proof of this proposition is similar to the proof in (cf. [11]).

Let $\eta : \partial\Omega \to \mathbb{R}$ be a measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{\eta(x)}(\partial\Omega) = \Big\{u : \partial\Omega \to \mathbb{R} \text{ is measurable and}$$

$$\int_{\partial\Omega} |\eta(x)||u|^{p(x)} d\sigma < \infty\Big\},$$

with the norm

$$|u|_{(p(x),a(x))} = \inf\{\tau > 0; \int_{\partial\Omega} |\eta(x)||\frac{u}{\tau}|^{p(x)} d\sigma \leq 1\},$$

where $d\sigma$ is the measure on the boundary.

Lemma 2.8. *(cf. [5]) Let* $\rho(x) = \int_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma$ *for* $u \in L^{p(x)}_{a(x)}(\partial\Omega)$ *we have*

$$|u|_{(p(x),\eta(x))} \geq 1 \Rightarrow |u|^{p^-}_{(p(x),\eta(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),\eta(x))},$$

$$|u|_{(p(x),\eta(x))} \leq 1 \Rightarrow |u|^{p^+}_{(p(x),\eta(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),\eta(x))}.$$

For $A \subseteq \bar{\Omega}$ denote by $\inf_{x\in A} p(x) = p^-$, $\sup_{x\in A} p(x) = p^+$. Define

$$p^\partial(x) = (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & p(x) < N, \\ +\infty & p(x) \geq N, \end{cases} \tag{2.2}$$

$$p^\partial(x)_{r(x)} := \frac{r(x)-1}{r(x)} p^\partial(x),$$

where $x \in \partial\Omega, r \in C(\partial\Omega, \mathbb{R})$ and $r(x) > 1$.

Proposition 2.9. *(cf. [16], [22]) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*_\partial(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.*

Here, we review the definitions and basic properties from the theory of generalized differentiation for locally Lipschitz functions.
Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $< \cdot, \cdot >$ the duality brackets for the pair $(X, X^\star)$. A function $h : X \to \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$ there exists a neighbourhood U of x and a constant $K > 0$ depending on U such that $|h(y) - h(z)| \leq K\|y - z\|$ for all $y, z \in U$.
For a locally Lipschitz function $h : X \to \mathbb{R}$ we define the *generalized directional derivative* of h at $u \in X$ in the direction $\gamma \in X$ is defined by

$$h^0(u;\gamma) = \limsup_{w \to u, t \to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The *generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \{x^\star \in X^\star \ : < x^\star, \gamma >_X \leq h^0(u;\gamma), \ \forall \gamma \in X\}.$$

It is nonempty, convex and $w^\star$−compact subset of $X^\star$, where $< \cdot, \cdot >_X$ is the duality pairing between $X^\star$ and X, see (cf. [4]).

Proposition 2.10. *(cf. [4]) Let $h, g : X \to \mathbb{R}$ be a locally Lipschitz function. Then*
(i) $h^0(u;\cdot)$ is subadditive and positively homogeneous.
(ii) $(-h)^0(u;v) = h^0(u;-v), \quad \forall u, v \in X$.
(iii) $h^0(u;v) = \max\{< \xi, v > : \ \xi \in \partial h(u)\}, \ \forall u, v \in X$.
(iv) $(h + g)^0(u;v) \leq h^0(u;v) + g^0(u;v), \ \forall u, v \in X$.

Proposition 2.11. *(cf. [4])(Lebourg's mean value theorem) Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional. Then, for every $u, v \in X$ there exists $w \in [u, v]$, $w^* \in \partial h(u)$ such that $h(u) - h(v) = \langle w^*, u - v \rangle$.*

Definition 2.12. *(cf. [24]) Let X be a Banach space, $\mathcal{I} : X \to (-\infty, +\infty]$ is called a Motreanu-Panagiotopoulos-type functional, if $\mathcal{I} = h + \chi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.*

Definition 2.13. *(cf. [24]) An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \chi$ if*

$$h^0(u; v - u) + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X.$$

In most applications, the following special case is considered: Let $h : X \to \mathbb{R}$ be a locally Lipschitz functional and we assume it is also given a functional $\chi : X \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous and proper whose restriction to the set $dom(\chi) = \{x \in X \ : \ \chi(u) < \infty\}$ is continuous. The indicator of $\mathcal{B}$ is the function $\chi_{\mathcal{B}} : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\chi_{\mathcal{B}} = \begin{cases} 0 & u \in \mathcal{B} \\ +\infty & u \notin \mathcal{B} \end{cases} \tag{2.3}$$

(it is easily seen that $\chi_{\mathcal{B}}$ is proper, convex and lower semicontinuous), while its restriction to $dom(\chi_{\mathcal{B}}) = \mathcal{B}$ is the constant 0; clearly $u \in X$ is a critical point for the Motreanu-Panagiotopoulos functional $h + \chi_{\mathcal{B}}$ iff $u \in \mathcal{B}$ and the following condition holds

$$h^0(u; v - u) \geq 0, \qquad \forall v \in \mathcal{B}.$$

Definition 2.14. *(cf. [17]) The functional $I : X \to X^\star$ verifies the $(\mathcal{S}_+)$ property if for any weakly convergence sequence $\{u_n\}_n \subset X$ to u in X in which*

$$\limsup_{n\to\infty} < I(u_n), u_n - u > \leq 0,$$

then $\{u_n\}_n$ converges strongly to u in X.

3. Main Results

For the reader's convenience, we recall the non-smooth three critical points theorem.

Theorem 3.1. *(cf. [19]) Let X be a separable and reflexive Banach space, Λ a real interval and $\mathcal{B}$ a nonempty, closed, convex subset of X. $\phi \in C^1(X, \mathbb{R})$ a sequentially weakly l.s.c. functional and bounded on any bounded subset of X such that ϕ' is of type $(S)_+$, suppose that $\mathcal{F} : X \to \mathbb{R}$ is a locally Lipschitz functional with compact gradient. Assume that:*
(i) $lim_{\|u\|\to+\infty}[\phi - \lambda\mathcal{F}] = +\infty, \quad \forall \lambda \in \Lambda$,
(ii) There exists $\rho_0 \in \mathbb{R}$ such that

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} [\phi + \lambda(\rho_0 - \mathcal{F}(u))] < \inf_{u\in X} \sup_{\lambda\in\Lambda} [\phi + \lambda(\rho_0 - \mathcal{F}(u))].$$

Then, there exist $\lambda_1, \lambda_2 \in \Lambda$ $(\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every locally Lipschitz functional $\mathcal{G} : X \to \mathbb{R}$ with compact derivative, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ the functional $\phi - \lambda\mathcal{F} + \mu\mathcal{G}$ has at least three critical points whose norms are less than σ.

Let us introduce the following conditions of our problem.
We assume that $F : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable and satisfies the following properties:

(F_1) $|\xi| \leq K(|s|^{t(x)-1} + |s|^{z(x)-1})$ for all $\xi \in \partial F(x,s)$ with $(x,s) \in \partial\Omega \times \mathbb{R}$ $(1 \leq p^- \leq p(x) \leq p^+ < z^- \leq z(x) \leq z^+ < t^- \leq t(x) \leq t^+ < p^\partial(x))$;
(F_2) $|F(x,s)| \leq H(|s|^{\alpha(x)} + |s|^{\beta(x)})$ for all $(x,s) \in \partial\Omega \times \mathbb{R}$ $(H > 0,\ 1 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < \beta^- \leq \beta(x) \leq \beta^+ < p^- \leq p(x) \leq p^+ < p^\partial(x))$;
(F_3) there exists $\hat{u} \in W_0^{1,p(x)}(\Omega)$ such that $\int_{\partial\Omega} F(x,\hat{u})d\sigma > 0$ for a.e. $x \in \partial\Omega$,
(G) let $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ such that $|g(x,s)| \leq N(1 + |s|^{q(x)-1})$ $\forall (x,s) \in \bar{\Omega} \times \mathbb{R}$, where $q \in C(\bar{\Omega})$, $N > 0$ and $1 \leq q(x) < p^*(x)$, $\forall x \in \bar{\Omega}$.

The next lemma displays some properties of ϕ (cf. [10]).

Lemma 3.2. *Let* $\phi(u) = \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}]dx$. *Then*
(i) $\phi : X \to \mathbb{R}$ *is sequentially weakly lower semicontinuous,* $\phi \in C^1(X, \mathbb{R})$.
(ii) ϕ' *is of* (S_+) *type.*
(iii) ϕ' *is a homeomorphism.*

We need the following lemmas in the proof of our main result.

Lemma 3.3. *Let* (F_1) *be satisfied. Then* $\mathcal{F} : X \to \mathbb{R}$ *is locally Lipschitz functional with compact gradient.*

Proof: *First we prove that* $\mathcal{F}$ *is Lipschitz continuous on each bounded subset of* X. *Let* $u, v \in B(0, M)$ $(M > 0)$, *and* $\|u\|, \|v\| \geq 1$. *Utilizing Proposition 2.11 , from the Hölder inequality, and the embedding of* X *in* $L^{t(x)}(\partial\Omega)$ *and* $L^{z(x)}(\partial\Omega)$

$$\begin{aligned} |\mathcal{F}(u) - \mathcal{F}(v)| &\leq \int_{\partial\Omega} |F(x,u(x)) - F(x,v(x))| d\sigma \\ &\leq \int_{\partial\Omega} K\left(|u(x)|^{t(x)-1} + |v(x)|^{t(x)-1}\right. \\ &\quad \left. + |u(x)|^{z(x)-1} + |v(x)|^{z(x)-1}\right)|u(x) - v(x)| d\sigma \\ &\leq K\left(\|u\|^{t^+-1} + \|v\|^{t^+-1}\right)\|u - v\| + K\left(\|u\|^{z^+-1} + \|v\|^{z^+-1}\right)\|u - v\| \\ &\leq 2K\left(c_1 M^{t^+-1} + c_2 M^{z^+-1}\right)\|u - v\|, \end{aligned}$$

where c_1, c_2 *are positive constants.*
We prove that $\partial\mathcal{F}$ *is compact. Let* $\{u_n\}$ *be a sequence in* X *such that* $\|u_n\| \leq M$ *and choose* $u_n^* \in \partial\mathcal{F}(u_n)$ *for any* $n \in \mathbb{N}$. *From* (F_1) *it follows that for any* $n \in \mathbb{N}$, $v \in X$,

$$\begin{aligned} < u_n^*, v > &\leq \int_{\partial\Omega} |u_n^*(x)||v(x)| d\sigma \\ &\leq \int_{\partial\Omega} K(|u(x)|^{t(x)-1} + |u(x)|^{z(x)-1})|v(x)| d\sigma \\ &\leq (c_3 M^{t^+-1} + c_4 M^{z^+-1})\|v\|, \end{aligned}$$

where c_3, c_4 are positive constants.
Consequently,

$$\|u_n^*\|_{X^*} \leq (c_3 M^{t^+-1} + c_4 M^{z^+-1}).$$

The sequence (u_n^*) is bounded and hence, up to a subsequence, $u_n^* \rightharpoonup u^*$.
Suppose on the contrary; we assume there exists $\epsilon > 0$ for which $\|u_n^* - u^*\|_{X^*} > \epsilon$ (choose a subsequence if necessary). For every $n \in \mathbb{N}$, we can find $v_n \in X$ with $\|v_n\| < 1$ and

$$\langle u_n^* - u^*, v_n \rangle > \epsilon. \tag{3.1}$$

Then, (v_n) is a bounded sequence and up to a subsequence, $v_n \rightharpoonup v$, $\|v_n - v\|_{L^{t(x)}(\partial\Omega)} \to 0$ and $\|v_n - v\|_{L^{z(x)}(\partial\Omega)} \to 0$.
Therefore

$$\begin{aligned}\langle u_n^* - u^*, v_n \rangle &\leq \langle u_n^*, v_n - v \rangle + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \\ &\leq \int_{\partial\Omega} |u_n^*(x)||v_n(x) - v(x)| d\sigma + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle\end{aligned}$$

$$\leq K(c_3 M^{t^+-1}\|v_n - v\|_{L^{t(x)}} + c_4 M^{z^+-1}\|v_n - v\|_{L^{z(x)}}) + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \to 0,$$

which contradicts (3.1).
For $\|u\|, \|v\| \leq 1$ the proof is similar. □

Lemma 3.4. *Let G be satisfied. Then $\mathcal{G}$ is a locally Lipschitz functional with compact derivative.*

Proof: *$\mathcal{G}(u) = \int_\Omega G(x,u)dx$ is locally continuous on each bounded subset of X. Indeed, let $u, v \in B(0,M)(M > 0)$ and apply Theorem 2.5, the Hölder inequality and mean value Theorem there is a functional $\omega(x)$ in which $\int_v^u g(x,s)ds = g(x,\omega(x))(u-v)$. Then*

$$\begin{aligned}|\mathcal{G}(u) - \mathcal{G}(v)| &= |\int_\Omega G(x,u)dx - \int_\Omega G(x,v)dx| \\ &= |\int_\Omega (\int_0^u g(x,s)ds - \int_0^v g(x,s)ds)dx| = |\int_\Omega (\int_v^u g(x,s)ds)dx| \\ &\leq \int_\Omega |g(x,w(x))||(u(x)-v(x))|dx \leq (\frac{1}{p^-} + \frac{1}{p'^-})\|g(x,w(x))\|_{p(x)}\|u-v\|_{p'(x)} \\ &\leq \int_\Omega (1 + |w(x)|^{q(x)-1})^{p(x)} dx \|u-v\|_X \leq c_5(1 + M^{q^+-1})^{p^+}\|u-v\|_X,\end{aligned}$$

where c_5 is positive constant. Hence, $\mathcal{G}$ is locally Lipschitz.
It remains to show that $\mathcal{G}'$ is compact. Let $(u_n) \subset X$ be a sequence such that $u_n \rightharpoonup u$. From compact embedding of X into $L^{q(x)}(\Omega)$, we can assume up to subsequence $u_n \to u$ in $L^{q(x)}(\Omega)$. According to the Krasnoselki's theorem, the Nemytskii

operator $N_g : u(x) \to g(x, u(x))$ *is a continuous bounded operator from* $L^{q(x)}(\Omega)$ *to* $L^{\frac{q(x)}{q(x)-1}}(\Omega)$.
Using Hölder's inequality and the continuous embedding of X *to* $L^{q(x)}(\Omega)$, *it follows that*

$$|\langle D\mathcal{G}(u_n) - D\mathcal{G}(u), v\rangle| \leq 2c_6 \|N_g(u_n) - N_g(u)\|_{\frac{q(x)}{q(x)-1}} \|v\|_X.$$

This inequality shows that the operator $A : L^{\frac{q(x)}{q(x)-1}}(\Omega) \to X^*$ *defined by* $A(g(x, u)) = D\mathcal{G}(u)$ *is continuous. Then the composite operator* $D\mathcal{G} = AoN_g oI : u \to D\mathcal{G}(u)$ *from* X *into* X^* *is continuous. Therefore,* $\mathcal{G}$ *is Frèchet differentiable and its Frèchet derivative* $\mathcal{G}'(u) = D\mathcal{G}(u)$. *Hence,* $\mathcal{G}(u) \in C^1(X, \mathbb{R})$ *and* $\mathcal{G}'$ *is compact.* □

Proposition 3.5. *(cf. [4]) Let* $F : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *be a locally Lipschitz function which satisfies* (F_1). *Then* $\mathcal{F}$ *is well-defined and*

$$\mathcal{F}^0(u; v) \leq \int_\Omega F^0(x, u; v)d\sigma, \qquad \forall u, v \in X.$$

The next lemma points out the relationship between the critical points of $\mathcal{I}(u)$ and solutions of Problem (1.2).

Lemma 3.6. *Every critical point of the functional* $\mathcal{I}$ *is a solution of Problem* (1.1).

Proof: *According to the Definition 2.13,* $\mathcal{I} = \phi - \lambda\mathcal{F} + \mu\mathcal{G} + \chi$ *is a Motreanu-Panagiotopoulos type functional. Let* $u \in X$ *be a critical point of* $\mathcal{I}(u) = \phi(u) - \lambda\mathcal{F}(u) + \mu\mathcal{G} + \chi(u)$. *Then* $u \in \mathcal{B}$ *and by Definition 2.13*

$$\langle \phi' u, v - u\rangle + \lambda(-\mathcal{F})^0(u; v - u) + \mu\langle \mathcal{G}' u, v - u\rangle \geq 0, \quad \forall v \in X.$$

Using Proposition 3.5 and the property (ii) of Proposition 2.10, we obtain the desired inequality. □

Lemma 3.7. *If* (F_2) *holds, then for any* $\lambda \in (0, +\infty)$

$$\lim_{\|u\| \to +\infty} [\phi - \lambda\mathcal{F}] = +\infty. \tag{3.2}$$

Proof: *For* $u \in X$ *such that* $\|u\| \geq 1$ *and using* (F_2),

$$\mathcal{F}(u) = \int_{\partial\Omega} F(x, u)d\sigma \leq \int_{\partial\Omega} H(|u|^{\alpha(x)} + |u|^{\beta(x)})d\sigma \leq H(\|u\|^{\alpha^+}_{L^{\alpha(x)}(\partial\Omega)} + \|u\|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}).$$

By the embedding theorem for suitable positive constant c_7, c_8 *it implies that*

$$\mathcal{F}(u) \leq H(c_7\|u\|^{\alpha^+}_X + c_8\|u\|^{\beta^+}_X).$$

On the other hand from Proposition 2.7,

$$\phi(u) = \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}]dx \geq \frac{1}{p^+}\|u\|^{p^-}_X.$$

This implies that for any $\lambda > 0$,

$$\phi(u) - \lambda \mathcal{F}(u) \geq \frac{1}{p^+}\|u\|_X^{p^-} - H(c_7\|u\|_X^{\alpha^+} + c_8\|u\|_X^{\beta^+}).$$

Since $p^- > \min\{\alpha^+, \beta^+\}$, *it follows that*

$$\lim_{\|u\|\to+\infty} [\phi - \lambda\mathcal{F}] = +\infty, \quad \forall u \in X, \ \lambda \in [0, +\infty).$$

□

Theorem 3.8. *Let* Ω, p, F *be as mentioned and* F_1, F_2, F_3 *are satisfied. Then there exist* $\lambda_1, \lambda_2 > 0(\lambda_1 < \lambda_2)$ *and* $\sigma > 0$ *such that for every* $\lambda \in [\lambda_1, \lambda_2]$ *and every* $\mathcal{G}$ *satisfying* G, *there exists* $\mu_1 > 0$ *such that for every* $\mu \in]0, \mu_1[$ *problem* (1.1) *admits at least three solutions whose norms are less than* σ.

Proof: *According to Lemma 3.6, it is sufficient to prove the existence of a critical point of functional* $\mathcal{I}$. *For this, we check if* $\mathcal{I}$ *satisfies the conditions of the non-smooth three critical points Theorem 3.1. First, we note that Lemma 3.2 guarantees that* ϕ *satisfies the weakly sequentially lower semicontinuous property and* ϕ' *is of type* (S_+). *Besides, Due to Lemma (3.3), the functional* $\mathcal{F}$ *is weakly sequentially continuous.*
Lemma 3.7, implies that $\phi - \lambda\mathcal{F}$ *is coercive on* X *for all* $\lambda \in \Lambda =]0, +\infty[$; *the assumption (i) of Theorem 3.1, verified.*
For assumption (ii), let us consider two cases.
Case 1. *Let us assume that* $\|u\| < 1$.
Put for every $r > 0$,

$$\theta_1(r) = \sup\{\mathcal{F}(u); u \in X, \frac{1}{p^-}\|u\|^{p^-} \leq r\},$$

we prove that

$$\lim_{r\to 0^+} \frac{\theta_1(r)}{r} = 0. \tag{3.3}$$

In view of (F_1), *it is follows that for every* $\epsilon > 0$, *there exists* $c(\epsilon) > 0$ *such that for every* $x \in \partial\Omega, u \in \mathbb{R}$ *and* $\xi \in \partial F(x, u)$

$$|\xi| \leq \epsilon|u|^{t(x)-1} + c(\epsilon)|u|^{z(x)-1}. \tag{3.4}$$

It implies that for every $u \in X$ *by the Sobolev embedding theorem, there exist suitable positive constants* c_9 *and* c_{10}

$$\mathcal{F}(u) = \int_{\partial\Omega} F(x, u)d\sigma \leq \int_{\partial\Omega} K(|u|^{t(x)} + |u|^{z(x)})d\sigma \leq K(\|u\|^{t^+}_{L^{t(x)}(\partial\Omega)} + \|u\|^{z^+}_{L^{z(x)}(\partial\Omega)})$$

$$\leq Kc_9(\|u\|_X^{t^+} + \|u\|_X^{z^+}) \leq Kc_{10}(r^{\frac{t^+}{p^-}} + r^{\frac{z^+}{p^-}}).$$

It follows from $\min\{t^+, z^+\} > p^-$ *that*

$$\lim_{r\to 0^+} \frac{\theta_1(r)}{r} = 0.$$

From (F_3) $\hat{u} \neq 0$. *Hence, due to* (3.3), *there is* $r \in \mathbb{R}$ *in which*

$0 < r < \frac{1}{p^-}\|\hat{u}\|^{p^-}$, $\quad 0 < \frac{\theta_1(r)}{r} < \frac{\mathcal{F}(\hat{u})}{\frac{1}{p^-}\|\hat{u}\|^{p^-}}$.

Choose $\rho_0 > 0$ *such that*

$$\theta_1(r) < \rho_0 < \frac{r\mathcal{F}(\hat{u})}{\frac{1}{p^-}\|\hat{u}\|^{p^-}}, \tag{3.5}$$

especially, $\rho_0 < \mathcal{F}(\hat{u})$.

We claim that

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] < r. \tag{3.6}$$

It is obvious that the mapping

$$\lambda \mapsto \sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\phi(u) + \lambda(\rho_0 - \mathcal{F}(u))]$$

is upper semicontinuous on Λ *and*

$$\lim_{\lambda\to+\infty} \inf_{u\in\mathcal{B}} [\phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] \leq \lim_{\lambda\to+\infty} [\frac{1}{p^-}\|\hat{u}\|^{p^-} + \lambda(\rho_0 - \mathcal{F}(\hat{u}))] = -\infty.$$

Therefore, there exists $\bar{\lambda} \in \Lambda$ *in which*

$$\sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{B}} [\phi(u) + \lambda(\rho_0 - \mathcal{F}(u))] = \inf_{u\in\mathcal{B}} [\frac{1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))].$$

We consider two cases:
(I) If $\bar{\lambda}\rho_0 < r$, *we obtain*

$$\inf_{u\in\mathcal{B}} [\frac{1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \bar{\lambda}\rho_0 < r$$

(II) If $\bar{\lambda}\rho_0 \geq r$, *from* (3.5) *we obtain*

$$\inf_{u\in\mathcal{B}} [\frac{1}{p^-}\|u\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(u))] \leq \frac{1}{p^-}\|\hat{u}\|^{p^-} + \bar{\lambda}(\rho_0 - \mathcal{F}(\hat{u})) \leq$$

$$\leq \frac{1}{p^-}\|\hat{u}\|^{p^-} + \frac{r}{\rho_0}(\rho_0 - \mathcal{F}(\hat{u})) \leq r.$$

We claim that

$$\inf_{u\in\mathcal{B}}\sup_{\lambda\in\Lambda}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]\geq r. \tag{3.7}$$

Infact, for every $u\in\mathcal{B}$ *there are two cases:*
(I) *If* $\mathcal{F}(u)<\rho_0$,

$$\sup_{\lambda\in\Lambda}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]=+\infty.$$

(II) *If* $\mathcal{F}(u)\geq\rho_0$, *by* (3.5)

$$\sup_{\lambda\in\Lambda}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]=\phi(u)\geq\frac{1}{p^+}\|u\|^{p^+}\geq r.$$

From (3.6), (3.7) *and the assumption* (ii) *of Theorem* 3.1, *this case verified.*

Case 2. *Assume that* $\|u\|>1$.
In a similar way like the case 1:
Put for every $r>0$

$$\theta_2(r)=\sup\{\mathcal{F}(u);u\in X,\frac{1}{p^-}\|u\|^{p^+}\leq r\}.$$

We claim that

$$\lim_{r\to0^+}\frac{\theta_2(r)}{r}=0. \tag{3.8}$$

In order to (3.4), *for every* $u\in X$ *for continuous and compact embedding, it implies the existence of* c_{11} *and* c_{12} *such that*

$$\mathcal{F}(u)=\int_{\partial\Omega}F(x,u)d\sigma\leq\int_{\partial\Omega}K(|u|^{t(x)}+|u|^{z(x)})d\sigma\leq K(\|u\|^{t^+}_{L^{t(x)}(\partial\Omega)}+\|u\|^{z^+}_{L^{z(x)}(\partial\Omega)})$$

$$\leq Kc_{11}(\|u\|^{t^+}_X+\|u\|^{z^+}_X)\leq Kc_{12}(r^{\frac{t^+}{p^+}}+r^{\frac{z^+}{p^+}}).$$

It follows from $\min\{t^+,z^+\}>p^+$ *that*

$$\lim_{r\to0^+}\frac{\theta_2(r)}{r}=0.$$

From (F_3) $\hat{u}\neq0$, *so, due to* (3.3), *there is some* $r\in\mathbb{R}$ *such that*

$0<r<\frac{1}{p^-}\|\hat{u}\|^{p^+}$, $\qquad 0<\frac{\theta_2(r)}{r}<\frac{\mathcal{F}(\hat{u})}{\frac{1}{p^-}\|\hat{u}\|^{p^+}}$.

Let $\rho_0>0$ *such that*

$$\theta_2(r)<\rho_0<\frac{r\mathcal{F}(\hat{u})}{\frac{1}{p^-}\|\hat{u}\|^{p^+}}. \tag{3.9}$$

We claim that

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{B}}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]<r. \tag{3.10}$$

Because of the mapping

$$\lambda\mapsto\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{B}}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]$$

is upper semicontinuous on Λ, *so*

$$\lim_{\lambda\to+\infty}\inf_{u\in\mathcal{B}}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]\le\lim_{\lambda\to+\infty}[\frac{1}{p^-}\|\hat{u}\|^{p^+}+\lambda(\rho_0-\mathcal{F}(\hat{u}))]=-\infty.$$

Therefore, there exists $\bar{\lambda}\in\Lambda$

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{B}}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]=\inf_{u\in\mathcal{B}}[\frac{1}{p^-}\|u\|^{p^+}+\bar{\lambda}(\rho_0-\mathcal{F}(u))].$$

We consider two cases:
(I) *If* $\bar{\lambda}\rho_0<r$, *we obtain*

$$\inf_{u\in\mathcal{B}}[\frac{1}{p^-}\|u\|^{p^+}+\bar{\lambda}(\rho_0-\mathcal{F}(u))]\le\bar{\lambda}\rho_0<r.$$

(II) *If* $\bar{\lambda}\rho_0\ge r$, *from* (3.9) *we obtain*

$$\inf_{u\in\mathcal{B}}[\frac{1}{p^-}\|u\|^{p^+}+\bar{\lambda}(\rho_0-\mathcal{F}(u))]\le\frac{1}{p^-}\|\hat{u}\|^{p^+}+\bar{\lambda}(\rho_0-\mathcal{F}(\hat{u}))\le$$

$$\le\frac{1}{p^-}\|\hat{u}\|^{p^+}+\frac{r}{\rho_0}(\rho_0-\mathcal{F}(\hat{u}))\le r.$$

Moreover, we claim that

$$\inf_{u\in\mathcal{B}}\sup_{\lambda\in\Lambda}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]\ge r. \tag{3.11}$$

For every $u\in\mathcal{B}$ *two cases can occure:*
(I) *If* $\mathcal{F}(u)<\rho_0$ *we have*

$$\sup_{\lambda\in\Lambda}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]=+\infty.$$

(II) *If* $\mathcal{F}(u)\ge\rho_0$ *we have by* (3.9)

$$\sup_{\lambda\in\Lambda}[\phi(u)+\lambda(\rho_0-\mathcal{F}(u))]=\phi(u)\ge\frac{1}{p^+}\|u\|^{p^-}\ge r.$$

From (3.10), (3.11) *and the assumption* (ii) *of Theorem* 3.1, *this case verified.*

For function $\mathcal{G}$ which satisfies (G), it follows from Lemma 3.4, that the functional $\mathcal{G}: X \to \mathbb{R}$ is locally Lipschitz with weakly sequentially continuous. Then according to Theorem 3.1 there exist $\lambda_1, \lambda_2 \in \Lambda$ (without loss of generality we may assume $0 < \lambda_1 < \lambda_2$) and $\sigma > 0$ with the following property that, for $\lambda \in [\lambda_1, \lambda_2]$ there exists $\mu_1 > 0$ in which: for every $\mu_1 \in]0, \mu[$, the functional $\phi - \lambda\mathcal{F} + \mu\mathcal{G}$ admits at least three critical points $u_0, u_1, u_2 \in \mathcal{B}$ with $\|u_i\| < \sigma$. So by Lemma 3.6 u_0, u_1, u_2 are three solutions of the problem (1.1). □

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