Remarks on Heron's cubic root iteration formula

S.K. Padhan*, S. Gadtia† and A.K. Pattanaik‡

*,†Department of Mathematics, ‡ Department of Physics

Veer Surendra Sai University of Technology, Burla-768018, India

Abstract

The existence as well as the computation of roots appears in number theory, algebra, numerical analysis and other areas. The present study illustrate the contributions of several authors towards the extraction of different order roots of real number. Different methods with several approaches are studied to extract the roots of real number. Some of the methods described earlier are equivalent as observed in the present study. Heron developed a general iteration formula to determine the cube root of a real number N i.e. $\sqrt[3]{N} = a + \dfrac{bd}{bd + aD}(b - a)$, where $a^3 < N < b^3$, $d = N - a^3$ and $D = b^3 - N$. Although the direct proof of the above method is not available in literature, some authors have proved the same with the help of conjectures. In the present investigation, the proof of Heron's method is explained and is generalized for any odd order roots. Thereafter it is observed that Heron's method is a particular case of the generalized method.

Keyword: Cube root; Higher order roots; Heron's method. Mathematics Subject Classifications 2010: 01A30; 01A35; 11A07.

1 Introduction

Recently, attempts have been made by many to find the cube and higher order roots of a real number in various methods with different approaches. Heron's iteration formula to determine the cube root of a number N was $\sqrt[3]{N} = a + \dfrac{bd}{bd + aD}(b - a)$, where $a^3 < N < b^3$, $d = N - a^3$ and $D = b^3 - N$ as stated by Deslauriers and Dubuc [13]. According to Heath [16], a conjecture on Heron's cubic root iteration formula was made by Wertheim [37] taking $b = a + 1$. Assuming some elementary considerations, Eneström [14] proved the Wertheim's conjecture. Again Taisbak [34] made a conjecture about Heron's method and provided possible proofs for it with the help of difference operators. Many researchers like Hess [17], Taisbak [35], Crisman and Veatch [11] are also continuing their research on Heron's work. Recently, Gadtia *et al.* [15] extended the Wertheim [37] conjecture to find the odd order roots of a number and suggested possible proofs for even order roots. Again they showed that Al-Samawal's and Lagrange's method are equivalent for nth root extraction of a real number.

A direct proof of Heron's general cubic root iteration formula is provided and it is extended for any odd order roots in the present work. Further, many counterexamples are discussed in support of the work. In addition to above some contributions made by different authors for extraction of roots are described.

2 Historical Background

In the middle of the tenth century, the book written by al-Udl$\bar{i}$dis$\bar{i}$ (Book of chapters on Hindu arithmetic, kit$\bar{a}$b al-fus$\bar{u}$l f$\bar{i}$ al-his$\bar{a}$b al-hind$\bar{i}$ [2]) described the earliest work in Arabic that treats the computation of cube roots. The oldest description of the extraction of cube roots was also found in China, in the

*Corresponding author. E-mail: skpadhan_math@vssut.ac.in

classical work of the Nine chapters on the Mathematical Art which was translated by Chemla and Guo [8] in French and Kangshen *et al.* [21] in English. Later, Jia Xian used five rows in computation of cube root of a real number in the early eleventh century. Jia's algorithm differs from the Nine chapters in surface structure. Again Jia gave two methods for cube root extraction and one method for fourth root extraction as quoted by Yang Hui in 1261.

The work presented in Principles of Hindu Reckoning (kit$\bar{a}$b al-fus$\bar{u}$l f$\bar{i}$ al-his$\bar{a}$b al-hind$\bar{i}$) written c.1000 by the Persian K$\bar{u}$shy$\bar{a}$r ibn Labb$\bar{a}$n, The Sufficient on Hindu Calculation (al-Muqni f$\bar{i}$ al-his$\bar{a}$b al-hind$\bar{i}$) by Al$\bar{i}$ ibn Ahmad al-Nasaw$\bar{i}$(text written before 1030 [3]) and The Completion of Arithmetic (al-Takmila f$\bar{i}$ al-his$\bar{a}$b) by Abd al-Q$\bar{a}$hir ibn T$\bar{a}$hir al-Baghd$\bar{a}$d$\bar{i}$ were roughly contemporary with Jia Xian. K$\bar{u}$shy$\bar{a}$r's work was translated into English by Levey and Petruck [23]. Al-Nasaw$\bar{i}$'s text on the cube root was translated into German by Paul [28] and the work by Ibn T$\bar{a}$hir was edited by Saidan [3]. The generalization of the algorithm for higher order roots was known in the twelfth century by al-Samawal al-Maghribi (Rashed[30]) and also by Nas$\bar{i}$r al-D$\bar{i}$n al-T$\bar{u}$s$\bar{i}$ in the thirteenth century [1]. Fifth and fourth order root of a number was formulated by Al-Samawal (1172) and Nasir al-Din al-T$\bar{u}$s$\bar{i}$ (1265), respectively using sexagesimal system. The algorithm described by Al-T$\bar{u}$s$\bar{i}$ was very close to the earlier proposed algorithm of K$\bar{u}$shy$\bar{a}$r. Their procedures coincide completely with the methods given by Jia Xian and Al-Samawal. This approach was later followed by N$\bar{i}$z$\bar{a}$m al-D$\bar{i}$n al- N$\bar{i}$s$\bar{a}$b$\bar{u}$r$\bar{i}$, who described the extraction of cube and higher roots elaborately in "The Epistle on Arithmetic", arround fourteenth century [24].

The earliest works on arithmetic from the Maghreb and Muslim Spain came from the twelfth and thirteenth centuries. During this period Al-Hass$\bar{a}$r, Ibn al-Y$\bar{a}$sam$\bar{i}$n and Ibn Mun'im all included cube extraction in some of their treatises. With the help of complete binomial expansions Ibn Mun'im extracted the fifth and seventh roots of a number. The fifth root of a 13 digits number was extracted by a Mathematician in Kairouan, Tunisia, before 1241 as described by Rashed [31] using binomial expansion. According to Lamrabet [22], Ibn al-Y$\bar{a}$sam$\bar{i}$n proposed an algorithm for the extraction of cube root on the development of $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In 1427, Jamshid al-Kashi extracted the fifth root of a decimal number and sixth root of a sexagesimal number. The same algorithm was described by al-Qatraw$\bar{a}$n$\bar{i}$ [22] in fifteenth century.

Later, many researchers have studied the cube and other roots of a real number in different ways (Ruffini [32], Horner [18, 19]). For the first time Paul [28] studied, the extraction of higher roots in Arabic (Islamic) mathematics, which was focused on the work of Jamshid al-Kashi. Burr [6] developed several iteration methods for computing cube roots, when a fast square root was available. Some methods are superior to the conventional Newton's method in particular situations was observed by him. Padro and Saez [26] generalized the algorithms established by Shanks [33] and Peralta [29] for computing square roots modulo of a prime to algorithms for computing cube roots, which played an important role in cryptosystems. Ahmadi *et al.* [4] calculated the number of nonzero coefficients (Hamming weight) in the polynomial representation of $x^{\frac{1}{3}}$ in $\mathbb{F}_3[x]/(f)$, where $f \in \mathbb{F}_3[x]$ is an irreducible trinomial. Cho *et al.* [10] found that the shifted polynomial basis and variation of polynomial basis reduces Hamming weight of $x^{\frac{1}{3}}$ and $x^{\frac{2}{3}}$ and suggested a suitable shifted polynomial basis which was eliminating modular reduction process in cube roots computation. Tapia [36] presented an algorithm which extracts the principal *Q-th* root of any real number for any given number system, which was an extension of the square root algorithm. Newton's method is a first way to compute *nth* root of a number and has quadratic convergence. Chen and Hsieh [9] proposed a new class of iterative methods based on Pade approximation to Taylor's series of the differentiable function for computing the roots which was faster than Newton's method. The fourth degree algorithm in the work of Chen and Hsieh [9] converges two times faster than Newton's method. Dubeau [12] analyzed a double iteration process to find *nth* root of a positive real number which was equivalent to the Newton's method. Higher order methods were also mentioned for finding *nth* roots. The study of Johansson [20] supported the hypotheses formulated by Paul [28] and Chemla [7] on an early scientific connection between China and Persia. Parakh [27] observed that, the methods adopted today for teaching in schools are the extension of Aryabhata's root extraction methods.

Recently, Padhan *et al.* [25] gave a general method to determine the cube and higher order roots of any real number, which is very simple and similar to the method proposed by Black [5]. This method has better performance in delay, area and power consumption than the method adopted by Black [5] while implementing on FPGA.

3 Iteration formulae for odd order roots

In this section, the Heron's cubic root iteration formula is proved and extended this for any odd order roots.

3.1 Heron's root iteration formula

Let us state the Heron's cubic root iteration formula as Theorem 3.1.

Theorem 3.1 If $a^3 < N < b^3$, then cube root of N is defined as

$$\sqrt[3]{N} = a + \frac{bd}{bd + aD}(b - a),$$

where $d = N - a^3$ and $D = b^3 - N$.

Proof: Let x be the cube root of N. Assume that $(x - a)^3 = \delta_1$ and $(b - x)^3 = \delta_2$.

$$\begin{aligned} & (x-a)^3 = \delta_1 \\ \Rightarrow\ & x^3 - 3x^2a + 3xa^2 - a^3 = \delta_1 \\ \Rightarrow\ & 3x^2a - 3xa^2 = x^3 - a^3 - \delta_1 \\ \Rightarrow\ & 3xa(x-a) = d - \delta_1, \text{ where } d = x^3 - a^3. \end{aligned} \tag{3.1}$$

Similarly, taking $D = b^3 - x^3$, we get

$$3bx(b - x) = D - \delta_2. \tag{3.2}$$

Dividing eqn. (3.2) by eqn. (3.1), we have

$$\begin{aligned} \frac{D - \delta_2}{d - \delta_1} &= \frac{3bx(b-x)}{3xa(x-a)} \\ &= \frac{b(b-x)}{a(x-a)}. \end{aligned} \tag{3.3}$$

As the value of δ_1 and δ_2 are very small, from eqn. (3.6), we get

$$\begin{aligned} & \frac{D}{d} = \frac{b(b-x)}{a(x-a)} \\ \Rightarrow\ & \frac{aD}{bd} = \frac{b-a}{x-a} - 1 \\ \Rightarrow\ & x - a = \frac{bd}{bd + aD}(b - a) \\ \Rightarrow\ & x = a + \frac{bd}{bd + aD}(b - a) \\ \Rightarrow\ & \sqrt[3]{N} = a + \frac{bd}{bd + aD}(b - a). \end{aligned}$$

$\square$

3.2 Generalization of Heron's root iteration formula

In the Theorem 3.2, the general formula for any odd order root of a number N is discussed.

Theorem 3.2 If $a^n < N < b^n$, $n = 2m+1$, then approximate nth root of N is defined as

$$\sqrt[n]{N} = a + \frac{b^m d}{b^m d + a^m D}(b-a),$$

where $d = N - a^n$ and $D = b^n - N$.

Proof: Let x be the nth root of N. Assume that $(x-a)^{2m+1} = \delta_1$ and $(b-x)^{2m+1} = \delta_2$.

$$\begin{aligned}
& (x-a)^{2m+1} = \delta_1 \\
\Rightarrow\ & x^{2m+1} - (2m+1)_{C_1} x^{2m} a + (2m+1)_{C_2} x^{2m-1} a^2 - ... + (2m+1)_{C_{2m}} x a^{2m} - a^{2m+1} = \delta_1 \\
\Rightarrow\ & (2m+1)_{C_1} x^{2m} a - (2m+1)_{C_2} x^{2m-1} a^2 + ... - (2m+1)_{C_{2m}} x a^{2m} = x^{2m+1} - a^{2m+1} - \delta_1 \\
\Rightarrow\ & (2m+1)_{C_1} xa(x^{2m-1} - m x^{2m-2} a + ... - a^{2m-1}) = d - \delta_1, \qquad (3.4) \\
& \text{where } d = x^{2m+1} - a^{2m+1}.
\end{aligned}$$

Similarly, taking $b^{2m+1} - x^{2m+1} = D$, we have

$$(2m+1)_{C_1} bx\{b^{2m-1} - m b^{2m-2} x + ... - x^{2m-1}\} = D - \delta_2. \qquad (3.5)$$

Dividing eqn. (3.5) by eqn. (3.4), we get

$$\begin{aligned}
\frac{D-\delta_2}{d-\delta_1} &= \frac{(2m+1)_{C_1} bx\{b^{2m-1} - m b^{2m-2} x + ... - x^{2m-1}\}}{(2m+1)_{C_1} xa(x^{2m-1} - m x^{2m-2} a + ... - a^{2m-1})} \\
&= \frac{b\{b^{2m-1} - m b^{2m-2} x + ... - x^{2m-1}\}}{a(x^{2m-1} - m x^{2m-2} a + ... - a^{2m-1})} \\
& \quad \big(\text{neglecting the very small terms } (x-a)^3,\ (x-a)^5, ..., (x-a)^{2m-1} \\
& \quad \text{ and } (b-x)^3,\ (b-x)^5, ..., (b-x)^{2m-1} \text{ and simplifying } \big) \\
&= \frac{b^m}{a^m}\Big(\frac{b-a}{x-a} - 1\Big). \qquad (3.6)
\end{aligned}$$

As the values of δ_1 and δ_2 are very small, from eqn. (3.6), we get

$$\begin{aligned}
& \frac{D}{d} = \frac{b^m}{a^m}\Big(\frac{b-a}{x-a} - 1\Big) \\
\Rightarrow\ & \frac{b-a}{x-a} = \frac{a^m D}{b^m d} + 1 \\
\Rightarrow\ & x - a = \frac{b^m d}{b^m d + a^m D}(b-a) \\
\Rightarrow\ & x = a + \frac{b^m d}{b^m d + a^m D}(b-a) \\
\Rightarrow\ & \sqrt[n]{N} = a + \frac{b^m d}{b^m d + a^m D}(b-a).
\end{aligned}$$

□

Remark 3.1 When $m = 1$, Theorem 3.2 reduces to Heron's cubic root iteration formula.

Example 3.1 Evaluation of $5th$ root of 100.
It is clear that $2.5^5 < 100 < 2.6^5$ that is $97.65625 < 100 < 118.81376$. According to Theorem 3.2,

$a = 2.5,\ b = 2.6,\ d = 100 - 97.65625 = 2.34375,\ D = 118.81376 - 100 = 18.81376.$
Therefore,

$$\begin{aligned}\sqrt[5]{100} &= 2.5 + \frac{2.6^2 \times 2.34375}{2.6^2 \times 2.34375 + 2.5^2 \times 18.81376}(2.6 - 2.5) \\ &= 2.5118742259504.\end{aligned}$$

It can be easily verified that $(2.5118742259504)^5 = 99.997572463296$ and the error is very minimum that is 0.002429536704.

Example 3.2 Evaluation of $7th$ root of 100.
It is clear that $1.9^7 < 100 < 2^7$ that is $89.3871739 < 100 < 128$. According to Theorem 3.2, $a = 1.9,\ b = 2,\ d = 100 - 89.3871739 = 10.6128261,\ D = 128 - 100 = 28.$
Therefore,

$$\begin{aligned}\sqrt[7]{100} &= 1.9 + \frac{2^3 \times 10.6128261}{2^3 \times 10.6128261 + 1.9^3 \times 28}(2 - 1.9) \\ &= 1.9306557847757.\end{aligned}$$

It can be easily checked that $(1.9306557847757)^7 = 99.984793600089$ and the error is 0.015206399911.

Remark 3.2 The error can be made as minimum as required by taking the values of a and b closed enough to the root.

4 Conclusion

A direct proof of Heron's general cubic root iteration formula is described and extended for any odd order roots. It is observed that the Heron's general cubic root iteration formula is a particular case of the present study. Counterexamples are discussed in support of the present investigation.

References

[1] A.S. Saidan, Nas$\bar{i}$r al-D$\bar{i}$n al-T$\bar{u}$s$\bar{I}$: Jaw$\bar{a}$mi‘ al-his$\bar{a}$b bi al-takht wa al-tur$\bar{a}$b. (The Comprehensive Work on Computation with Board and Earth of Nasir al-Din al-Tusi). Al-Abhath, XX (2) 91-163, (1967) 213-292.

[2] A.S. Saidan, The Arithmetic of Al-Uql$\bar{i}$dis$\bar{i}$: The Story of HinduArabic Arithmetic as Told in Kit$\bar{a}$b al-Fus$\bar{u}$l f$\bar{i}$ al-His$\bar{a}$b al-Hind$\bar{i}$. Reidel, Dordrecht, 1978.

[3] A.S. Saidan, ‘Abd al-Q$\bar{a}$hir Ibn T$\bar{a}$hir al-Baghd$\bar{a}$d$\bar{I}$: Al-Takmila f$\bar{i}$ al-his$\bar{a}$b. Mansh$\bar{u}$r$\bar{a}$t Ma‘had al-Makht$\bar{u}$t$\bar{a}$t al-‘Arabiyya, Kuwait, 1985.

[4] O. Ahmadi, D. Hankerson and A. Menzes, Formulas for cube roots in $\mathbb{F}_{3^m}$, Disdrete Applied Mathematics, 155 (2007) 260-270.

[5] P.E. Black, cube root, in dictionary of algorithms and data structures, Vreda Pieterse and Paul E. Black, eds., 2009.

[6] S.A. Burr, Computing cube roots when a fast square root is available, Computer and Mathematics with Applications, 8 (1982) 181-183.

[7] K. Chemla, Similarities between Chinese and Arabic mathematical writings. I. Root extraction, Arabic Sciences and Philosophy, 4 (1994) 207-266.

[8] K. Chemla and S. Guo, Les neuf chapitres. Le Classique mathmatique de la Chine ancienne et ses commentaires. Dunod, Paris, 2004.

[9] S.G. Chen and P.Y. Hsieh, Fast computation of the nth root, Computer and Mathematics with Applications, 17 (1989) 1423-1427.

[10] Y.I. Cho, N.S. Chang and S. Homg, Formulas for cube roots in $\mathbb{F}_{3^n}$ using shifted polynomial basis, Information Processing Letters, 114 (2014) 331-337.

[11] K.D. Crisman and M.H. Veatch, Reinventing Heron, The college Mathematics Journal, 45 (2014) 191-197.

[12] F. Dubeau, nth root extraction: double iteration process and Newton's method, Journal of Computational and Applied Mathematics, 91 (1998) 191-198.

[13] G. Deslauriers and S. Dubuc, Le calcul de la racine cubique selon Hron, Elemente der Mathematik, 51 (1996) 28-34.

[14] G. Eneström, Bibliotheca Mathematica, VIII (1907-8) 412-413.

[15] S. Gadtia, S.K. Padhan and R.N. Mohapatra, Extension of Heron's cubic root iteration method, Personally communicated, 2014.

[16] T. Heath, A history of Greek Mathematics, Vol.-II, 1923.

[17] A. Hess, A highway from Heron to Brahmagupta, Forum Geom, 12 (2012) 191-192.

[18] W. Horner, On popular methods of approximation, Mathematical Repository, 4 (1819) 131-136.

[19] W. Horner, A new method of solving numerical equations of all orders, by continuous approximation, Philosophical Transactions of the Royal Society of London, Part II (1819), 308-335.

[20] B.G. Johansson, Cube root extraction in medieval mathematics, Historia Mathematica, 38 (2011) 338-367.

[21] S. Kangshen, C. John and L. Anthony, The Nine Chapters on the Mathematical Art. Oxford University Press and Science Press, Beijing, 1999.

[22] D. Lamrabet, Introduction á l'Histoire des Mathématiques maghrébines, Rabat, 1994.

[23] M. Levey and M. Petruck, Kūshyār Ibn Labban. Principles of Hindu Reckoning. The University of Wisconsin Press, Madison and Milwaukee, 1965

[24] R.G. Morrison, Islam and Science. The intellectual carrier of Nīzām al-Dīn al- Nīsābūrī, Routledge, London and New York, 2007.

[25] S.K. Padhan, S. Gadtia and B. Bhoi, FPGA based implementation for extracting the roots of real number, Personally communicated, 2014.

[26] C. Padro and G. Saez, Taking cube roots in $\mathbb{Z}_m$, Applied Mathematics Letters, 15 (2002) 703-708.

[27] A. Parakh, $\bar{A}$ryabhata's root extraction methods, Indian Journal of History of Science, 42.2 (2007) 149-161.

[28] L. Paul, Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der isla-mischen Mathematik, Mathemathische Annalen, 120 (1948) 217-274.

[29] R.C. Peralta, A simple and fast probabilistic algorithm for computing square roots modulo a prime number, IEEE transactions on information theory, 6 (1986) 846-847.

[30] R. Rashed, L'Extraction de la racine $n^{iéme}$ et l'invention des fraction d$\acute{e}$cimales (XI^e XII^e si$\acute{e}$cles)., Arcive for History of Exact Sciences, 18 (1978) 191-243.

[31] R. Rashed, Encyclopedia of the history of Arabic science, Routledge, London, 2 (1996) 387-388.

[32] P. Ruffini, Sopra la determinazione delle radici nelle equazioni numeriche di qualunque grado, Modena: Societa' Italiana delle Scienze, 1804.

[33] D. Shanks, Five number theoritical algorithms, In proceeding Second Manitoba Conference on Numerical Mthematics, University of Manitoba, Winnipeg, Manitoba, Canada, 1972.

[34] C.M. Taisbak, Cube roots of integers. A conjecture about Herons method in Metrika III. 20, Historia Mathematica, 41 (2014) 103-106

[35] C.M. Taisbak, An algorithm for extracting the principal $Q-th$ root of any real number to any positive base, Natural and Applied Science Bulletin, 38 (1986) 317-325.

[36] C.G. Tapia, An Archimedean proof of Heron's formula for the area of a triangle: heuristics reconstructed. From Alexandria, through Bagdad, Springer, Heidelberg, (2014) 189-198.

[37] G. Wertheim, Zeitschr. f. Math. u. Physik, Hist. Lit. Abt., XLIV (1899) 1-3.