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Common Fixed Point Theorems Without Continuity and Compatible Property of Maps

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abstract: In this paper, without assuming continuity, commutativity and compatibility of self maps, some common fixed point theorems for weak contraction of integral type in complete metric spaces are proved. An example and some remarks are also given to justify that our contraction is new and weaker than other existing contractions.

Key Words: Common fixed point, continuity, compatibility, integral type.

Contents

1. Introduction

Fixed-point theory is one of the most fruitful and effective tool in mathematics. It has number of applications within as well as outside the mathematics. The first important result on fixed point for contractive type mapping is the Banach contraction principle [13].

Theorem 1.1. *[13] Let (X, d) be a complete metric space, $\alpha \in (0, 1)$, and F be a self-maps of X such that for all $x, y \in X$,*

$$d(Fx, Fy) \leq \alpha d\,(x, y)\,,$$

then F has a unique fixed point $z \in X$.

This principle is a forceful tool in nonlinear analysis. It has many applications in solving nonlinear equations.

In 1975, Dass and Gupta [2] proved an extension of the Banach contraction principle through rational expression.

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Theorem 1.2. *[2] Let (X, d) be a metric space and F be a self-maps of X such that*

$$d(Fx, Fy) \leq \frac{\alpha d\,(y, Fy)\,[1 + d(x, Fx)]}{[1 + d(x, y)]} + \beta d(x, y)$$

for all $x, y \in X$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$, and for some $x_0 \in X$, the sequence of iterates $\{F^n(x_0)\}$ has a subsequence $\{F^{n_k}(x_0)\}$ with $z = \lim F^{n_k}(x_0)$. Then F has a unique fixed point $z \in X$.

After this result, several authors have proved common fixed point theorems for contractive type conditions satisfying rational inequalities (see [3,4,6,12]). In 2002, Branciari [1] introduced an integral type contractive mapping to analyze the existence of fixed points for self mappings and then generalized the result of Banach [13].

Theorem 1.3. *[1] Let (X, d) be a complete metric space and F be a self-maps of X such that for each $x, y \in X$*

$$\int_0^{d(Fx,Fy)} \varphi(t)dt \leq \beta \int_0^{d(x,y)} \varphi(t)dt,$$

where $\beta \in (0, 1)$, and $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable function which is summable on each compact subset of R^+ and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

then F has a unique fixed point $z \in X$ such that for each $w \in X$, $F^n w \to z$ as $n \to +\infty$.

In 2003, Rhoades [5] extended the result of Branciari [1] by replacing more stronger contractive condition. Since then lot of work have been done by various authors satisfying a general contractive condition of integral type. Some of them are noted in ([1,5,11,16,18,19,20,10,15]). In 2011, Samet and Yazidi [6] proved the following fixed point theorem, as an extension of the result of Dass and Gupta [2], satisfying integral type rational inequalities in Hausdorff spaces.

Theorem 1.4. *[6] Let X be a Hausdorff space and $H : X \times X \to [0, +\infty)$ be a continuous mapping such that*

$$H(x, y) \neq 0, \quad \forall \quad x, y \in X \quad \text{and} \quad x \neq y.$$

Let F be self-maps of X satisfying the contractive condition such that for each $x, y \in X$

$$\int_0^{H(Fx,Fy)} \varphi(t)dt \leq \alpha \int_0^{M(x,y)} \varphi(t)dt + \beta \int_0^{H(x,y)} \varphi(t)dt,$$

and

$$M\left(x,y\right)=\frac{H\left(y,Fy\right)\left[1+H(x,Fx)\right]}{\left[1+H(x,y)\right]},$$

where $\alpha,\beta>0$ *are constants such that* $\alpha+\beta<1$ *and* $\varphi:[0,\infty)\to[0,\infty)$ *is a Lebesgue-integrable function which is summable on each compact subset of* R^+*, and such that for each* $\epsilon>0$,

$$\int_0^{\epsilon}\varphi(t)dt>0,$$

then F *admits a fixed point* z *if for some* $x_0\in X$ *the sequence of iterates* $\{F^n x_0\}$ *has a subsequence* $\{F^{n_k}x_0\}$ *converging to* $z\in X$.

In 1976, Jungck [7] introduced the concept of commuting maps. He also generalized the Banach fixed point theorem from single self map to two self maps. In 1982, Sessa [14] defined the concept of weak commutativity and proved some common fixed point theorems. Again in 1988, Jungck [8] generalized this idea, first to compatible mappings [8] and then to weakly compatible mappings [9]. There are many examples which shows that these generalizations of commutativity are proper extension of the previous definitions.

The main aim of our paper is to give a new contraction to prove some common fixed point theorems for pair of self-maps satisfying integral type contractive condition without using the concept of continuity, commutativity and compatibility. An example is given at the end to show that the contractions used in main results are weaker than the previous existing contractions.

2. Main results

The following lemma plays important role in this paper.

Lemma 2.1. *Let* $\varphi:[0,\infty)\to[0,\infty)$ *be a Lebesgue-integrable function which is summable on each compact subset of* R^+*, and such that for each* $\epsilon>0$, $\int_0^{\epsilon}\varphi(t)dt>0$ *and* $\left(d_n\right)_{n\in N}$ *be non- negative sequence with* $\lim_{n\to\infty}d_n=a$*, then*

$$\lim_{n\to\infty}\int_0^{d_n}\varphi(t)dt=0\quad iff\quad \lim_{n\to\infty}d_n=0.$$

Proof: Follows directly from Lemma - 2.1 in Mocanu and Popa [17]. □

Now we prove our main result.

Theorem 2.2. *Let* (X,d) *be a complete metric space, and* F *and* G *be self-maps of* X *such that for each* $x,y\in X$

$$\int_0^{d(Fx,Gy)}\varphi(t)dt\le\alpha\int_0^{M(x,y)}\varphi(t)dt+\beta\int_0^{d(x,y)}\varphi(t)dt \tag{2.1}$$

and

$$M\,(x,y)=\frac{d\,(y,Gy)\,[1+d(x,Fx)]}{[1+d(x,y)]}, \tag{2.2}$$

where $\alpha,\beta>0$ *are constants such that* $\alpha+\beta<1$ *and* $\varphi:[0,\infty)\to[0,\infty)$ *is a Lebesgue-integrable function which is summable on each compact subset of* R^+ *and such that for each* $\epsilon>0$,

$$\int_0^{\epsilon}\varphi(t)dt>0, \tag{2.3}$$

then F *and* G *have a unique common fixed point.*

Proof: Choose $x_0\in X$ such that $Fx_0=x_1$ and $Gx_1=x_2$. Define a sequence $\{x_n\}$ in X such that $x_{2n+1}=Fx_{2n}$ and $x_{2n+2}=Gx_{2n+1}$, where $n=0,1,2,\cdots$. Let $\{y_n\}=\int_0^{d(Fx_{2n},Gx_{2n+1})}\varphi(t)dt$.
Consider,

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt&=\int_0^{d(Fx_{2n},Gx_{2n+1})}\varphi(t)dt\\&\le\alpha\int_0^{M(x_{2n},x_{2n+1})}\varphi(t)dt+\beta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt.\end{aligned} \tag{2.4}$$

From (2.2)

$$M\,(x_{2n},x_{2n+1})=\frac{d(x_{2n+1},Gx_{2n+1})\,[1+d\,(x_{2n},Fx_{2n})]}{[1+d\,(x_{2n},x_{2n+1})]}=d(x_{2n+1},x_{2n+2}). \tag{2.5}$$

Hence from (2.4),

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt&\le\alpha\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt+\beta\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt\\&\le\frac{\beta}{1-\alpha}\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt\\&<\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt,\end{aligned}$$

continuing this way we get,

$$\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt<\int_0^{d(x_{2n},x_{2n+1})}\varphi(t)dt<\cdots<\int_0^{d(x_0,x_1)}\varphi(t)dt.$$

It follows that $\{y_n\}$ is a monotone decreasing and lower bounded sequence of numbers, and consequently there exists a $r\ge 0$ such that

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}\int_0^{d(x_{2n+1},x_{2n+2})}\varphi(t)dt=r. \tag{2.6}$$

Suppose $r > 0$. Taking limit as $n \to \infty$ on both side of (2.4), we get

$$r \leq \alpha r + \beta \lim_{n\to\infty} \inf \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \leq \alpha r + \beta r < r,$$

this is a contradiction. Therefore $r = 0$.
Hence from (2.6),

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt = 0. \tag{2.7}$$

Now we prove that $\{y_n\}$ is a Cauchy sequence. To prove this, suppose that $\{y_n\}$ is not a Cauchy sequence. Therefore there exists $\epsilon > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$ and

$$d(y_{m_i}, y_{n_i}) \geq \epsilon \quad and \quad d(y_{m_i}, y_{n_{i-1}}) < \epsilon. \tag{2.8}$$

Consider,

$$\int_0^{\epsilon} \varphi(t)dt \leq \lim_{i\to\infty} \int_0^{d(y_{m_i},y_{n_i})} \varphi(t)dt \leq \lim_{i\to\infty} \int_0^{d(y_{m_i},y_{n_{i-1}})+d(y_{n_{i-1}},y_{m_i})} \varphi(t)dt$$
$$< \lim_{i\to\infty} \int_0^{\epsilon+d(y_{n_{i-1}},y_{m_i})} \varphi(t)dt < \int_0^{\epsilon} \varphi(t)dt.$$

Thus,

$$\lim_{i\to\infty} \int_0^{d(y_{m_i},y_{n_i})} \varphi(t)dt = \int_0^{\epsilon} \varphi(t)dt. \tag{2.9}$$

By using triangle inequality,

$$d(y_{m_{i-1}}, y_{n_{i-1}}) \leq d(y_{m_{i-1}}, y_{m_i}) + d(y_{m_i}, y_{n_i}) + d(y_{n_i}, y_{n_{i-1}})$$

and so

$$\int_0^{d(y_{m_{i-1}},y_{n_{i-1}})} \varphi(t)dt < \int_0^{d(y_{m_{i-1}},y_{m_i})+d(y_{m_i},y_{n_i})+d(y_{n_i},y_{n_{i-1}})} \varphi(t)dt.$$

Therefore,

$$\lim_{i\to\infty} \int_0^{d(y_{m_{i-1}},y_{n_{i-1}})} \varphi(t)dt = \int_0^{\epsilon} \varphi(t)dt. \tag{2.10}$$

Now

$$\int_0^{\epsilon} \varphi(t)dt \leq \int_0^{d(y_{m_i},y_{n_i})} \varphi(t)dt$$
$$\leq \alpha \int_0^{M(y_{m_{i-1}},y_{n_{i-1}})} \varphi(t)dt + \beta \int_0^{d(y_{m_{i-1}},y_{n_{i-1}})} \varphi(t)dt, \tag{2.11}$$

where

$$M(y_{m_{i-1}}, y_{n_{i-1}}) = \frac{d(y_{n_{i-1}}, y_{n_i})\left[1 + d(y_{m_{i-1}}, y_{n_{i-1}})\right]}{\left[1 + d(y_{m_{i-1}}, y_{n_{i-1}})\right]} = d(y_{n_{i-1}}, y_{n_i}). \tag{2.12}$$

Taking limit as $i \to \infty$ in (2.11) and using (2.9), (2.10), (2.12) we have,

$$\int_0^{\epsilon} \varphi(t)dt \leq \beta \int_0^{\epsilon} \varphi(t)dt,$$

since $\beta \leq 1$, so we get a contradiction. Hence the sequence $\{y_n\}$ is a Cauchy sequence and therefore there exists a $z \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \int_0^{d(Fx_{2n}, Gx_{2n+1})} \varphi(t)dt = z.$$

Furthermore, by Lemma (2.1) we get,

$$\lim_{n\to\infty} Fx_{2n} = z \quad \textit{and} \quad \lim_{n\to\infty} Gx_{2n+1} = z. \tag{2.13}$$

Now we show that z is a fixed point of F and G. First claim that z is a fixed point of G.
Consider,

$$\int_0^{d(Fx_{2n}, Gz)} \varphi(t)dt \leq \alpha \int_0^{M(x_{2n}, z)} \varphi(t)dt + \beta \int_0^{d(x_{2n}, z)} \varphi(t)dt, \tag{2.14}$$

where

$$M(x_{2n}, z) = \frac{d(z, Gz)\left[1 + d(x_{2n}, Fx_{2n})\right]}{\left[1 + d(x_{2n}, z)\right]} = d(z, Gz). \tag{2.15}$$

Taking limit $n \to \infty$ on both side of (2.14) and using (2.15), we get

$$\int_0^{d(z,Gz)} \varphi(t)dt \leq \alpha \int_0^{d(z,Gz)} \varphi(t)dt,$$

this is a contradiction. Therefore $\int_0^{d(z,Gz)} \varphi(t)dt = 0$. Thus z is a fixed point of G.

Now, we shall show that any fixed point of G is also a fixed point of F.
Using (2.2),

$$M(z, z) = \frac{d(z, Gz)\left[1 + d(z, Fz)\right]}{\left[1 + d(z, z)\right]} = 0.$$

Now from (2.1),

$$\int_0^{d(Fz, x_{2n})} \varphi(t)dt \leq \alpha \int_0^{M(z, x_{2n})} \varphi(t)dt + \beta \int_0^{d(z, x_{2n})} \varphi(t)dt. \tag{2.16}$$

Taking limit as $n \to \infty$

$$\int_0^{d(Fz,z)} \varphi(t)dt \leq 0,$$

which is a contradiction. Thus $d(Fz,z) = 0$. Therefore z is also a fixed point of F. In general, we can say that F and G have common fixed point.
For uniqueness, suppose that there exist some other point $w \neq z$ such that $Fw = Gw = w$.
Again from (2.1),

$$\int_0^{d(w,z)} \varphi(t)dt = \int_0^{d(Fw,Gz)} \varphi(t)dt \leq \alpha \int_0^{M(w,z)} \varphi(t)dt + \beta \int_0^{d(w,z)} \varphi(t)dt, \tag{2.17}$$

where

$$M(w,z) = \frac{d(z,Gz)\,[1 + d(w,Fw)]}{[1 + d(w,z)]} = 0. \tag{2.18}$$

Hence

$$\int_0^{d(w,z)} \varphi(t)dt \leq \beta \int_0^{d(w,z)} \varphi(t)dt,$$

we reaches at contradiction. This established the uniqueness and hence the result. □

In our next result, on omitting the completeness of metric space, we still get unique common fixed point for pair of self maps without using continuity and compatibility.

Theorem 2.3. *Let (X,d) be a metric space and F and G be self-maps of X such that for each $x, y \in X$*

$$\int_0^{d(Fx,Gy)} \varphi(t)dt \leq \alpha \int_0^{M(x,y)} \varphi(t)dt \tag{2.19}$$

and

$$M(x,y) = \frac{d(y,Gy)\,[1 + d(x,Fx)]}{[1 + d(x,y)]}, \tag{2.20}$$

where $0 \leq \alpha < 1$ and $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable function which is summable on each compact subset of R^+ and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

then F and G have a unique common fixed point.

Proof: Define a sequence $\{x_n\}$ in X such that for each n = 0,1,2,...

$$x_{2n+1} = Fx_{2n} \text{ and } x_{2n+2} = Gx_{2n+1}. \tag{2.21}$$

Suppose that

$$x_{2n+1} = x_{2n+2} \text{ for no } n \in N. \tag{2.22}$$

Consider,

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt &= \int_0^{d(Fx_{2n},Gx_{2n+1})} \varphi(t)dt \\ &\leq \alpha \int_0^{M(x_{2n},x_{2n+1})} \varphi(t)dt. \end{aligned} \tag{2.23}$$

From (2.20)

$$M\left(x_{2n}, x_{2n+1}\right) = \frac{d(x_{2n+1}, Gx_{2n+1})\left[1 + d\left(x_{2n}, Fx_{2n}\right)\right]}{\left[1 + d\left(x_{2n}, x_{2n+1}\right)\right]} = d(x_{2n+1}, x_{2n+2}). \tag{2.24}$$

. Hence from (2.23),

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \alpha \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt.$$

Since $\alpha < 1$, so this is a contradiction. Thus (2.22) is false and so $x_{2n+1} = x_{2n+2}$ for some $n \in N$, say $n = k$. Consequently, with $x_{2k+1} = x_{2k+2}$ and $z = x_{2k+1}$,by using (2.21) we get $Gz = z$. This proves that z is a fixed point of G.
Now we prove that any fixed point of G is also a fixed point of F.
Using (2.20),

$$M(z, z) = \frac{d(z, Gz)\left[1 + d(z, Fz)\right]}{\left[1 + d(z, z)\right]} = 0.$$

Hence from (2.19),

$$\int_0^{d(Fz,z)} \varphi(t)dt \leq \alpha \int_0^{M(z,z)} \varphi(t)dt = 0. \tag{2.25}$$

Thus $d(Fz, z) = 0$. Therefore z is also a fixed point of F. In general, we can say that F and G have common fixed point.
For uniqueness, suppose that there exist some other fixed point $w \neq z$ such that $Fw = Gw = w$.
Again from (2.19),

$$\begin{aligned}\int_0^{d(w,z)} \varphi(t)dt &= \int_0^{d(Fw,Gz)} \varphi(t)dt \\ &\leq \alpha \int_0^{M(w,z)} \varphi(t)dt, \end{aligned} \tag{2.26}$$

where

$$M(w,z)=\frac{d(z,Gz)\,[1+d(w,Fw)]}{[1+d(w,z)]}=0. \tag{2.27}$$

Hence

$$\int_0^{d(w,z)}\varphi(t)dt\le 0.$$

This is possible only if $d(w,z)=0$. This established the uniqueness and hence the result. □

Corollary 2.4. *Let (X,d) be a complete metric space and F be a self-maps of X such that for each $x,y\in X$*

$$\int_0^{d(Fx,Fy)}\varphi(t)dt\le\alpha\int_0^{M(x,y)}\varphi(t)dt+\beta\int_0^{d(x,y)}\varphi(t)dt,$$

and

$$M\,(x,y)=\frac{d\,(y,Fy)\,[1+d(x,Fx)]}{[1+d(x,y)]},$$

where $\alpha,\beta>0$ are constants such that $\alpha+\beta<1$ and $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue-integrable function which is summable on each compact subset of R^+ such that for each $\epsilon>0$,

$$\int_0^{\epsilon}\varphi(t)dt>0,$$

then F has a unique fixed point.

Proof: By taking $F=G$ in Theorem 2.2, we get the required result. □

3. Remarks and Example

In this section, we give some remarks and an example in support and existence of our result.

Remark 3.1. *In above Theorem 2.2, if we take $F=G$ and $\alpha=0$, then the result of Branciari [1] is retrieved.*

Remark 3.2. *In above Corollary 2.4, if we take $\varphi(t)=1$, then we get the result of Dass and Gupta [2].*

Remark 3.3. *It is important to note that result given in Theorem 2.3 is new and unique one. It can not be derived from any of the previous results. Also, if we take $G=F$ in (2.20), existence of fixed point without completeness of space and without continuity of map F is doubtful and is open problem for future research.*

Here we give an example in support of our main result (Theorem 2.2).

Example 3.4. *Let $X = R$ be endowed with usual metric $d(x, y) = |x - y|$ and let $E = \{0, \frac{1}{4}, 1\}$. Let $F, G : E \to E$ be defined by*

$$F(0) = F(\frac{1}{4}) = 0 \quad and \quad F(1) = \frac{1}{4}$$
$$G(x) = 0, \quad for\ all \quad x \in E.$$

Let $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable function which is summable on each compact subset of R^+ and be defined as $\varphi(t) = 2t$ for all $t \in R^+$, such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)d = \epsilon^2 > 0.$$

If we define constants $\alpha = \frac{1}{3} > 0$ and $\beta = \frac{1}{2} > 0$ such that $\alpha + \beta < 1$, then by a careful calculation, we can see that all the conditions of Theorem 2.2 holds. Hence the self maps F and G have a unique common fixed point $x = 0$. i.e $F(0) = G(0) = 0$. More importantly, we can see that this example can not satisfy the condition of main result of Branciari [1].

4. Conclusion

In this paper, contraction given in Theorem 2.2 and in Theorem 2.3 is new and specially constructed for pair of self of maps such that we get unique common fixed point without using continuity, compatibility and commutativity. Remarks- 3.1, 3.2 showed that our main result generalize the result of Branciari [1] and Dass and Gupta [2], while Remark- 3.3 assured that contractions used in main results are new and unique one. An example is given in support of our result.

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