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Numerical study of the Benjamin-Bona-Mahony-Burgers equation

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ABSTRACT: In this paper, the quadratic B-spline collocation method is implemented to find numerical solution of the Benjamin-Bona-Mahony-Burgers (BBMB) equation. Applying the Von-Neumann stability analysis technique, we show that the method is unconditionally stable. Also the convergence of the method is proved. The method is applied on some test examples, and numerical results have been compared with the exact solution. The numerical solutions show the efficiency of the method computationally.

Key Words: Quadratic B-spline, Benjamin-Bona-Mahony-Burgers equation, Convergence analysis, Finite difference, Collocation method.

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1. Introduction

In this paper we consider the solution of the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u u_x = 0, \quad x \in [a, b], \ t \in [0, T], \tag{1.1}$$

with the initial condition

$$u(x, 0) = f(x), \quad x \in [a, b], \tag{1.2}$$

and boundary conditions

$$u(a, t) = u(b, t) = 0, \tag{1.3}$$

where α and β are positive constants. For $\alpha = 0$, Eq. (1.1) is called the Benjamin-Bona-Mahony (BBM) equation. The BBMB equation has been proposed as a model for propagation of long waves. This equation incorporates dispersive and dissipative effects. The dissipative term can be found in $-\alpha u_{xx}$. For more details

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on this topic, see [1,2]. In recent years, many different methods have been used to estimate the solution of the Benjamin-Bona-Mahony-Burgers equation and the BBM equation, for example, see [3,4,5,7,11].
The paper is organized as follows. In Section 2, quadratic B-spline collocation method is explained. In Section 3, is devoted to stability analysis and convergence analysis of the method. In Section 4, examples are presented. A summary is given at the end of the paper in Section 5.

2. Quadratic B-spline collocation method

Our numerical treatment for BBMB equation using the collocation method with quadratic B-spline is to find an approximate solution $U(x,t)$ to the exact solution $u(x,t)$ in the form

$$U(x,t) = \sum_{i=-1}^{N} c_i(t) B_i(x), \tag{2.1}$$

where $c_i(t)$ are time-dependent quantities to be determined from the boundary conditions and collocation form of the differential equations. Also $B_i(x)$ are the quadratic B-spline basis functions at knots, given by [6,12]

$$B_i(z) = \frac{1}{h^2} \begin{cases} (z - z_{i-1})^2, & z \in [z_{i-1}, z_i), \\ 2h^2 - (z_{i+1} - z)^2 - (z - z_i)^2, & z \in [z_i, z_{i+1}), \\ (z_{i+2} - z)^2, & z \in [z_{i+1}, z_{i+2}), \\ 0, & otherwise. \end{cases} \tag{2.2}$$

The solution domain $a \le z \le b$ partitioned into a mesh of uniform length $h = \frac{b-a}{N}$, by the knots z_j where $j = 0, 1, 2, \ldots, N$ such that $a = z_0 < z_1 \ldots z_{N-1} < z_N = b$ and $z_j = z_0 + jh$. The values of $B_i(z)$ and its first and second derivatives at the mid knots points are given in Table 1. Also numerical solutions are given at mid points. We note that the mid points are $x_i = \frac{z_i + z_{i+1}}{2}$.

Table 1: B_i, $B_i^{'}$, $B_i^{''}$ at mid points.

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
B_i	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	0
$hB_i^{'}$	0	1	0	-1	0
$h^2 B_i^{''}$	0	2	-4	2	0

By using approximate function (2.1) and Table 1, we have

$$u(x_i, t_n) \approx U_i^n = \frac{1}{4} c_{i-1}^n + \frac{3}{2} c_i^n + \frac{1}{4} c_{i+1}^n, \tag{2.3}$$

Table 2: B_i and $B_i^{'}$at node points.

z	z_{i-2}	z_{i-1}	z_i	z_{i+1}	z_{i+2}
B_i	0	0	1	1	0
$hB_i^{'}$	0	0	2	-2	0

$$hu_x(x_i,t_n) \approx h(U^{'})_i^n = -c_{i-1}^n + c_{i+1}^n, \tag{2.4}$$

$$h^2 u_{xx}(x_i,t_n) \approx h^2(U^{''})_i^n = 2c_{i-1}^n - 4c_i^n + 2c_{i+1}^n, \tag{2.5}$$

where $c_i^n := c_i(t_n)$. In this step by using the finite difference method, we can write

$$\frac{u^{n+1}-u^n}{\Delta t} - \frac{u_{xx}^{n+1}-u_{xx}^n}{\Delta t} - \alpha\frac{u_{xx}^{n+1}+u_{xx}^n}{2} + \beta\frac{u_x^{n+1}+u_x^n}{2} + \frac{(uu_x)^{n+1}+(uu_x)^n}{2} = 0. \tag{2.6}$$

The nonlinear term in (2.6) can be approximated by using the following formula [9]:

$$(uu_x)^{n+1} = u^{n+1}u_x^n + u^n u_x^{n+1} - (uu_x)^n. \tag{2.7}$$

Substituting the approximate solution U for u and putting the values of the mid values U, its derivatives using (2.3),(2.4) and (2.5) at the knots in (2.6) yield the following difference equation with the variables c_i , $i = 0,1,\dots,N$,

$$\acute{a}c_{i-1}^{n+1} + \acute{b}c_i^{n+1} + \acute{c}c_{i+1}^{n+1} = \Psi_i^n \ , \ i = 0,1,...,N-1, \tag{2.8}$$

where

$$\Psi_i^n = U_i^n + (-1+\alpha\frac{\Delta t}{2})(U^{''})_i^n - \beta\frac{\Delta t}{2}(U^{'})_i^n, \tag{2.9}$$

and
$\acute{a} := \frac{\dot{a}}{4} - \frac{\dot{b}}{h} + \frac{2\dot{c}}{h^2}$, $\acute{b} := \frac{3\dot{a}}{2} - \frac{4\dot{c}}{h^2}$, $\acute{c} := \frac{\dot{a}}{4} + \frac{\dot{b}}{h} + \frac{2\dot{c}}{h^2}$,

with $\dot{a} := 1 + \frac{\Delta t}{2}(U^{'})_i^n$, $\dot{b} := \beta\frac{\Delta t}{2} + \frac{\Delta t}{2}U_i^n$, $\dot{c} := -1 - \alpha\frac{\Delta t}{2}$, $\acute{d} := \frac{\alpha\Delta t}{2} - 1$, $\acute{e} := -\frac{\beta\Delta t}{2}$.

The system (2.8) consists of N linear equations in $N+2$ unknowns $\{c_{-1}, c_0, ..., c_{N-1} \ , c_N\}$. To obtain a unique solution for $c = \{c_{-1}, c_0, ..., c_{N-1}, c_N\}$, we must use the boundary conditions. From the boundary conditions and Table 2, we can write

$$c_{-1}^{n+1} + c_0^{n+1} = 0, \tag{2.10}$$

$$c_{N-1}^{n+1} + c_N^{n+1} = 0. \tag{2.11}$$

Associating (2.10) and (2.11) with (2.8), we obtain a $(N+2)\times(N+2)$ system of equations in the following form

$$AC = Q, \tag{2.12}$$

where

$$A := \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ \acute{a} & \acute{b} & \acute{c} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \dots & \acute{a} & \acute{b} & \acute{c} \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix}, \tag{2.13}$$

$$C := (c_{-1}^{n+1}, c_0^{n+1}, \dots, c_{N-1}^{n+1}, c_N^{n+1})^T, \tag{2.14}$$

$$Q := \begin{cases} \Big(0, u(x_0,t) + (-1+\alpha\frac{\Delta t}{2})u_{xx}(x_0,t) - \beta\frac{\Delta t}{2}u_x(x_0,t), \dots, \\ u(x_{N-1},t) + (-1+\alpha\frac{\Delta t}{2})u_{xx}(x_{N-1},t) - \beta\frac{\Delta t}{2}u_x(x_{N-1},t), 0\Big)^T, if \;\; t = \Delta t, \\ \Big(0, \Psi_0^n, \dots, \Psi_{N-1}^n, 0\Big)^T, \qquad if \;\; t > \Delta t. \end{cases} \tag{2.15}$$

3. Stability and convergence analysis

We present the stability of the quadratic B-spline approximation (2.8) using the Von Numann method [8,10]. According to the Von-Neumann method, we have

$$c_i^n = \xi^n exp(\lambda khi), \qquad \lambda^2 = -1, \tag{3.1}$$

where k is the mode number and h is the element size. To apply this method, we have linearized the nonlinear term uu_x by consider u as a constant ϖ in term (2.6). We obtain the equation

$$\begin{aligned} \widehat{a}\,\xi^{n+1}exp(\lambda kh(i-1)) + \widehat{b}\,\xi^{n+1}exp(\lambda kh(i)) + \widehat{c}\,\xi^{n+1}exp(\lambda kh(i+1)) = \\ \widehat{d}\,\xi^{n}exp(\lambda kh(i-1)) + \widehat{e}\,\xi^{n}exp(\lambda kh(i)) + \widehat{f}\,\xi^{n}exp(\lambda kh(i+1)), \end{aligned} \tag{3.2}$$

where

$\widehat{a} := \frac{1}{4} + \frac{2x}{h^2} - \frac{y}{h}$, $\widehat{b} := \frac{3}{2} - \frac{4x}{h^2}$, $\widehat{c} := \frac{1}{4} + \frac{2x}{h^2} + \frac{y}{h}$,
$\widehat{d} := \frac{1}{4} + \frac{2z}{h^2} + \frac{y}{h}$, $\widehat{e} := \frac{3}{2} - \frac{4z}{h^2}$, $\widehat{f} := \frac{1}{4} + \frac{2z}{h^2} - \frac{y}{h}$,

with $x := -1 - \frac{\alpha\Delta t}{2}$, $y := \beta\frac{\Delta t}{2} + \frac{\Delta t\varpi}{2}$, $z := -1 + \frac{\alpha\Delta t}{2}$.

Dividing both sides of (3.2) by $exp\,(i\lambda kh)$, we can obtain

$$\xi^{n+1}\Big(\widehat{a}\,exp(-\lambda kh) + \widehat{b} + \widehat{c}\,exp(\lambda kh)\Big) = \xi^n\Big(\widehat{d}\,exp(-\lambda kh) + \widehat{e} + \widehat{f}\,exp(\lambda kh)\Big), \tag{3.3}$$

(3.3) can be rewritten in a simple form as

$$\xi = \frac{X - \lambda Y}{X_1 + \lambda Y}, \tag{3.4}$$

where

$X := (\frac{1}{2} + \frac{4z}{h^2})\cos(kh) + (\frac{3}{2} - \frac{4z}{h^2}),$
$X_1 := (\frac{1}{2} + \frac{4x}{h^2})\cos(kh) + (\frac{3}{2} - \frac{4x}{h^2}),$
$Y := (\frac{2y}{h})\sin(kh).$

X and X_1 can be rewritten in the form:

$X = (\frac{1}{2} - \frac{4}{h^2})\cos(kh) + (\frac{3}{2} + \frac{4}{h^2}) - \frac{4\alpha\Delta t}{2h^2}(1 - \cos(kh)),$
$X_1 = (\frac{1}{2} - \frac{4}{h^2})\cos(kh) + (\frac{3}{2} + \frac{4}{h^2}) + \frac{4\alpha\Delta t}{2h^2}(1 - \cos(kh)).$

We note that $X \leq X_1$, so $|\xi|^2 = \xi\bar{\xi} = \frac{X^2+Y^2}{X_1^2+Y^2} \leq 1$. Therefore, the linearized numerical scheme for the BBMB equation is unconditionally stable.

Now we discuss the convergence of the collocation method.

Theorem 3.1. *Suppose that $f(x) \in C^4[a,b]$ and $| f^4(x) | \leq L$ for $x \in [a,b]$. Let Δ be a partition $\Delta = \{a = x_0 < x_1 < \cdots < x_N = b\}$ be the equally spaced partition of $[a,b]$ with step size h. If $S(x)$ be the unique spline function interpolate $f(x)$ at knots $x_0, x_1, \cdots, x_N \in \Delta$ then there exist a constant λ_j such that*

$$\| f^{(j)} - S^{(j)} \|_\infty \leq \lambda_j L h^{4-j}\ ,\ j = 0, 1, 2. \tag{3.5}$$

Proof: For the proof see [14]. □

Lemma 3.2. *The B-splines $\{B_{-1}, \cdots, B_N\}$ satisfy the following inequality:*

$$\sum_{i=-1}^{N} |B_i(x)| \leq \frac{7}{2}, \quad (a \leq x \leq b). \tag{3.6}$$

Proof: At any nodal point $x = x_i$, we can write

$$\sum_{i=-1}^{N} |B_i(x)| = |B_{i-1}(x)| + |B_i(x)| + |B_{i+1}(x)| = 2,$$

also for any $x \in [x_{i-1}, x_i]$ we have

$$\sum_{i=-1}^{N} |B_i(x)| \leq \frac{7}{2}.$$

which completes the proof. □

Theorem 3.3. *Suppose that $u(x,t)$ be the exact solution of (1.1) and assume that $|\frac{\partial^4 u(x,t)}{\partial x^4}| \leq L$ and $U(x,t)$ be the approximate solution of BBMB (1.1) given by our approach, then*

$$\|u(x,t) - U(x,t)\|_\infty \leq \varrho(h^2 + \Delta t), \tag{3.7}$$

where ϱ is a constant and independent of h.

Proof: At the $(n+1)$th time step, we assume that S be the unique spline interpolate to the exact solution u of (1.1)-(1.3) given by

$$S(x) = \sum_{i=-1}^{N} c^* B_i(x), \tag{3.8}$$

To continue, we note that matrix A is strictly diagonally dominant matrix, and from [13] we can find $\acute{M}$ independent of h, such that $||A^{-1}||_\infty \leq \acute{M}$. Also from Theorem 3.1, we can write

$$\|u(x) - S(x)\|_\infty \leq \lambda_0 L h^4, \tag{3.9}$$

we substituting $S(x)$ in (2.6), we have the following result

$$AC^* = Q^*. \tag{3.10}$$

Subtracting (2.12), (3.10) and taking the infinity norm, we can write

$$\|C^* - C\|_\infty = \|A^{-1}\|_\infty \|Q^* - Q\|_\infty. \tag{3.11}$$

From (2.9) and using Theorem 3.1, we get

$$\|Q^* - Q\|_\infty \leq \widehat{M} h^2, \tag{3.12}$$

where $\widehat{M} = \lambda_0 L h^2 + |\frac{\beta \Delta t}{2}| \lambda_1 L h + | -1 + \frac{\alpha \Delta t}{2} | \lambda_2 L$. Then we have

$$\|C^* - C\|_\infty \leq \widehat{M} \acute{M} h^2. \tag{3.13}$$

Applying (3.13) and Lemma 3.2, we get the result as

$$\|U - S\|_\infty \leq \overline{M} h^2, \tag{3.14}$$

where $\overline{M} = 7\widehat{M}\acute{M}/6$. From (3.9) and (3.14),we have

$$\|u - U\|_\infty \leq M h^2. \tag{3.15}$$

In the next step, suppose that $\varepsilon_i = u(t_i) - U(t_i)$ be the local truncation error for (2.6) at the ith level of time. By using the truncation error , we get

$$| \varepsilon_i | \leq v_i \Delta t^2 \ , \ i \geq 1, \tag{3.16}$$

where v_i is some finite constant. We can write the following global error estimate at $n+1$ level

$$E_{n+1} = \sum_{i=1}^{n} \varepsilon_i, \ (\Delta t \leq T/n), \tag{3.17}$$

thus with the help of (3.16), we can write

$$\mid E_{n+1} \mid = \mid \sum_{i=1}^{n} \varepsilon_i \mid \leq \sum_{i=1}^{n} v_i \Delta t^2 \leq n v \Delta t^2 \leq n v \frac{T}{n} \Delta t = \rho \Delta t, \tag{3.18}$$

where $\rho = vT$ and $v = max\{v_1, ..., v_n\}$. Which completes the proof. □

4. Numerical examples

In order to illustrate the performance of the quadratic B-spline collocation method in solving BBMB equation and justify the accuracy and efficiency of the present method, we consider the following examples. To show the efficiency of the present method for our problem in comparison with the solution, we report L_∞ and L_2 using formulae
$L_\infty = \max_i \mid U(x_i, t) - u(x_i, t) \mid, \ L_2 = (h \sum_i \mid U(x_i, t) - u(x_i, t) \mid^2)^{\frac{1}{2}}$,
where U is numerical solution and u denotes exact solution. Note that we have computed the numerical results by Mathematica-9 programming.

Example 4.1. *Consider the BBMB equation with $\alpha = 0$ and $\beta = 1$ in the interval $[-40, 60]$, with the solution $u(x,t) = 3csech^2(k(x - vt - x_0))$. We have taken $c = 0.03, 0.1$, $v = 1 + c$, $x_0 = 0$ and $k^2 = \frac{c}{4v}$. The initial condition is taken from the solution. Also the solution satisfy three conservation laws:*
$C_1 = \int_{-\infty}^{\infty} u dx = h \sum_i u_i^n$, $C_2 = \int_{-\infty}^{\infty} (u^2 + u_x^2) dx = h \sum_i \left((u_i^n)^2 + ((u_x)_i^n)^2\right)$,
$C_3 = \int_{-\infty}^{\infty} u^3 + 3u^2 dx = h \sum_i \left((u_i^n)^3 + 3(u_i^n)^2\right)$.
Table 3 and Table 6 give C_1, C_2, C_3, L_∞ and L_2 found by our method in different times for $c = 0.1, 0.03$ and Table 4 and Table 7 give numerical results from method in [4,5]. Figure. 1 and Figure. 2 show that solution obtained by our method is closed to the solutions. In addition, in Table 5 and Table 8 we see that L_2 decreases with decreasing in Δt or increasing in N. Also from Figure 3 we can see that numerical solutions show the same behavior as solutions.

Table 3: Numerical results for Example 4.1 with $h = 0.1$, $N = 1000$ and $c = 0.1$.

Time	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
2	3.97993	0.810461	2.579	0.0325248	0.0122345
4	3.97993	0.810461	2.579	0.0646491	0.0249644
6	3.97993	0.810461	2.579	0.0966147	0.0378028
8	3.97993	0.810461	2.579	0.12823	0.0504925
10	3.97993	0.810461	2.579	0.159403	0.0628118
12	3.97993	0.810461	2.579	0.190073	0.0747027
14	3.97992	0.810461	2.579	0.220198	0.0861316
16	3.97992	0.810461	2.579	0.249754	0.0970962
18	3.97991	0.810461	2.579	0.278738	0.107622
20	3.97988	0.810461	2.579	0.307172	0.117734

Table 4: Numerical results for Example 4.1 with $h = 0.1$, $N = 1000$ and $c = 0.1$ with the algorithim of [4,5].

Time	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
2	4.41677	0.899303	2.86167	19.92	6.817
4	4.42017	0.899873	2.86339	39.82	13.74
6	4.41922	0.899559	2.86226	59.67	20.71
8	4.41822	0.899236	2.86106	79.46	27.66
10	4.41722	0.898919	2.85986	99.17	34.55
12	4.41623	0.898601	2.85863	118.8	41.35
14	4.41523	0.898283	2.85739	138.3	48.04
16	4.41423	0.897967	2.85613	157.7	54.60
18	4.41321	0.897653	2.85487	176.9	61.04
20	4.41219	0.897342	2.85361	196.1	67.35

Table 5: Comparison of L_2 for Example 4.1 at different Δt with $c = 0.1$.

Partition\Time	1	1.5	2
$N = 1000, \Delta t = 0.25$	0.0722912×10^{-3}	0.108617×10^{-3}	0.144949×10^{-3}
$N = 1000, \Delta t = 0.1$	0.0166866×10^{-3}	0.0245446×10^{-3}	0.0325248×10^{-3}
$N = 1000, \Delta t = 0.01$	7.46198×10^{-6}	9.71695×10^{-6}	0.012213×10^{-3}

Example 4.2. *Consider the BBMB equation in the interval* $[-10, 10]$ *with* $\alpha = 1$ *and* $\beta = 1$ *and the initial condition* $u(x, 0) = exp(-x^2)$. *Table 9 and Table 10 give numerical results found by our method in different times. Also Figure. 4 shows approximate solution graphs. In addition, we can see that the graph shows the same behavior as in [7].*

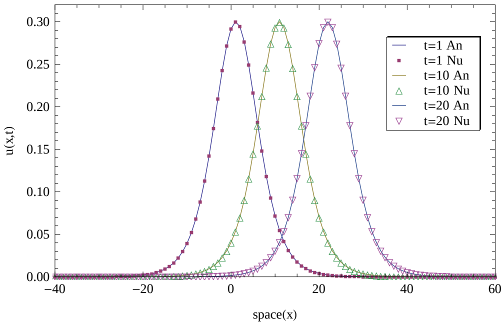

Figure 1: Numerical and analytical solutions for Example 4.1 with $c = 0.1, N = 1000$, and $\Delta t = 0.1$.

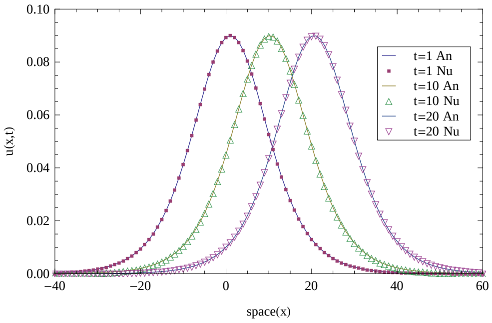

Figure 2: Numerical and analytical solutions for Example 4.1 with $c = 0.03, N = 1000$, and $\Delta t = 0.1$.

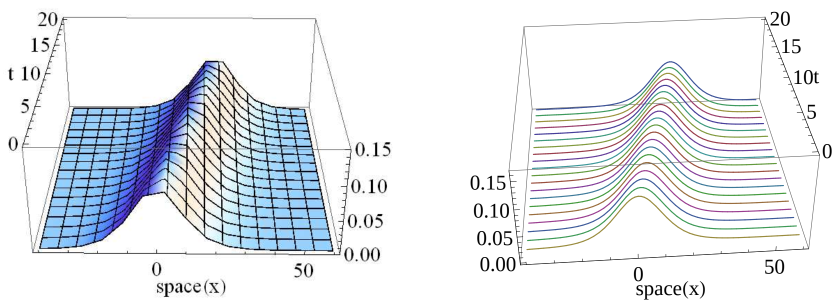

Figure 3: Analytical solution (left) and numerical solution (right) using $\Delta t = 0.1$ and $N = 500$ with $c = 0.03$ of Example 4.1.

Table 6: Numerical results for Example 4.1 with $h = 0.1$, $N = 1000$ and $c = 0.0.03$.

$Time$	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
2	2.10701	0.127301	0.388804	0.333312	0.276642
4	2.1071	0.127301	0.388804	0.409707	0.229933
6	2.10703	0.127301	0.388804	0.472601	0.22496
8	2.1069	0.127301	0.388804	0.510916	0.221015
10	2.10674	0.127301	0.388804	0.528822	0.216744
12	2.10655	0.127301	0.388804	0.535957	0.212506
14	2.1063	0.127301	0.388804	0.539885	0.208489
16	2.10593	0.127301	0.388804	0.544644	0.212056
18	2.10539	0.127301	0.388803	0.552954	0.301244
20	2.10461	0.127301	0.388802	0.568417	0.427854

Table 7: Numerical results for Example 4.1 with $h = 0.1$, $N = 1000$ and $c = 0.03$ with the algorithim of [4,5].

$Time$	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
2	2.338	0.141245	0.431389	1.487	0.392
4	2.340	0.141322	0.431621	2.928	0.786
6	2.340	0.141259	0.431427	4.373	1.183
8	2.339	0.141195	0.431231	5.816	1.582
10	2.338	0.141131	0.431031	7.257	1.983
12	2.337	0.141067	0.430834	8.698	2.384
14	2.337	0.141004	0.430636	10.14	2.787
16	2.336	0.140940	0.430440	11.58	3.190
18	2.335	0.140877	0.430245	13.01	3.593
20	2.333	0.140815	0.430052	14.45	3.996

Table 8: Comparison of L_2 for Example 4.1 at different N with $c = 0.1$.

$Partition \backslash Time$	3	5	7
$N = 200, \Delta t = 0.1$	0.438353×10^{-3}	0.730055×10^{-3}	1.01786×10^{-3}
$N = 400, \Delta t = 0.1$	0.133167×10^{-3}	0.22177×10^{-3}	0.309324×10^{-3}
$N = 1000, \Delta t = 0.1$	0.0485834×10^{-3}	0.080668×10^{-3}	0.112473×10^{-3}

Table 9: Numerical results for Example 2 with $\Delta t = 0.1$ and $N = 200$.

$x \backslash t$	1	5	7	10
-5	-0.00109485	-0.000342832	-0.000133937	-0.0000298046
0	0.572752	0.0202368	-0.00091068	-0.00220189
5	0.0396263	0.234568	0.179672	0.073734

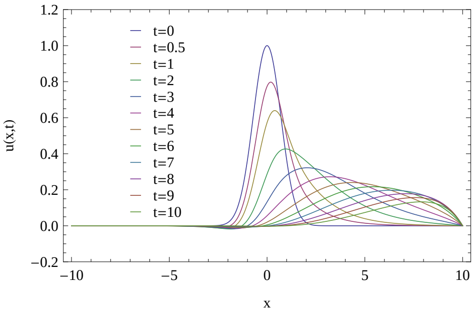

Figure 4: Approximate solution graphs of Example 4.2 in different times with $\Delta t = 0.1$ and $N = 200$.

Table 10: Numerical results for Example 2 with $\Delta t = 0.1$ and $N = 400$.

$x\backslash t$	1	5	7	10
-5	-0.00109258	-0.00034219	-0.000133763	-0.0000298188
0	0.572783	0.0201537	-0.000921405	-0.00219315
5	0.039628	0.2346	0.179665	0.0737031

5. Conclusion

The quadratic B-spline collocation method is used to solve the Benjamin-Bona-Mahony-Burgers(BBMB) equation with initial and boundary conditions. We study the stability analysis and the convergence analysis of the method. The numerical results given in the previous section demonstrate the good accuracy and stability of the proposed scheme in this research.

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