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On the second eigencurve for the p-laplacian operator with weight

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abstract: In this paper we establish the existence of the second eigencurves of the p-laplacian with indefinite weights. we obtain also their asymptotic behavior and variational formulation.

Key Words: eigencurves; p-laplacian; asymptotic behavior; variational formulation.

Contents

1. Introduction

We consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u &= \lambda m(x)|u|^{p-2}u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian, $1 < p < +\infty$ and $m(.) \in M^+(\Omega) = \{u \in L^\infty(\Omega) : \text{meas}\{x \in \Omega : m(x) > 0\} > 0\}$ is a weight function which can change sign.

The spectrum of p-laplacian operator with indefinite weight is defined as the set $\sigma_p(-\Delta_p, m, \Omega)$ of $\lambda = \lambda(m,\Omega)$ for which there exists a nontrivial solution $u \in W_0^{1,p}(\Omega)$ of problem (1.1), this values are called eigenvalues and the corresponding solutions are called eigenfunctions.
We will denote $\sigma_p^+(-\Delta_p, m, \Omega)$ the set of all positive eigenvalues.

For $p = 2$ ($\Delta_p = \Delta$ Laplacian Operator) it is well known (see [10]) that $\sigma_2^+(-\Delta, m, \Omega) = \{\mu_k(m,\Omega), k = 1, 2\ldots\}$, with $0 < \mu_1(m,\Omega) < \mu_2(m,\Omega) \leq \mu_3(m,\Omega)\ldots \to +\infty$, $\mu_k(m,\Omega)$ repeated according to its multiplicity.

For $p \neq 2$ (nonlinear problem), the critical point theory of Ljusternik Schnirelman (see [11]) provides that $\sigma_p^+(-\Delta_p, m, \Omega)$ contains an infinite sequence of eigenvalues for these problems given by $\lambda_1(m,\Omega) < \lambda_2(m,\Omega) \leq \lambda_3(m,\Omega)\ldots\lambda_n(m,\Omega) \to$

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$+\infty$ and formulated as follows

$$\frac{1}{\lambda_n(m)} = \sup_{K\in\Gamma_n} \min_{u\in K} \int_\Omega m|u|^p \tag{1.1}$$

where $\Gamma_n = \{K \subset S : K \text{ is symmetrical, compact and } \quad \xi(K) \geq n\}$, S is the sphere unity of $W_0^{1,p}(\Omega)$ and ξ is the genus function.

We may also define the negative spectrum when $-m \in M^+(\Omega)$ by $-\sigma^+(-\Delta_p, -m, \Omega)$ which contains an infinite sequence $\lambda_{-1}(m,\Omega) > \lambda_{-2}(m,\Omega) \geq \lambda_{-3}(m,\Omega)\ldots \geq \lambda_{-n}(m,\Omega) \to -\infty$, such that $\lambda_{-n}(m,\Omega) = -\lambda_n(-m,\Omega)$ (See [1], [2], [3], [7]...). Whether or not this sequence denoted $\lambda_k(m,\Omega)$ constitutes the set of all eigenvalues is an open question when $N > 1$ and $p \neq 2$.

We denote $C_2 = \{(\alpha,\beta) \in \mathbb{R}^2 : \lambda_2(\alpha m + \beta m') = 1\}$, where m and m' satisfies the condition:

$$m, m' \in M^+(\Omega) \text{ and } m' \geq 0 \quad a.e,\ x \in \Omega. \tag{H_0}$$

The purpose of this article is to study the following problem: For $\alpha \in \mathbb{R}$ Find the existence of real numbers $\beta(\alpha)$ such that $(\alpha,\beta(\alpha)) \in C_2$ and the asymptotic behavior of $\beta(\alpha)$ as $|\alpha| \to +\infty$,

Many results have been obtained on this kind of problems (see; [4], [5], [8], in [5] the authors proved some properties related to the first eigencurve C_1 such as concavity, differentiability and the asymptotic behavior, this last property can not be adapted to the other eigencurves, in [8] the authors have studied this class of problems under the following assumptions

$$m, m' \in M^+(\Omega) \text{ and } \operatorname{ess\,inf}_\Omega m' > 0. \tag{H'}$$

Several applications can be found in the bifurcation domain, we refer the reader to [6].

This article is organized as follows, in section 2 we recall some basic result, in section 3 we study the existence of the eigencurve C_2 and in section 4 we study the asymptotic behavior of C_2.

2. Preliminary results

Firstly we recall the following results which will be used later.

Proposition 2.1. *([3], [8]). Let $n \in \{1,2,3\ldots\}$.*

1. *For m, $m' \in M^+(\Omega)$, if $m \leq m'$ (resp $m < m'$), then $\lambda_n(m) \geq \lambda_n(m')$ (resp $\lambda_n(m) > \lambda_n(m')$).*
2. *$\lambda_n : m \mapsto \lambda_n(m)$ is continuous in $(M^+(\Omega), d(m,m') = ||m - m'||_\infty)$.*

Proposition 2.2. . *Let (m_k) be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$, then we have for every $n \in \mathbb{N}^\star$, we have:*

$$\lim_{k\to+\infty} \lambda_n(m_k) = +\infty \Leftrightarrow m \leq 0\, a.e.x \in \Omega.$$

Proof: Let (m_k) be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$.
Assume first that $\lim_{k\to+\infty} \lambda_n(m_k) = +\infty$, we claim that $m \leq 0$ almost everywhere in Ω, indeed, if meas$\{x \in \Omega : m(x) > 0\} \neq 0$, we get by proposition2.1

$$\lim_{k\to+\infty} \lambda_n(m_k) = \lambda_n(m)$$

is a finite, which gives a contradiction. Inversely, if $m \leq 0$ almost everywhere in Ω, suppose by contradiction that there exists $\lambda > 0$ and a subsequence $(m_{i(k)})$ of (m_k) such that

$$\lambda_n(m_{i(k)}) \leq \lambda.$$

Let $r = \dfrac{2\lambda}{\lambda_n(2)}$, Since $m_{i(k)} \to m$ in $L^\infty(\Omega)$, there exists $N \in \mathbb{N}$, such that $\forall k \geq N$, we have:

$$||m_{i(k)} - m||_\infty \leq \frac{2}{r},$$

hence

$$m_{i(k)} \leq m + \frac{2}{r} \quad a.e.x \in \Omega.$$

So, using the fact that $m \leq 0 \quad a.e.x \in \Omega$, we conclude that

$$m_k \leq \frac{2}{r} \quad a.e.x \in \Omega.$$

It follows that

$$\lambda_n(m_{i(k)}) \geq \lambda_n(\frac{2}{r}) = r\lambda_n(2) = 2\lambda.$$

Which is a contradiction. The proof is complete. □

3. Existence of the eigencurve C_2

For $m \in M^+(\Omega)$, we denote by $\Omega_m^- = \{x \in \Omega : m(x) < 0\}$, $\Omega_m^+ = \{x \in \Omega : m(x) > 0\}$ and $\Omega_m^* = \{x \in \Omega : m(x) \neq 0\}$.

Theorem 3.1. . *Assume (H_0) holds, then we have:*

1. *For all $\alpha \in [0, \lambda_2(m)]$, there exists $\beta(\alpha) \in \mathbb{R}^+$ such that $(\alpha, \beta(\alpha)) \in C_2$.*
2. *If meas $(\Omega_m^-) > 0$, then for all $\alpha \in [\lambda_{-2}(m), 0[$, there exists $\beta(\alpha) \in \mathbb{R}^+$ such that $(\alpha, \beta(\alpha)) \in C_2$.*

Proof: To prove the first result, we consider the real function $h_\alpha(.)$ defined by $h_\alpha(t) = \lambda_2(\alpha m + tm')$, h_α is well defined and continuous in $[0, +\infty[$ (see proposition 2.1), on the other hand, if $\alpha \in]0, \lambda_2(m)]$, we have:

$$\begin{aligned} h_\alpha(0) &= \lambda_2(\alpha m) \\ &= \tfrac{\lambda_2(m)}{\alpha} \\ &\geq 1. \end{aligned}$$

and for $t > 0$, we have:

$$\begin{array}{rcl} h_\alpha(t) & = & \lambda_2(\alpha m + tm') \\ & = & \frac{1}{t}\lambda_2(\frac{\alpha m}{t} + m'), \end{array}$$

hence $\lim_{t\to+\infty} h_\alpha(t) = 0$, since h_α is continuous, we deduce that there exists a real $\beta(\alpha) \in [0, +\infty[$ such that $h_\alpha(\beta(\alpha)) = 1$ that is $(\alpha, \beta(\alpha)) \in C_2$. If $\alpha = 0$, we take $\beta_2(\alpha) = \lambda_2(m')$.
The proof of the second result is similar to the first: For $\lambda_{-2}(m) < \alpha < 0$, h_α is well defined, continuous in $[0, +\infty[$, in other hand we have: $h_\alpha(0) = \frac{\lambda_{-2(m)}}{\alpha} > 1$ and $\lim_{t\to+\infty} h_\alpha(t) = 0$, so we conclude the existence of a real $\beta(\alpha) \in [0, +\infty[$ such that $(\alpha, \beta(\alpha)) \in C_2$. If $\alpha = \lambda_{-2}(m)$ we take $\beta(\alpha) = 0$. □

Remark 3.2. . *Let $(\alpha, \beta) \in C_2$*

1. *If $\alpha > \lambda_2(m)$ then we have $\beta < 0$.*
2. *If meas $(\Omega_m^-) > 0$ and $\alpha < \lambda_{-2}(m)$ then we have $\beta < 0$.*

Indeed, if $\alpha > \lambda_2(m)$ and $\beta \geq 0$, we have:

$$\alpha m \leq \alpha m + \beta m',$$

so

$$\lambda_2(\alpha m + \beta m') \leq \lambda_2(\alpha m) = \frac{\lambda_2(m)}{\alpha} < 1,$$

hence, if $\lambda_2(\alpha m + \beta m') = 1$ necessarily we have $\beta < 0$. The case $\alpha < \lambda_{-2}(m)$ is similar.

Theorem 3.3. . *Assume H_0, we have:*

1. *For each $\beta \in \mathbb{R}^{\star -}$, there exists a unique $\alpha_2^+(\beta)$ in $\mathbb{R}^{\star +}$ such that $(\alpha_2^+(\beta), \beta) \in C_2$.*
2. *If meas $(\Omega_m^-) > 0$, then for each $\beta \in \mathbb{R}^{\star -}$, there exists a unique $\alpha_2^-(\beta) \in \mathbb{R}^{\star -}$ such that $(\alpha_2^-(\beta), \beta) \in C_2$.*
3. *If meas $(\Omega_m^-) = 0$, then for each $\alpha \in \mathbb{R}^{\star -}$, there exists a unique $\beta_2(\alpha) \in \mathbb{R}$ such that $(\alpha, \beta_2(\alpha)) \in C_2$.*

Proof: To prove the first result, we take $\beta < 0$ and we define $\alpha_2^+(\beta)$ as follows:

$$\frac{1}{\alpha_2^+(\beta)} = \sup_{K\in\Gamma_2} \inf_{u\in K} \frac{\int_\Omega m|u|^p}{1 - \beta\int_\Omega m'|u|^p}.$$

By definition of $\alpha_2^+(\beta)$ and the property of $\lambda_2(m)$ (see [3]), we deduce that there exists eigenfunction u which change sign in Ω such that:

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla w - \beta m'|u|^{p-2}uw) = \int_\Omega \alpha_2^+(\beta) m|u|^{p-2}uw, \qquad (3.1)$$

$\forall w \in W_0^{1,p}(\Omega)$, we deduce also that, if $\varphi \in W_0^{1,p}(\Omega)$ is eigenfunction of $-\Delta_p(.) - \beta m^{'}|(.)|^{p-2}(.)$ which change singe in Ω with the corresponding eigenvalue $\lambda > 0$ then $\lambda \geq \alpha_2^+(\beta)$. From (3.1), we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w = \int_\Omega (\alpha_2^+(\beta)m + \beta m^{'})|u|^{p-2}uw \quad \forall w \in W_0^{1,p}(\Omega), \tag{3.2}$$

hence the real 1 is eigenvalue of $-\Delta_p$ with weight $(\alpha_2^+(\beta)m + \beta m^{'})$, since the corresponding eigenfunction u change singe in Ω, we conclude that:

$$\lambda_2(\alpha_2^+(\beta)m + \beta m^{'}) \leq 1. \tag{3.3}$$

In other hand, let $K \in \Gamma_2$, we have

$$\min_{v\in K} \frac{\int_\Omega m|v|^p}{1 - \beta\int_\Omega m^{'}|v|^p} \leq \frac{1}{\alpha_2^+(\beta)},$$

and

$$\min_{v\in K} \frac{\int_\Omega m|v|^p}{1 - \beta\int_\Omega m^{'}|v|^p} = \frac{\int_\Omega m|v_k|^p}{1 - \beta\int_\Omega m^{'}|v_k|^p} \qquad \text{for some } v_k \in K,$$

so we deduce

$$\min_{v\in K} \int_\Omega (\alpha_2^+(\beta)m|v|^p + \beta m^{'}|v|^p) \leq 1,$$

taking account that $K \in \Gamma_2$ is arbitrary, we get

$$\frac{1}{\lambda_2(\alpha_2^+(\beta)m + \beta m^{'})} = \sup_{K\in\Gamma_2} \min_{v\in K} \int_\Omega (\alpha_2^+(\beta)m + \beta m^{'})|v|^p \leq 1,$$

so

$$\lambda_2(\alpha_2^+(\beta)m + \beta m^{'}) \geq 1 \tag{3.4}$$

(3.3) and (3.4) gives

$$\lambda_2(\alpha_2^+(\beta)m + \beta m^{'}) = 1.$$

Let $\gamma > 0$ such that $\lambda_2(\gamma m + \beta m^{'}) = 1$, there exists eigenfunction θ change singe in Ω and

$$\int_\Omega |\nabla\theta|^{p-2}\nabla\theta\nabla w = \int_\Omega (\gamma m + \beta m^{'})|\theta|^{p-2}\theta w \quad \forall w \in W_0^{1,p}(\Omega),$$

hence

$$\int_\Omega |\nabla\theta|^{p-2}\nabla\theta\nabla w - \beta m^{'}|\theta|^{p-2}\theta w = \gamma\int_\Omega m|\theta|^{p-2}\theta w \quad \forall w \in W_0^{1,p}(\Omega), \tag{3.5}$$

from (3.5), we conclude that γ is eigenvalue of the operator $-\Delta_p(.) - \beta m^{'}|(.)|^{p-2}(.)$ with weight m, since the eigenfunction θ change singe, we conclude that:

$$\gamma \geq \alpha_2^+(\beta).$$

Assume by contradiction that $\gamma > \alpha_2^+(\beta)$.
So

$$\frac{1}{\gamma} < \frac{1}{\alpha_2^+(\beta)} = \sup_{K\in\Gamma_2}\min_{v\in K}\frac{\int_\Omega m|v|^p}{1-\beta\int_\Omega m'|v|^p},$$

by the inequality above we deduce that there exists $K_0 \in \Gamma_2$ such that

$$\frac{1}{\gamma} < \min_{v\in K_0}\frac{\int_\Omega m|v|^p}{1-\beta\int_\Omega m'|v|^p},$$

since K_0 is compact, we conclude that

$$\frac{1}{\gamma} < \frac{\int_\Omega m|v_0|^p}{1-\beta\int_\Omega m'|v_0|^p}, \quad \text{for some}\, v_0 \in K_0,$$

hence

$$1 < \min_{K_0}\int_\Omega(\gamma m + \beta m^{'})|v|^p,$$

it follows that

$$1 < \sup_{K\in\Gamma_2}\min_{v\in K}\int_\Omega(\gamma m + \beta m^{'})|v|^p = \frac{1}{\lambda_2(\gamma m + \beta m')} = 1,$$

which is a contradiction, hence we have $\gamma = \alpha_2^+(\beta)$.
The conclusion (2) of the theorem is similar to that of 1, we define $\alpha_2^-(\beta)$ as follows

$$\frac{1}{\alpha_2^-(\beta)} = \inf_{K\in\Gamma_2}\max_{u\in K}\frac{\int_\Omega m|u|^p}{1-\beta\int_\Omega m'|u|^p}.$$

To prove conclusion (3), we consider the coercive operator $-\Delta_p(.) - \alpha m|(.)|^{p-2}(.)$, we define $\beta_2(\alpha)$ as follows

$$\frac{1}{\beta_2(\alpha)} = \sup_{K\in\Gamma_2}\inf_{u\in K}\frac{\int_\Omega m^{'}|u|^p}{1-\alpha\int_\Omega m|u|^p},$$

by the same proof as the first, we deduce the result, taking account that, for $\beta < 0$, $\alpha m + \beta m^{'} \notin M^+(\Omega)$. □

4. Asymptotic behavior of C_2

Theorem 4.1. . *Assume* (H_0) *holds, then we have:*

1. $\lim_{\beta\to-\infty}\frac{\alpha_2^+(\beta)}{\beta} = -ess\,inf_{\Omega_m^+}\frac{m^{'}}{m}$.

2. *If* $mes(\Omega_m^-) > 0$, *then*
   $\lim_{\beta\to-\infty}\frac{\alpha_2^-(\beta)}{\beta} = -ess\,sup_{\Omega_m^-}\frac{m^{'}}{m}$.

Proof: Verification of the first result, indeed for each $\beta < 0$ there exists $\alpha_2^+(\beta) > 0$ such that:

$$\lambda_2(\alpha_2^+(\beta)m + \beta m^{'}) = 1,$$

so we have

$$\alpha_2^+(\beta)m + \beta m^{'} > 0 \text{ in } \Omega_\alpha \subset \Omega \quad \text{with meas}\,(\Omega_\alpha) > 0,$$

hence necessarily $\Omega_\alpha \subset \Omega_m^+$, if follows that:

$$\frac{-\alpha_2^+(\beta)}{\beta} > \frac{m^{'}}{m} \quad \text{in } \Omega_\alpha \subset \Omega_m^+,$$

thus

$$\liminf_{\beta\to-\infty} \frac{-\alpha_2^+(\beta)}{\beta} \geq \text{ess inf}_{\Omega_m^+} \frac{m^{'}}{m}. \tag{4.1}$$

Let $k = \limsup_{\beta\to-\infty} \dfrac{-\alpha_2^+(\beta)}{\beta}$, for a subsequence $(\beta_n)\,(\beta_n \to -\infty)$, we have:

$$\lim_{n\to+\infty} \frac{-\alpha_2^+(\beta_n)}{\beta_n} = k \quad \text{and} \quad \lambda_2(\frac{-\alpha_2^+(\beta_n)m}{\beta_n} - m^{'}) = -\beta_n$$

since $\frac{-\alpha_2^+(\beta_n)m}{\beta_n} - m^{'} \to km - m^{'}$ in $L^\infty(\Omega)$, and $-\beta_n \to +\infty$, from proposition 2.2 we conclude that: $km - m^{'} \leq 0$ almost every where in Ω, hence

$$k \leq \text{ess inf}_{\Omega_m^+} \frac{m^{'}}{m} \tag{4.2}$$

By (4.1) and (4.2), we deduce that

$$\lim_{\beta\to-\infty} \frac{\alpha_2^+(\beta)}{\beta} = -\text{ess inf}_{\Omega_m^+} \frac{m^{'}}{m}.$$

The proof of conclusion 2 is similar to that of the previous. □

Theorem 4.2. . *Assume* (H_0) *and* $m \geq 0$ *in* Ω, *then we have:*

$$\lim_{\alpha\to-\infty} \frac{\beta_2(\alpha)}{\alpha} = -\textit{ess inf}_{\Omega^\star_{m'}} \frac{m}{m^{'}}.$$

Proof: For each $\alpha < 0$ there exists $\beta_2(\alpha) > 0$ such that $\lambda_2(\alpha m + \beta_2(\alpha)m^{'}) = 1$ (see theorem 3.3), thus we have

$$\alpha m + \beta_2(\alpha)m^{'} > 0 \text{ in } \Omega_\alpha \text{ with meas}\,(\Omega_\alpha) > 0,$$

so necessarily $\Omega_\alpha \subset \Omega^\star_{m'}$, hence we deduce that:

$$\liminf_{\alpha\to-\infty} \frac{-\beta_2(\alpha)}{\alpha} \geq \text{ess inf}_{\Omega^\star_{m'}} \frac{m}{m^{'}}. \tag{4.3}$$

Let $k = \limsup_{\alpha \to -\infty} \dfrac{-\beta_2(\alpha)}{\alpha}$. Using the same argument as in the proof of theorem4.1, we get

$$k \leq \operatorname{ess\,inf}_{\Omega^\star_{m'}} \frac{m}{m'}.$$

So by (4.3), we conclude that:

$$\lim_{\alpha \to -\infty} \frac{-\beta_2(\alpha)}{\alpha} = \operatorname{ess\,inf}_{\Omega^\star_{m'}} \frac{m}{m'}.$$

□

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