On the second eigencurve for the p-laplacian operator with weight

A.DAKKAK[(1)], M.MOUSSAOUI[(2)]

1. *Laboratory-LSI- sidi Mohamed ben Abdellah University, polydisciplinary, Faculty of TAZA, MOROCCO*

2. *Regional Center of Trads Education and Training, TAZA, MOROCCO*

dakkakahmed@hotmail.com

mmoussaoui_2@yahoo.fr

Abstract. In this paper we establish the existence of the second eigencurves of the p-laplacian with indefinite weights. we obtain also their asymptotic behavior and variational formulation.

Submitted August 2015.

2000 Mathematic Subject Classification 35j48 ; 35j65 ; 35j60 ; 47j30 ; 58E05......

Key words : eigencurves ; p-laplacian ; asymptotic behavior ; variational formulation. [1]

1 Introduction

We consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u &= \lambda m(x)|u|^{p-2}u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian, $1 < p < +\infty$ and $m(.) \in M^+(\Omega) = \{u \in L^\infty(\Omega) : \text{meas}\{x \in \Omega : m(x) > 0\} > 0\}$ is a weight function which can change sign.

The spectrum of p-laplacian operator with indefinite weight is defined as the set $\sigma_p(-\Delta_p, m, \Omega)$ of $\lambda = \lambda(m, \Omega)$ for which there exists a nontrivial solution $u \in W_0^{1,p}(\Omega)$ of problem (1.1), this values are called eigenvalues and

1. 2000 Mathematic Subject Classification 35j48 ; 35j65 ; 35j60 ; 47j30 ; 58E05.....

the corresponding solutions are called eigenfunctions.
We will denote $\sigma_p^+(-\Delta_p, m, \Omega)$ the set of all positive eigenvalues.

For $p = 2$ ($\Delta_p = \Delta$ Laplacian Operator) it is well known (see [11]) that $\sigma_p^+(-\Delta_p, m, \Omega) = \{\mu_k(m, \Omega), k = 1, 2 \ldots\}$, with $0 < \mu_1(m, \Omega) < \mu_2(m, \Omega) \leq \mu_3(m, \Omega) \ldots \to +\infty$, $\mu_k(m, \Omega)$ repeated according to its multiplicity.

For $p \neq 2$ (nonlinear problem), the critical point theory of Ljusternik Schnirelman (see [12]) provides that $\sigma_p^+(-\Delta_p, m, \Omega)$ contains an infinite sequence of eigenvalues for these problems given by $\lambda_1(m, \Omega) < \lambda_2(m, \Omega) \leq \lambda_3(m, \Omega) \ldots \lambda_n(m, \Omega) \to +\infty$ and formulated as follows

$$\frac{1}{\lambda_n(m)} = \sup_{K \in \Gamma_n} \min_{u \in K} \int_\Omega m|u|^p \tag{1}$$

where Γ_n is defined by :

$$\Gamma_n = \{K \subset S : K \text{ is symmetrical, compact and } \quad \xi(K) \geq n\},$$

S is the sphere unity of $W_0^{1,p}(\Omega)$ and ξ is the genus function.
We may also define the negative spectrum when $-m \in M^+(\Omega)$ by $-\sigma^+(-\Delta_p, -m, \Omega)$ which contains an infinite sequence $\lambda_{-1}(m, \Omega) > \lambda_{-2}(m, \Omega) \geq \lambda_{-3}(m, \Omega) \ldots \geq \lambda_{-n}(m, \Omega) \to -\infty$, such that $\lambda_{-n}(m, \Omega) = -\lambda_n(-m, \Omega)$ (See [1], [2], [3], [7]...).
Whether or not this sequence denoted $\lambda_k(m, \Omega)$ constitutes the set of all eigenvalues is an open question when $N > 1$, $m \neq 1$ and $p \neq 2$.

The purpose of this article is to study the following problem :
Find the real numbers α, $\beta_2(\alpha)$ *such that* $\lambda_2(\alpha m_1 + \beta_2(\alpha) m_2) = 1$ *and the asymptotic behavior of the eigencurve* $C_2 = \{(\alpha, \beta_2(\alpha)) : \lambda_2(\alpha m_1 + \beta_2(\alpha) m_2) = 1\}$, *where* m_1 *and* m_2 *satisfies only the condition :*

(H_0) $\qquad m_1, m_2 \in M^+(\Omega)$ *and* $m_2 \geq 0$ *in* Ω.

Several applications can be found in the bifurcation domain, we refer the reader to [6].
Many results have been obtained on this kind of problems (see ;[4], [5], [8], [9]), in [5] the authors proved some properties related to the first eigencurve C_1 such as concavity, defferentiability and the asymptotic behavior, this last property can not be adapted to the other eigencurves, in [8] the authors have studied this class of problems under the following assumptions

(H') $\qquad m_1, m_2 \in M^+(\Omega)$ and $\operatorname{ess\,inf}_\Omega m_2 > 0$.

In [9], the authors proved some results under the assumptions :

$$(H^{''}) \qquad m_1, m_2 \in M^+(\Omega) \text{ and } \operatorname{ess\ inf}_{\Omega^\star_{m_1}} m_2 > 0,$$

where $\Omega^\star_{m_1} = \{x \in \Omega : m_1(x) \neq 0\}$.

This article is organized as follows, in section 2 we recall some basic result, in section 3 we study the existence of the eigencurve C_2 and in section 4 we study the asymptotic behavior of C_2.

2 Preliminary results

Firstly we recall the following results which will be used later.

Proposition 2.1 *([3], [8])*

1. *Let m, $m^{'} \in M^+(\Omega)$. If $m \leq m^{'}$ (resp $m < m^{'}$), then $\lambda_n(m) \geq \lambda_n(m^{'})$ (resp $\lambda_n(m) > \lambda_n(m^{'})$).*
2. *$\lambda_n : m \to \lambda_n(m)$ is continuous in $(M^+(\Omega), ||.||_\infty)$.*

Proposition 2.2 *Let (m_k) be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$, then we have :*

$$\lim_{k\to+\infty} \lambda_n(m_k) = +\infty \text{ if and only if } m \leq 0 \text{ almost everywhere in } \Omega.$$

Proof.
Let (m_k) be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$.
Assume first that $\lim_{k\to+\infty} \lambda_n(m_k) = +\infty$, we claim that $m \leq 0$ almost everywhere in Ω, indeed, if meas$\{x \in \Omega : m(x) > 0\} \neq 0$, we get

$$\lim_{k\to+\infty} \lambda_n(m_k) = \lambda_n(m)$$

is a finite, which gives a contradiction.
Inversely, if $m \leq 0$ almost everywhere in Ω, suppose by contradiction that there exists $\lambda > 0$ such that

$$\lambda_n(m_k) \leq \lambda \quad \forall k \in \mathbb{N}^\star$$

Let $r = \dfrac{2\lambda}{\lambda_n(2)}$, Since $m_k \to m$ in $L^\infty(\Omega)$, there exists $N \in \mathbb{N}$, such that $\forall k \geq N$, we have :

$$||m_k - m||_\infty \leq \frac{2}{r},$$

hence

$$m_k \leq m + \frac{2}{r} \quad p.p.x \in \Omega.$$

So, using the fact that $m \leq 0 \quad p.p.x \in \Omega$, we conclude that

$$m_k \leq \frac{2}{r} \quad p.p.x \in \Omega.$$

It follows that

$$\lambda_n(m_k) \geq \lambda_n(\frac{2}{r}) = r\lambda_n(2) = 2\lambda.$$

Which is a contradiction. The proof is complete. ■

3 Existence of the eigencurve C_2

For $m \in M^+(\Omega)$, we denote by $\Omega_m^- = \{x \in \Omega : m(x) < 0\}$ and $\Omega_m^+ = \{x \in \Omega : m(x) > 0\}$.

Theorem 3.1 *Assume (H_0) holds, then we have :*

1. *For all $\alpha \in [0, \lambda_2(m_1)]$, there exists $\beta_2(\alpha) \in \mathbb{R}^+$ such that $\lambda_2(\alpha m_1 + \beta_2(\alpha)m_2) = 1$.*
2. *If $\alpha > \lambda_2(m_1)$, we have,*

$$\lambda_2(\alpha m_1 + \beta m_2) = 1 \Rightarrow \beta < 0.$$

3. *for all $\beta < 0$ there exists $\alpha_2^+(\beta) > 0$ such that :*
   (i) $\lambda_2(\alpha_2^+(\beta)m_1 + \beta m_2) = 1$.
   (ii) if $\gamma > 0$ and $\lambda_2(\gamma m_1 + \beta m_2) = 1$ then $\gamma = \alpha_2^+(\beta)$.
4. *Assume meas $(\Omega_{m_1}^-) > 0$, we have :*
   (i) if $\alpha < \lambda_{-2}(m_1)$ then, $\lambda_2(\alpha m_1 + \beta m_2) = 1 \Rightarrow \beta < 0$.
   (ii) For all $\beta < 0$ there exists $\alpha_2^-(\beta)$ such that :
   (a) $\lambda_2(\alpha_2^-(\beta)m_1 + \beta m_2) = 1$.
   (b) if $\gamma < 0$ and $\lambda_2(\gamma m_1 + \beta m_2) = 1$ then $\gamma = \alpha_2^-(\beta)$.
5. *Assume meas $(\Omega_{m_1}^-) = 0$, then for all $\alpha < 0$ there exists $\beta^+(\alpha)$ such that*

$$\lambda_2(\alpha m_1 + \beta m_2) = 1 \Leftrightarrow \beta = \beta^+(\alpha).$$

Proof.
1. We consider the real function $h_\alpha(.)$ defined by $h_\alpha(t) = \lambda_2(\alpha m_1 + t m_2)$, h_α

is decreasing and continuous in $[0,+\infty[$ (see proposition 2.1), in other hand :
If $\alpha \in]0, \lambda_2(m_1)]$, we have :

$$\begin{array}{rcl} h_\alpha(0) & = & \lambda_2(\alpha m_1) \\ & = & \frac{\lambda_2(m_1)}{\alpha} \\ & \geq & 1. \end{array}$$

and for $t > 0$, we have :

$$\begin{array}{rcl} h_\alpha(t) & = & \lambda_2(\alpha m_1 + t m_2) \\ & = & \frac{1}{t}\lambda_2(\frac{\alpha m_1}{t} + m_2), \end{array}$$

hence

$$\lim_{t\to+\infty} h_\alpha(t) = 0.$$

Thus, since h_α is continuous, we deduce that there exists a real $\beta_2(\alpha) \in [0,+\infty[$ such that $h_\alpha(\beta_2(\alpha)) = 1$.
If $\alpha = 0$, we take $\beta_2(\alpha) = \lambda_2(m_2)$.
2. Assume that $\alpha > \lambda_2(m_1)$.
For $\beta \geq 0$, we have :

$$\alpha m_1 \leq \alpha m_1 + \beta m_2,$$

so

$$\lambda_2(\alpha m_1 + \beta m_2) \leq \lambda_2(\alpha m_1) = \frac{\lambda_2(m_1)}{\alpha} < 1,$$

hence, if $\lambda_2(\alpha m_1 + \beta m_2) = 1$ necessarily we have $\beta < 0$.
3. (**i**). We denote by $\Gamma_2 = \{K \in S, \quad K \text{ is compact, symmetric and } \xi(K) \geq 2\}$, where ξ is the genus function and $S = \{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p = 1\}$.
For $\beta < 0$ we define $\alpha_2^+(\beta)$ as follows :

$$\frac{1}{\alpha_2^+(\beta)} = \sup_{K\in\Gamma_2} \inf_{u\in K} \frac{\int_\Omega m_1|u|^p}{1 - \beta\int_\Omega m_2|u|^p}.$$

By definition of $\alpha_2^+(\beta)$ and the property of $\lambda_2(m)$ (see [3]), we deduce that there exists eigenfunction u which change sign in Ω such that :

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla w - \beta m_2|u|^{p-2}uw = \int_\Omega \alpha_2^+(\beta) m_1|u|^{p-2}uw \quad \forall w \in W_0^{1,p}(\Omega), \tag{2}$$

we deduce also that, if $\varphi \in W_0^{1,p}(\Omega)$ is eigenfunction of $-\Delta_p - \beta m_2$, change singe in Ω with the corresponding eigenvalue $\lambda > 0$ that is :

$$\int_\Omega |\nabla \varphi|^{p-2}\nabla \varphi\nabla w - \beta m_2|\varphi|^{p-2}\varphi w = \lambda\int_\Omega m_1|\varphi|^{p-2}\varphi w \quad \forall w \in W_0^{1,p}(\Omega), \tag{3}$$

then $\lambda \geq \alpha_2^+(\beta)$.
From (2), we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w = \int_\Omega (\alpha_2^+(\beta)m_1 + \beta m_2)|u|^{p-2}uw \quad \forall w \in W_0^{1,p}(\Omega), \qquad (4)$$

hence the real 1 is eigenvalue of $-\Delta_p$ with weight $(\alpha_2^+(\beta)m_1 + \beta m_2)$, since the corresponding eigenfunction u change singe in Ω, we conclude that :

$$\lambda_2(\alpha_2^+(\beta)m_1 + \beta m_2) \leq 1. \qquad (5)$$

In other hand, let $K \in \Gamma_2$, we have

$$\min_{v\in K} \frac{\int_\Omega m_1|v|^p}{1-\beta\int_\Omega m_2|v|^p} \leq \frac{1}{\alpha_2^+(\beta)},$$

and

$$\min_{v\in K} \frac{\int_\Omega m_1|v|^p}{1-\beta\int_\Omega m_2|v|^p} = \frac{\int_\Omega m_1|v_k|^p}{1-\beta\int_\Omega m_2|v_k|^p} \qquad \text{for some } v_k \in K,$$

so we deduce :

$$\min_{v\in K} \int_\Omega (\alpha_2^+(\beta)m_1|v|^p + \beta m_2|v|^p) \leq 1,$$

taking cont that $K \in \Gamma_2$ is arbitrary, we get

$$\frac{1}{\lambda_2(\alpha_2^+(\beta)m_1 + \beta m_2)} = \sup_{K\in\Gamma_2} \min_{v\in K} \int_\Omega (\alpha_2^+(\beta)m_1 + \beta m_2)|v|^p \leq 1,$$

so

$$\lambda_2(\alpha_2^+(\beta)m_1 + \beta m_2) \geq 1 \qquad (6)$$

(5) and (6) gives

$$\lambda_2(\alpha_2^+(\beta)m_1 + \beta m_2) = 1.$$

3. (**ii**). Let $\gamma > 0$ such that $\lambda_2(\gamma m_1 + \beta m_2) = 1$, there exists eigenfunction θ change singe in Ω and

$$\int_\Omega |\nabla \theta|^{p-2}\nabla\theta\nabla w = \int_\Omega (\gamma m_1 + \beta m_2)|\theta|^{p-2}\theta w \quad \forall w \in W_0^{1,p}(\Omega),$$

hence

$$\int_\Omega |\nabla \theta|^{p-2}\nabla\theta\nabla w - \beta m_2|\theta|^{p-2}\theta w = \gamma \int_\Omega m_1|\theta|^{p-2}\theta w \quad \forall w \in W_0^{1,p}(\Omega), \qquad (7)$$

from (7), we conclude that γ is eigenfunction of the operator $(-\Delta_p - \beta m_2)$ with weight m_1, since the eigenfunction θ change singe, we conclude that :

$$\gamma \geq \alpha_2^+(\beta).$$

Assume by contradiction that $\gamma > \alpha_2^+(\beta)$.
So

$$\frac{1}{\gamma} < \frac{1}{\alpha_2^+(\beta)} = \sup_{K\in\Gamma_2} \min_{v\in K} \frac{\int_\Omega m_1|v|^p}{1-\beta\int_\Omega m_2|v|^p},$$

by the inequality above we deduce that there exists $K_0 \in \Gamma_2$ such that

$$\frac{1}{\gamma} < \min_{v\in K_0} \frac{\int_\Omega m_1|v|^p}{1-\beta\int_\Omega m_2|v|^p},$$

since K_0 is compact, we conclude that

$$\frac{1}{\gamma} < \frac{\int_\Omega m_1|v_0|^p}{1-\beta\int_\Omega m_2|v_0|^p}, \quad \text{for some } v_0 \in K_0,$$

hence

$$1 < \min_{K_0} \int_\Omega (\gamma m_1 + \beta m_2)|v|^p,$$

it follows that

$$1 < \sup_{K\in\Gamma_2} \min_{v\in K} \int_\Omega (\gamma m_1 + \beta m_2)|v|^p = \frac{1}{\lambda_2(\gamma m_1 + \beta m_2)} = 1,$$

which is a contradiction, hence we have $\gamma = \alpha_2^+(\beta)$.
4 (i) $\alpha < \lambda_{-2}(m_1)$. We consider the real function $h_\alpha(.)$ defined by $h_\alpha(t) = \lambda_2(\alpha m_1 + t m_2)$, since $h_\alpha(0) = \frac{\lambda_{-2}(m_1)}{\alpha} < 1$, then necessarily we have

$$h_\alpha(\beta) = 1 \Rightarrow \beta < 0,$$

that is

$$\lambda_2(\alpha m_1 + \beta m_2) = 1 \Rightarrow \beta < 0.$$

4 (ii). We define $\alpha_2^-(\beta)$ as follows

$$\frac{1}{\alpha_2^-(\beta)} = \inf_{K\in\Gamma_2} \max_{u\in K} \frac{\int_\Omega m_1|u|^p}{1-\beta\int_\Omega m_2|u|^p},$$

the proof is similar to that of 3.
5. In this case we consider the coercive operator $-\Delta_p - \alpha m_1$, we define $\beta^+(\alpha)$ as follows

$$\beta^+(\alpha) = \sup_{K\in\Gamma_2} \inf_{u\in K} \frac{\int_\Omega m_2|u|^p}{1-\alpha\int_\Omega m_1|u|^p},$$

taking cont that, for $\beta < 0$, $\alpha m_1 + \beta m_2 \notin M^+(\Omega)$, by the same proof as 3 of theorem , we deduce the result.

4 Asymptotic behavior of C_2

Theorem 4.1 *Assume (H_0) holds, then we have :*

1. $\lim_{\beta\to-\infty} \frac{\alpha_2^+(\beta)}{\beta} = -inf\, ess_{\Omega^+_{m_1}} \frac{m_2}{m_1}$.
2. *If $mes(\Omega^-_{m_1}) > 0$, then*
   $\lim_{\beta\to-\infty} \frac{\alpha_2^-(\beta)}{\beta} = -sup\, ess_{\Omega^-_{m_1}} \frac{m_2}{m_1}$.

Proof.
1. For each $\beta < 0$ there exists $\alpha_2^+(\beta) > 0$ such that :

$$\lambda_2(\alpha_2^+(\beta)m_1 + \beta m_2) = 1,$$

so we have

$$\alpha_2^+(\beta)m_1 + \beta m_2 > 0 \text{ in}\,\Omega_\alpha \subset \Omega \text{ with meas}\,(\Omega_\alpha) > 0,$$

hence necessarily $\Omega_\alpha \subset \Omega^+_{m_1}$, if follows that :

$$\frac{-\alpha_2^+(\beta)}{\beta} > \frac{m_2}{m_1} \quad \text{in}\,\Omega_\alpha \subset \Omega^+_{m_1},$$

thus

$$\liminf_{\beta\to-\infty} \frac{-\alpha_2^+(\beta)}{\beta} \geq \inf \text{ess}_{\Omega^+_{m_1}} \frac{m_2}{m_1}. \tag{8}$$

Let $k = \limsup_{\beta\to-\infty} \dfrac{-\alpha_2^+(\beta)}{\beta}$, for a subsequence $(\beta_n)\,(\beta_n \to -\infty)$, we have :

$$\lim_{n\to+\infty} \frac{-\alpha_2^+(\beta_n)}{\beta_n} = k,$$

and

$$\lambda_2(\alpha_2^+(\beta_n)m_1 + \beta_n m_2) = 1,$$

then

$$\lambda_2(\frac{-\alpha_2^+(\beta_n)m_1}{\beta_n} - m_2) = -\beta_n,$$

since $\frac{-\alpha_2^+(\beta_n)m_1}{\beta_n} - m_2 \to km_1 - m_2$ in $L^\infty(\Omega)$, and $-\beta_n \to +\infty$, from proposition 2.2 we conclude that :

$$km_1 - m_2 \leq 0 \quad \text{almost every where in}\,\Omega,$$

hence

$$k \leq \inf \text{ess}_{\Omega^+_{m_1}} \frac{m_2}{m_1} \tag{9}$$

By (8) and (9), we deduce that

$$\lim_{\beta\to-\infty}\frac{\alpha_2^+(\beta)}{\beta}=-\inf\mathrm{ess}_{\Omega^+_{m_1}}\frac{m_2}{m_1}.$$

The proof of 2) is similar to that of the previous. ■

Theorem 4.2 *Assume* (H_0) *and* $m_1\geq 0$ *in* Ω, *then we have :*

$$\lim_{\alpha\to-\infty}\frac{\beta_2^+(\alpha)}{\alpha}=-inf\,ess_{\Omega^\star_{m_2}}\frac{m_1}{m_2}.$$

Proof.
For each $\alpha<0$ there exists $\beta_2^+(\alpha)>0$ such that $\lambda_2(\alpha m_1+\beta_2^+(\alpha)m_2)=1$ (see theorem 3.1), thus we have

$$\alpha m_1+\beta_2^+(\alpha)m_2>0\ \text{in}\,\Omega_\alpha\,\text{with meas}\,(\Omega_\alpha)>0,$$

so necessarily $\Omega_\alpha\subset\Omega^\star_{m_2}$, hence we deduce that :

$$\liminf_{\alpha\to-\infty}\frac{-\beta_2^+(\alpha)}{\alpha}\geq\inf\mathrm{ess}_{\,\Omega^\star_{m_2}}\frac{m_1}{m_2}. \tag{10}$$

Let $k=\limsup_{\alpha\to-\infty}\dfrac{-\beta_2^+(\alpha)}{\alpha}$.
Using the same argument as in the proof of theorem4.2, we get

$$k\leq\inf\mathrm{ess}_{\,\Omega^\star_{m_2}}\frac{m_1}{m_2}.$$

So by (10), we conclude that :

$$\lim_{\alpha\to-\infty}\frac{-\beta_2^+(\alpha)}{\alpha}=\inf\mathrm{ess}_{\,\Omega^\star_{m_2}}\frac{m_1}{m_2}.$$

■

Acknowledjement. The authors wish to think the knowledjeable referees for their corrections and remarks.

Références

[1] A.Anane. O.Chakrone. M.Moussa, Spectrum of one dimensional p-laplacian operator with indefinite weight, EJQTDE, (2002), No 17, p1-11.

[2] A.Anane, Simplicité et isolation de la première valeur propre du p-laplacen avec poids, C.R.A.S, Paris, t. 305(1987), pp. 725-728.

[3] A.Anane. N.Tsouli, On the second eigenvalue of the p-laplacian, Nonlinear Partial Differential Equations, Pitman Research, Notes 343(1996), 1-9.

[4] P. A. Binding. P. J. Browne. Y. X. Huang. R. H. Picard, On eigencurves of elliptic boundary value problems, Proc. Roy. Soc. Edim. 118A, (1991) pp. 161-171

[5] P. A. Binding. Y. X. Huang, The principal eigenvalue of the plaplacian, Diff. Int. Equs. 8(1995), 405-414.

[6] P. A. Binding. Y. X. Huang, Bifurcation from eigencurves of the p-laplacian, Diff. Int. Equs. 8(1995), 415-428.

[7] M. Cuesta. Eigenvalue problems for the p-laplcian with indefinite weights. EJDE, V(2001), 33, PP. 1-9. (2001).

[8] A. Dakkak. M. Hadda, Eigencurves of the p-laplacian weights and their asymptotic behavior, EJDE. 35(2007), pp. 1-7.

[9] A. Dakkak. M. Moussaoui, Existence and asymptotic behavior of the eigencurves for the p-laplacian operator with weight, submitted.

[10] D.G. De Figueiredo, Positive solution of semi-linear elliptic equation, Lecture Notes in Mathematics, 957, (1982), pp 34-87.

[11] D.G. De Figueiredo and J.P.Gossez, Strict monotonicity of eigenvalues and unique continuation, Comm. Par. Diff. Equat, 17,(1992), pp 339-346.

[12] A. Szulkin, Ljusternik-Schnirelmann theory on C^1-manifolds, Ann. I. H. Poincaré, Anal. non linéaire, 5,(1988), pp.119-139.

[13] N. Tsouli, Etude de l'ensemble nodal des fonctions propres et de la non-résonance por l'opérateur p-laplacien, Tèse de Doctorat d'Etat Es-sciences, Faculté des sciences Oujda Maroc, (1995).