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Approximate mixed type additive and quartic functional equation

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ABSTRACT: In the current work, we introduce a general form of a mixed additive and quartic functional equation. We determine all solutions of this functional equation. We also establish the generalized Hyers-Ulam stability of this new functional equation in quasi-β-normed spaces.

Key Words: Additive functional equation; Hyers-Ulam stability; Quartic functional equation; Quasi-β-normed space.

Contents

1. Introduction

In 1940, Ulam [26] proposed the following stability problem:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?". Hyers [16] has given an affirmative answer to a question of Ulam by proving the stability of additive Cauchy equations in Banach spaces. Then, Aoki [1] and Th. M. Rassias [22] considered the stability problem with unbounded Cauchy differences for additive and linear mappings, respectively (see also [15]. This phenomenon is called generalized Ulam-Hyers stability and has been extensively investigated for different functional equations (for instance, [5], [7], [11] and [24]). It is worth mentioning that almost all proofs used the idea conceived by Hyers which is called the direct method or Hyers method. Cădariu and Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [10] and for the quadratic functional equation [9] (for more applications of this method, refer to [6], [8] and [20]).

The quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y) \tag{1.1}$$

was introduced by J. M. Rassias in [21], and then was studied by other authors. Rassias [21] investigated stability properties of the quartic functional equation

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(1.1). Other versions of a quartic functional equation can be found in [3], [17], [18] and [19].

In [14], Eshaghi Gordji introduced and obtained the general solution of the following mixed type additive and quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) &= 4\{(f(x+y)+f(x-y)\} \\ &\quad -\frac{3}{7}(f(2y)-2f(y))+2f(2x)-8f(x). \end{aligned} \tag{1.2}$$

He also proved the Hyers-Ulam Rassias stability of the functional equation (1.2) in real normed spaces. Recently, Bodaghi [4] presented a new form of the additive-quartic functional equation which is different from (1.2) as follows:

$$f(x+2y)-4f(x+y)-4f(x-y)+f(x-2y)=\frac{12}{7}(f(2y)-2f(y))-6f(x) \tag{1.3}$$

In this paper, we consider the following functional equation which is a general form of (1.3):

$$\begin{aligned} f(x+ny)+f(x-ny) &= n^2\{f(x+y)+f(x-y)\} \\ &\quad +\frac{1}{7}n^2(n^2-1)(f(2y)-2f(y))-2(n^2-1)f(x) \end{aligned} \tag{1.4}$$

It is easily verified that the function $f(x)=\alpha x+\beta x^4$ is a solution of the functional equation (1.4). Our aim is to highlight generalized Ulam-Hyers stability results for the functional equation (1.4) in a single variable for mappings with values in quasi-β-normed spaces, obtained by a result of Xu et al. (Lemma 3.2 of this writing) based on the fixed point alternative theorem.

2. Solution of Equation (1.4)

To achieve our aim in this section, we need the following result.

Theorem 2.1. *Let X and Y be real vector spaces. Then, the mapping $f: X \longrightarrow Y$ satisfies the functional equation (1.3) if and only if it satisfies the functional equation (1.4) for all $n \geq 3$.*

Proof: Assume that $f: X \longrightarrow Y$ satisfies the functional equation (1.3). Putting $x=y=0$ in (1.3), we have $f(0)=0$. Replacing x by $x+y$ and $x-y$ in (1.3), respectively, and adding those equations, we obtain

$$f(x+3y)+f(x-3y)=9\{f(x+y)+f(x-y)\}+\frac{72}{7}(f(2y)-2f(y))-16f(x).$$

Similar to the above, we can deduce that

$$f(x+4y)+f(x-4y)=16\{f(x+y)+f(x-y)\}+\frac{240}{7}(f(2y)-2f(y))-30f(x).$$

Using the above method, we get

$$f(x+ny)+f(x-ny)=n^2\{f(x+y)+f(x-y)\}+a_n(f(2y)-2f(y))-b_nf(x)$$

where

$$\begin{cases} a_n=2a_{n-1}-a_{n-2}+\frac{12}{7}(n-1)^2, & a_2=\frac{12}{7},\ a_3=\frac{72}{7}, \\ b_n=-b_{n-2}+4(n-1)^2, & b_2=6,\ b_3=16. \end{cases}$$

Solving the above recurrence equations, we have

$$a_n=\frac{1}{7}n^2(n^2-1) \quad \text{and} \quad b_n=2(n^2-1)$$

for all $x,y\in X$ and all positive integers $n\geq 2$.

Conversely, suppose that f satisfies the functional equation (1.4) for any positive integer $n\geq n_0$ where n_0 is a fixed integer with $n_0>2$. So, f satisfies (1.4) for every positive integer $k\geq n_0$, in particular for $k=n(n-1)$. In other words, replacing y by $(n-1)y$ in (1.4), we have

$$\begin{aligned} f(x+n(n-1)y)+f(x-n(n-1)y)&=n^2\{f(x+(n-1)y)+f(x-(n-1)y)\}\\ &\quad+\frac{1}{7}n^2(n^2-1)(f(2(n-1)y)-2f((n-1)y))\\ &\quad-2(n^2-1)f(x) \end{aligned} \tag{2.1}$$

for all $x,y\in X$. On the other hand, replacing n by n^2-n in (1.4), we arrive at

$$\begin{aligned} f(x+(n^2-n)y)+f(x-(n^2-n)x)&=(n^2-n)^2\{f(x+y)+f(x-y)\}\\ &\quad+\frac{1}{7}(n^2-n)^2((n^2-n)^2-1)(f(2y)-2f(y))\\ &\quad-2((n^2-n)^2-1)f(x) \end{aligned} \tag{2.2}$$

for all $x,y\in X$. Using (2.1) and (2.2), we get

$$\begin{aligned} \frac{1}{7}n^2(n^2-1)(f(2(n-1)y)-2f((n-1)y))&=(n^2-n)^2\{f(x+y)+f(x-y)\}\\ &+\frac{1}{7}(n^2-n)^2((n^2-n)^2-1)(f(2y)-2f(y))-2((n^2-n)^2-1)f(x)\\ &-n^2\{f(x+(n-1)y)+f(x-(n-1)y)\}+2(n^2-1)f(x) \end{aligned} \tag{2.3}$$

for all $x,y\in X$. We have

$$\begin{aligned} f(x+(n+1)(n-1)y)+f(x-(n+1)(n-1)y)&=(n+1)^2\{f(x+(n-1)y)\\ &\quad+f(x-(n-1)y)\}\\ &\quad+\frac{1}{7}(n+1)^2((n+1)^2-1)(f(2(n-1)y)-2f((n-1)y))\\ &\quad-2((n+1)^2-1)f(x) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Also,

$$f(x+(n^2-1)y)+f(x-(n^2-1)x) = (n^2-1)^2\{f(x+y)+f(x-y)\} + \frac{1}{7}(n^2-1)^2((n^2-1)^2-1)(f(2y)-2f(y)) - 2((n^2-1)^2-1)f(x) \tag{2.5}$$

for all $x, y \in X$. Plugging (2.4) into (2.5), and using (2.3), we get

$$f(x+(n-1)y)+f(x-(n-1)y) = (n-1)^2\{f(x+y)+f(x-y)\} + \frac{1}{7}(n-1)^2((n-1)^2-1)(f(2y)-2f(y)) - 2((n-1)^2-1)f(x)$$

This means that f fulfilling (1.4) for all $n \geq n_0 - 1$. This completes the proof. □

Remark 2.2. *Let n be a positive integer. Then, the functional equation (1.4) holds if we replae n by $-n$. So, the mapping $f : X \longrightarrow Y$ satisfies the functional equation (1.3) if and only if it satisfies the functional equation (1.4) for all integer numbers n.*

Lemma 2.3. *Let X and Y be real vector spaces.*

(i) If an odd function $f : X \longrightarrow Y$ satisfies the functional equation (1.4), then f is additive;

(ii) If an even function $f : X \longrightarrow Y$ satisfies the functional equation (1.4), then f is quartic.

Proof: The result follows from Theorem 2.1 and [4, Lemma 2.1]. □

3. Stability of (1.4) in quasi-β normed spaces

We recall some basic facts concerning quasi-β-normed space.

Definition 3.1. *Let β be a fix real number with $0 < \beta \leq 1$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:*

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and $t \in \mathbb{K}$;

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\|+\|y\|)$ for all $x, y \in X$.

Note that the condition (iii) imlies that

$$\left\|\sum_{j=1}^{2n} x_j\right\| \leq K^n \sum_{j=1}^{2n} \|x_j\| \qquad \text{and} \qquad \left\|\sum_{j=1}^{2n+1} x_j\right\| \leq K^{n+1} \sum_{j=1}^{2n+1} \|x_j\|,$$

for all $n \geq 1$ and $x_1, x_2, ..., x_{2n+1} \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X . The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-β-normed space is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0 < p \leq 1)$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$, for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space.

Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [23] (see also [2]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, here and subsequently, we restrict our attention mainly to p-norms. Moreover in [25], Tabor has investigated a version of Hyers-Rassias-Gajda Theorem in quasi-Banach spaces.

From now on, let X be a linear space with norm $\|\cdot\|_X$ and Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$ and K be the modulus of concavity of $\|\cdot\|_Y$, unless otherwise explicitly stated. In this section, by using an idea of Găvruta [12] we prove the stability of (1.4) in the spirit of Hyers, Ulam, and Rassias.

For notational convenience, given a function $f : X \longrightarrow Y$, we define the difference operator

$$\begin{aligned}\Delta_{a,q} f(x,y) &= f(x+ny) + f(x-ny) - n^2\{f(x+y) + f(x-y)\} \\ &\quad - \frac{1}{7}n^2(n^2-1)(f(2y) - 2f(y)) + 2(n^2-1)f(x)\end{aligned}$$

for all $x, y \in X$.

Before obtaining the main results in this section, we bring the following lemma which is proved in [27, Lemma 3.1] (see also the fixed point alternative of [13]).

Lemma 3.2. *Let $j \in \{-1, 1\}$ be fixed, $a, s \in \mathbb{N}$ with $a \geq 2$ and $\psi : X \longrightarrow [0,\infty)$ a function such that there exists an $L < 1$ with $\psi(a^j x) < L a^{js\beta}\psi(x)$ for all $x \in X$. If $f : X \longrightarrow Y$ is a mapping satisfying*

$$\|f(ax) - a^s f(x)\|_Y \leq \psi(x)$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \longrightarrow Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}|1-L^j|}\psi(x)$$

for all $x \in X$.

In the upcoming result, we prove the stability for (1.4) in quasi-β-normed spaces.

Theorem 3.3. *Let $j \in \{-1,1\}$ be fixed, and let $\varphi : X \times X \longrightarrow [0,\infty)$ be a function such that there exists an $0 < L < 1$ with $\varphi(2^j x, 2^j y) \leqslant 2^{j\beta} L\varphi(x,y)$ for all $x \in X$. Let $f : X \longrightarrow Y$ be a odd mapping satisfying*

$$\|\Delta_{a,q} f(x,y)\|_Y \leq \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{1}{2^\beta |1-L^j|} \left(\frac{7}{n^2(n^2-1)}\right)^\beta \varphi(0,x) \tag{3.2}$$

for all $x \in X$.

Proof: Replacing (x,y) by $(0,x)$ in (3.1), we get

$$\|f(2x) - 2f(x)\|_Y \leqslant \left(\frac{7}{n^2(n^2-1)}\right)^\beta \varphi(0,x) \tag{3.3}$$

for all $x \in X$. By Lemma 3.2, there exists a unique mapping $A : X \to Y$ such that $A(2x) = 2A(x)$ and

$$\|f(x) - A(x)\|_Y \leqslant \frac{1}{2^\beta |1-L^j|} \left(\frac{7}{n^2(n^2-1)}\right)^\beta \varphi(0,x) \tag{3.4}$$

for all $x \in X$. It remains to show that A is an additive mapping. By (3.1), we have

$$\begin{aligned} \left\| \frac{\Delta_{a,q} f(2^{jn}x, 2^{jn}y)}{2^{jn}} \right\|_Y &\leqslant 2^{-jn\beta} \varphi(2^{jn}x, 2^{jn}y) \\ &\leqslant 2^{-jn\beta}(2^{j\beta}L)^n \varphi(x,y) = L^n \varphi(x,y) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we observe that $\Delta_{a,q} A(x,y) = 0$ for all $x, y \in X$. It follows from Lemma 2.3 that the mapping A is additive, as required. □

The following corollary is the direct consequence of Theorem 3.3 concerning the stability of (1.4).

Corollary 3.4. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ be a positive number with $\lambda \neq \frac{\beta}{\alpha}$. If $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\|\Delta_{a,q} f(x,y)\|_Y \leq \theta(\|x\|_X^\lambda + \|y\|_X^\lambda)$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\|_Y \leq \begin{cases} \left(\frac{7}{n^2(n^2-1)}\right)^\beta \frac{\theta}{2^\beta - 2^{\alpha\lambda}} \|x\|_X^\lambda & \lambda \in \left(0, \frac{\beta}{\alpha}\right) \\ \left(\frac{7}{n^2(n^2-1)}\right)^\beta \frac{2^{\alpha\lambda}\theta}{2^{\alpha\lambda} - 2^\beta} \|x\|_X^\lambda & \lambda \in \left(\frac{\beta}{\alpha}, \infty\right) \end{cases}$$

for all $x \in X$.

Proof: Taking $\varphi(x,y)=\theta(\|x\|_X^\lambda+\|y\|_X^\lambda)$ in Theorem 3.3, we can obtain the desired result. □

In the next result, we indicate the hyperstability of the functional equation (1.4) under some conditions. Recall that a functional equation is called hyperstable if every approximately solution is an exact solution of it.

Corollary 3.5. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ, r and s be positive numbers with $\lambda:=r+s\neq\frac{\beta}{\alpha}$. If $f:X\longrightarrow Y$ be an odd mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y\leqslant\theta\|x\|_X^r\|y\|_X^s$$

for all $x,y\in X$, then f is an additive mapping.

We have the following result which is analogous to Theorem 3.3 for the functional equation (1.4). We include the proof.

Theorem 3.6. *Let $j\in\{-1,1\}$ be fixed, and let $\varphi:X\times X\longrightarrow[0,\infty)$ be a function such that there exists an $0<L<1$ with $\varphi(2^jx,2^jy)\leqslant 2^{4j\beta}L\varphi(x,y)$ for all $x\in X$. Let $f:X\longrightarrow Y$ be an even mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y\leq\varphi(x,y) \tag{3.5}$$

for all $x,y\in X$. Then there exists a unique quartic mapping $Q:X\longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)\|_Y\leq\frac{1}{2^{4\beta}|1-L^j|}\widetilde{\varphi}(x) \tag{3.6}$$

for all $x\in X$ where

$$\begin{aligned}\widetilde{\varphi}(x):=&\,K^6\left(\frac{7}{n^2(n^2-1)}\right)^\beta[\varphi(nx,x)+n^{2\beta}\varphi(x,x)]\\&+K^5\left(\frac{49(n^2+1)}{n^4(n^2-1)}\right)^\beta\varphi(0,0)+K^4\left(\frac{7}{n^2}\right)^\beta\varphi(0,x)+K^3\left(\frac{98}{n^4}\right)^\beta\varphi(0,0)\\&+K\left(\frac{98}{n^4(n^2-1)}\right)^\beta\varphi(0,0)\end{aligned} \tag{3.7}$$

Proof: Putting $x=y=0$ in (3.5), we have

$$\|f(0)\|_Y\leqslant\left(\frac{7}{n^2(n^2-1)}\right)^\beta\varphi(0,0). \tag{3.8}$$

Replacing (x,y) by $(0,x)$ in (3.5) and using eveness of f, we get

$$\left\|2f(nx)-\frac{1}{7}n^2(n^2-1)f(2x)+\frac{2}{7}n^2(n^2-8)f(x)+2(n^2-1)f(0)\right\|_Y\leqslant\varphi(0,x) \tag{3.9}$$

for all $x \in X$. Interchanging (x,y) into (nx,x) in (3.5), we deduce that

$$\|f(2nx) + 2(n^2-1)f(nx) - n^2[f((n+1)x) + f((n-1)x)] - \frac{1}{7}n^2(n^2-1)f(2x) + \frac{2}{7}n^2(n^2-1)f(x) + f(0)\|_Y \leqslant \varphi(nx,x) \tag{3.10}$$

for all $x \in X$. Putting $x = y$ in (3.5), we obtain

$$\|f((n+1)x) + f((n-1)x) - \frac{1}{7}n^2(n^2+6)f(2x) + \frac{2}{7}(n^2-1)(n^2+7)f(x) - n^2 f(0)\|_Y \leqslant \varphi(x,x) \tag{3.11}$$

for all $x \in X$. Thus, multiply n^2 on both sides, we find

$$\|n^2[f((n+1)x) + f((n-1)x)] - \frac{1}{7}n^4(n^2+6)f(2x) + \frac{2}{7}n^2(n^2-1)(n^2+7)f(x) - n^4 f(0)\|_Y \leqslant n^{2\beta}\varphi(x,x) \tag{3.12}$$

for all $x \in X$. It follows from (3.8), (3.10) and (3.12) that

$$\begin{aligned}\|f(2nx) &+ 2(n^2-1)f(nx) - \frac{1}{7}n^2(n^4+7n^2-1)f(2x) \\ &+ \frac{2}{7}n^2(n^2-1)(n^2+8)f(x)\|_Y \\ &\leqslant K^2[\varphi(nx,x) + n^{2\beta}\varphi(x,x)] + K\left(\frac{7(n^2+1)}{n^2}\right)^{\beta}\varphi(0,0)\end{aligned} \tag{3.13}$$

for all $x \in X$. Multiplying both sides of (3.9) by $(n^2-1)^{\beta}$, we get

$$\|2(n^2-1)f(nx) - \frac{1}{7}n^2(n^2-1)^2 f(2x) + \frac{2}{7}n^2(n^2-1)(n^2-8)f(x) + 2(n^2-1)^2 f(0)\|_Y \leqslant (n^2-1)^{\beta}\varphi(0,x) \tag{3.14}$$

for all $x \in X$. By (3.8), (3.13) and (3.14), we have

$$\begin{aligned}\|f(2nx) &- \frac{1}{7}n^2(9n^2-2)f(2x) + \frac{32}{7}n^2(n^2-1)f(x)\|_Y \\ &\leqslant K^4[\varphi(nx,x) + n^{2\beta}\varphi(x,x)] + K^3\left(\frac{7(n^2+1)}{n^2}\right)^{\beta}\varphi(0,0) \\ &+ K^2(n^2-1)^{\beta}\varphi(0,x) + K\left(\frac{14(n^2-1)}{n^2}\right)^{\beta}\varphi(0,0)\end{aligned} \tag{3.15}$$

for all $x \in X$. On the other hand, (3.9) implies that

$$\left\|2f(2nx) - \frac{1}{7}n^2(n^2-1)f(4x) + \frac{2}{7}n^2(n^2-8)f(2x) + 2(n^2-1)f(0)\right\|_Y \leqslant \varphi(0,2x) \tag{3.16}$$

for all $x \in X$. Multiplying both sides of (3.15) by 2^β and then adding the result to (3.16), we obtain

$$\begin{aligned}\left(\frac{1}{7}n^2(n^2-1)\right)^\beta &\|f(4x)-20f(2x)+64f(x)\|_Y\\ &\le K^6[\varphi(nx,x)+n^{2\beta}\varphi(x,x)]+K^5\left(\frac{7(n^2+1)}{n^2}\right)^\beta\varphi(0,0)\\ &+K^4(n^2-1)^\beta\varphi(0,x)+K^3\left(\frac{14(n^2-1)}{n^2}\right)^\beta\varphi(0,0)\\ &+K\left(\frac{14}{n^2}\right)^\beta\varphi(0,0) \qquad (3.17)\end{aligned}$$

for all $x \in X$. Therefore

$$\|f(4x)-20f(2x)+64f(x)\|_Y \le \widetilde{\varphi}(x)$$

for all $x \in X$. The above relation implies that

$$\|g(2x)-16g(x)\|_Y \le \widetilde{\varphi}(x)$$

for all $x \in X$ in which $g(x)=f(2x)-4f(x)$. By Lemma 3.2, there exists a unique mapping $Q : X \longrightarrow Y$ such that $Q(2x)=16Q(x)$ and

$$\|g(x)-Q(x)\|_Y \le \frac{1}{2^{4\beta}|1-L^j|}\widetilde{\varphi}(x) \qquad (3.18)$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 3.3. □

In the next corollaries, we bring some consequences of Theorem 3.3 concerning the stability of (1.4) when f is an even mapping. Since the proofs are similar to the previous corollaries, we omit them.

Corollary 3.7. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ be a positive number with $\lambda \neq 4\frac{\beta}{\alpha}$. If $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y \le \theta(\|x\|_X^\lambda+\|y\|_X^\lambda)$$

for all $x,y \in X$, then there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)\|_Y \le \begin{cases}\frac{\theta\Lambda_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\|x\|_X^\lambda & \lambda\in\left(0,4\frac{\beta}{\alpha}\right)\\ \frac{2^{\alpha\lambda}\theta\Lambda_\lambda}{2^{\alpha\lambda}-2^{4\beta}}\|x\|_X^\lambda & \lambda\in\left(4\frac{\beta}{\alpha},\infty\right)\end{cases}$$

for all $x \in X$ where

$$\Lambda_\lambda = K^6\left(\frac{7}{n^2(n^2-1)}\right)^\beta[n^{\alpha\lambda}+2\cdot n^{2\beta}+1]+K^4\left(\frac{7}{n^2}\right)^\beta. \qquad (3.19)$$

Corollary 3.8. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ, r and s be positive numbers with $\lambda := r+s \neq 4\frac{\beta}{\alpha}$. If $f: X \longrightarrow Y$ be an even mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y \leqslant \theta\|x\|_X^r\|y\|_X^s$$

for all $x,y\in X$, then there exists a unique quartic mapping $Q: X\longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)\|_Y \leq \begin{cases} \frac{\theta\Gamma_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\|x\|_X^\lambda & \lambda\in\left(0,4\frac{\beta}{\alpha}\right) \\ \\ \frac{2^{\alpha\lambda}\theta\Gamma_\lambda}{2^{\alpha\lambda}-2^{4\beta}}\|x\|_X^\lambda & \lambda\in\left(4\frac{\beta}{\alpha},\infty\right) \end{cases}$$

for all $x\in X$ where

$$\Gamma_\lambda = K^6\left(\frac{7}{n^2(n^2-1)}\right)^\beta [n^{\alpha r}+n^{2\beta}]. \tag{3.20}$$

Corollary 3.9. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ, r and s be positive numbers with $\lambda := r+s \neq 4\frac{\beta}{\alpha}$. If $f: X \longrightarrow Y$ be an even mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y \leqslant \theta(\|x\|_X^\lambda+\|y\|_X^\lambda+\|x\|_X^r\|y\|_X^s)$$

for all $x,y\in X$, then there exists a unique quartic mapping $Q: X\longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)\|_Y \leq \begin{cases} \frac{\theta(\Lambda_\lambda+\Gamma_\lambda)}{2^{4\beta}-2^{\alpha\lambda}}\|x\|_X^\lambda & \lambda\in\left(0,4\frac{\beta}{\alpha}\right) \\ \\ \frac{2^{\alpha\lambda}\theta(\Lambda_\lambda+\Gamma_\lambda)}{2^{\alpha\lambda}-2^{4\beta}}\|x\|_X^\lambda & \lambda\in\left(4\frac{\beta}{\alpha},\infty\right) \end{cases}$$

for all $x\in X$ where Λ_λ and Γ_λ are defined in (3.19) and (3.20), respetively.

Theorem 3.10. *Let $\varphi: X\times X\longrightarrow[0,\infty)$ be a function such that there exists an $0<L<1$ with $\varphi(2x,2y)\leq 2^\beta L\varphi(x,y)$ and $\varphi\left(\frac{x}{2},\frac{y}{2}\right)\leq 2^{-4\beta}L\varphi(x,y)$ for all $x\in X$. Let $f: X\longrightarrow Y$ be a mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y \leq \varphi(x,y) \tag{3.21}$$

for all $x,y\in X$. Then there exist a unique additive mapping $A: X\longrightarrow Y$ and a unique quartic mapping $Q: X\longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)-A(x)\|_Y \leq \widetilde{\Phi}(x)$$

for all $x\in X$ where

$$\begin{aligned}\widetilde{\Phi}(x) &:= \frac{1}{2^{4\beta+1}|1-L^j|}(\widetilde{\varphi}(x)+\widetilde{\varphi}(-x)) \\ &\quad+\frac{1}{2^{\beta+1}|1-L^j|}\left(\frac{7}{n^2(n^2-1)}\right)^\beta(\varphi(0,2x)+\varphi(0,-2x)) \\ &\quad \frac{2}{2^\beta|1-L^j|}\left(\frac{7}{n^2(n^2-1)}\right)^\beta(\varphi(0,x)+\varphi(0,-x)).\end{aligned} \tag{3.22}$$

in which $j \in \{-1,1\}$.

Proof: We decompose f into the even part and odd part by setting

$$f_{\mathrm{e}}(x)=\frac{f(x)+f(-x)}{2},\qquad f_{\mathrm{o}}(x)=\frac{f(x)-f(-x)}{2}.$$

for all $x \in X$. Clearly, $f(x)=f_{\mathrm{e}}(x)+f_{\mathrm{o}}(x)$ for all $x \in X$. Then

$$\begin{aligned}\|\Delta_{a,q}f_{\mathrm{e}}(x,y)\|_Y &= \frac{1}{2}\|\Delta_{a,q}f(x,y)+\Delta_{a,q}f(-x,-y)\|_Y\\ &\le \frac{1}{2}\Big(\|\Delta_{a,q}f(x,y)\|_Y+\|\Delta_{a,q}f(-x,-y)\|_Y\Big)\\ &\le \frac{1}{2}\Big(\varphi(x,y)+\varphi(-x,-y)\Big)\end{aligned}$$

for all $x \in X$. Similarly,

$$\|\Delta_{a,q}f_{\mathrm{o}}(x,y)\|_Y \le \frac{1}{2}\Big(\varphi(x,y)+\varphi(-x,-y)\Big)$$

for all $x \in \mathcal{X}$. By Theorems 3.3 and 3.6, there exists a unique additive function $A_0 : X \longrightarrow Y$ and a unique quratic function $Q_0 : X \longrightarrow Y$ such that

$$\|f_{\mathrm{o}}(x)-A_0(x)\|_Y \le \frac{1}{2^{\beta+1}|1-L^j|}\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}(\varphi(0,x)+\varphi(0,-x)) \tag{3.23}$$

and

$$\|f_{\mathrm{e}}(2x)-4f_{\mathrm{e}}(x)-Q_0(x)\|_{\mathcal{Y}} \le \frac{1}{2^{4\beta+1}|1-L^j|}(\widetilde{\varphi}(x)+\widetilde{\varphi}(-x)) \tag{3.24}$$

for all $x \in X$ where $\widetilde{\varphi}(x)$ is defined in (3.7). Put $Q(x)=Q_0(x)$ and $A(x)=-2A_0(x)$. Since $A_0(x)$ is odd and satisfies the equation (1.4), it is easy to check that $A_0(2x)=2A_0(x)$. Thus we have

$$\begin{aligned}\|f(2x)-4f(x)-Q(x)-A(x)\|_Y &= \|f(2x)-4f(x)-Q_0(x)+2A_0(x)\|_Y\\ &= \|(f_{\mathrm{e}}(2x)-4f_{\mathrm{e}}(x)-Q_0(x))\\ &\quad+(f_{\mathrm{o}}(2x)-4f_{\mathrm{o}}(x)+2A_0(x))\|_Y\\ &\le \|f_{\mathrm{e}}(2x)-4f_{\mathrm{e}}(x)-Q_0(x)\|_Y\\ &+\|f_{\mathrm{o}}(2x)-2A_0(x)\|+4\|f_{\mathrm{o}}(x)-A_0(x)\|_Y\\ &= \|f_{\mathrm{e}}(2x)-4f_{\mathrm{e}}(x)-Q_0(x)\|_Y\\ &+\|f_{\mathrm{o}}(2x)-A_0(2x)\|_Y+4\|f_{\mathrm{o}}(x)-A_0(x)\|_Y\\ &\le \widetilde{\Phi}(x).\end{aligned}$$

This finishes the proof. □

Corollary 3.11. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ be a positive number with $\lambda\neq\frac{\beta}{\alpha},4\frac{\beta}{\alpha}$. If $f:X\longrightarrow Y$ be a mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y\leq\theta(\|x\|_X^\lambda+\|y\|_X^\lambda)$$

for all $x,y\in X$, then there exists a unique additive mapping $A:X\longrightarrow Y$ and a unique quartic mapping $Q:X\longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)-A(x)\|_Y\leq\begin{cases}\left[\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}\frac{4+2^{\alpha\lambda}}{2^{\beta}-2^{\alpha\lambda}}+\frac{\Lambda_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\right]\theta\|x\|_X^\lambda & \lambda\in\left(0,4\frac{\beta}{\alpha}\right)\\ \left[\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}\frac{(4+2^{\alpha\lambda})2^{\alpha\lambda}}{2^{\alpha\lambda}-2^{\beta}}+\frac{\Lambda_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\right]\theta\|x\|_X^\lambda & \lambda\in\left(\frac{\beta}{\alpha},4\frac{\beta}{\alpha}\right)\\ \left[\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}\frac{4+2^{\alpha\lambda}}{2^{\alpha\lambda}-2^{\beta}}+\frac{\Lambda_\lambda}{2^{\alpha\lambda}-2^{4\beta}}\right]\theta 2^{\alpha\lambda}\|x\|_X^\lambda & \lambda\in\left(4\frac{\beta}{\alpha},\infty\right)\end{cases}$$

for all $x\in X$ where Λ_λ is defined in (3.19).

Corollary 3.12. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ, r and s be positive numbers with $\lambda:=r+s\neq\frac{\beta}{\alpha},4\frac{\beta}{\alpha}$. If $f:X\longrightarrow Y$ be a mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y\leqslant\theta\|x\|_X^r\|y\|_X^s$$

for all $x,y\in X$, then there exists a unique additive mapping $A:X\longrightarrow Y$ and a unique quartic mapping $Q:X\longrightarrow Y$ such that

$$\|f(2x)-4f(x)-Q(x)-A(x)\|_Y\leq\begin{cases}\frac{\theta\Gamma_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\|x\|_X^\lambda & \lambda\in\left(0,\frac{\beta}{\alpha}\right)\bigcup\left(\frac{\beta}{\alpha},4\frac{\beta}{\alpha}\right)\\ \frac{2^{\alpha\lambda}\theta\Gamma_\lambda}{2^{\alpha\lambda}-2^{4\beta}}\|x\|_X^\lambda & \lambda\in\left(4\frac{\beta}{\alpha},\infty\right)\end{cases}$$

for all $x\in X$ where Γ_λ is defined in (3.20).

Corollary 3.13. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let θ, r and s be positive numbers with $\lambda:=r+s\neq\frac{\beta}{\alpha},4\frac{\beta}{\alpha}$. If $f:X\longrightarrow Y$ be a mapping satisfying*

$$\|\Delta_{a,q}f(x,y)\|_Y\leqslant\theta(\|x\|_X^\lambda+\|y\|_X^\lambda+\|x\|_X^r\|y\|_X^s)$$

for all $x,y\in X$, then there exists a unique additive mapping $A:X\longrightarrow Y$ and a *unique quartic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(2x)-4f(x)-Q(x)-A(x)\|_Y\leq\begin{cases}\left[\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}\frac{4+2^{\alpha\lambda}}{2^{\beta}-2^{\alpha\lambda}}+\frac{\Lambda_\lambda+\Gamma_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\right]\theta\|x\|_X^{\lambda} & \lambda\in\left(0,\frac{\beta}{\alpha}\right)\\ \left[\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}\frac{(4+2^{\alpha\lambda})2^{\alpha\lambda}}{2^{\alpha\lambda}-2^{\beta}}+\frac{\Lambda_\lambda+\Gamma_\lambda}{2^{4\beta}-2^{\alpha\lambda}}\right]\theta\|x\|_X^{\lambda} & \lambda\in\left(\frac{\beta}{\alpha},4\frac{\beta}{\alpha}\right)\\ \left[\left(\frac{7}{n^2(n^2-1)}\right)^{\beta}\frac{4+2^{\alpha\lambda}}{2^{\alpha\lambda}-2^{\beta}}+\frac{\Lambda_\lambda+\Gamma_\lambda}{2^{\alpha\lambda}-2^{4\beta}}\right]\theta 2^{\alpha\lambda}\|x\|_X^{\lambda} & \lambda\in\left(4\frac{\beta}{\alpha},\infty\right)\end{cases}$$

for all $x \in X$ *where* Λ_λ *and* Γ_λ *are defined in* (3.19) *and* (3.20)*, respetively.*

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