Bol. Soc. Paran. Mat. (3s.) **v. 35** 2 (2017): 19–37.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v35i2.29077

On generalized difference Zweier ideal convergent sequences space defined by Musielak-Orlicz functions

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abstract: Let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function. In this article, we introduce a new class of ideal convergent sequence spaces defined by Musielak-Orlicz function, using an infinite matrix, and a generalized difference Zweier matrix operator $B^p_{(i)}$. We investigate some topological structures and algebraic properties of these spaces. We obtain some relations related to these sequence spaces.

Key Words: ideal convergence; difference space; Musielak-Orlicz function.

Contents

1. Introduction

The notion of the ideal convergence is the dual (equivelant) to the notion of filter convergence introduced by Cartan in 1937 [4]. The notion of the filter convergence is a generalization of the classical notion of convergence of a sequence and it has been an important tool in general topology and functional analysis. Nowadays many authors to use an equivalent dual notion of the ideal convergence. Kostyrko et al. [24] and Nuray and Ruckle [30] independently studied in detalis about the notion of ideal convergence which is based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$. Later on it was further investigated by many authors, e.g. Tripathy and Hazarika [36,37], Hazarika [12], Hazarika and Savaş [11] and references therein. Hazarika [14] introduced the concept of generalized difference ideal convergent sequences of fuzzy numbers and studied some interesting properties. Esi [6] introduced strongly almost summable sequence spaces in 2-normed spaces defined by ideal convergence and an Orlicz function and proved some interesting results.

Before proceeding let us recall a few concepts, which we shall use throughout this paper.

2000 *Mathematics Subject Classification:* 40A05; 40B50; 46A19; 46A45.

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$ we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\phi \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and each $A \in F$ and each $B \supset A$ we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([24]). In this case we write $I - \lim x_k = \ell$.

Throughout the article w, ℓ_∞, c, c_0, denote for the classes of *all, bounded, convergent, null* sequences of complex numbers, respectively.

The notion of difference sequence space was introduced by Kizmaz [23], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$. The notion was further generalized by Et and Colak [8] introducing the sequence spaces $\ell_\infty(\Delta^p)$, $c(\Delta^p)$, $c_0(\Delta^p)$. For a non negative integer p, the generalized difference sequence spaces are defined as follows. For a given sequence space Z we have

$$Z(\Delta^p) = \{x = (x_k) \in w : (\Delta^p x_k) \in Z\},$$

where $\Delta^p x_k = \Delta^{p-1} x_k - \Delta^{p-1} x_{k+1}$, $\Delta^0 x_k = x_k$, for all $k \in \mathbb{N}$, the difference operator is equivalent to the following binomial representation:

$$\Delta^p x_k = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} x_{k+\nu} \text{ for all } k \in \mathbb{N}.$$

Taking $p = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$, introduced and studied by Kizmaz [23]. Tripathy and Esi [34] introduced and studied the new type of generalized difference sequence spaces

$$Z(\Delta_i) = \{(x_k) \in w : \Delta_i x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ where $\Delta_i x = (\Delta_i x_k) = (x_k - x_{k+i})$ for all $k, i \in \mathbb{N}$.

Tripathy, et al [35] further generalized this notion and introduced the following sequence spaces. For $p \geq 1$ and $i \geq 1$,

$$Z(\Delta_i^p) = \{(x_k) \in w : \Delta_i^p x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$.This generalized difference has the following binomial representation,

$$\Delta_i^p x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{p}{\nu} x_{k+i\nu} \quad \text{for all } k \in \mathbb{N}.$$

Dutta [5] introduced the following difference sequence spaces

$$Z(\Delta^p_{(i)}) = \{(x_k) \in w : \Delta^p_{(i)} x_k \in Z\} \text{ for all } p, i \in \mathbb{N},$$

for $Z = \ell_\infty, \overline{c}, \overline{c}_0$ where $\overline{c}, \overline{c}_0$ are the sets of statistically convergent and statistically null sequences, respectively, and $\Delta^p_{(i)} x = (\Delta^p_{(i)} x_k) = (\Delta^{p-1}_{(i)} x_k - \Delta^{p-1}_{(i)} x_{k-i})$ and $\Delta^0_{(i)} x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^p_{(i)} x_k = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} x_{k-i\nu}.$$

Basar and Altay [1] introduced the generalized difference matrix $B(r,s) = (b_{pk}(r,s))$ which is a generalization of $\Delta^1_{(1)}$-difference operator as follows:

$$b_{pk}(r,s) = \begin{cases} r, & \text{if } k = p; \\ s, & \text{if } k = p-1; \\ 0, & \text{if } 0 \leq k < p-1 \text{ or } k > p. \end{cases}$$

for all $k, p \in \mathbb{N}, r, s \in \mathbb{R} - \{0\}$.
Basarir and Kayikci [2] have defined the generalized difference matrix B^p of order p, which reduced the difference operator $\Delta^p_{(1)}$ in case $r = 1, s = -1$ and the binomial representation of this operator is

$$B^p x_k = \sum_{\nu=0}^{p} \binom{p}{\nu} r^{p-\nu} s^\nu x_{k-\nu},$$

where $r, s \in \mathbb{R} - \{0\}$ and $p \in \mathbb{N}$.
Recently Basarir et al., [3] introduced the following generalized difference sequence spaces

$$Z(B^p_{(i)}) = \{(x_k) \in w : B^p_{(i)} x_k \in Z\} \text{ for all } p, i \in \mathbb{N},$$

for $Z = \ell_\infty, \overline{c}, \overline{c}_0$ where $\overline{c}, \overline{c}_0$ are the sets of statistically convergent and statistically null sequences, respectively, and $B^p_{(i)} x = (B^p_{(i)} x_k) = (rB^{p-1}_{(i)} x_k + sB^{p-1}_{(i)} x_{k-i})$ and $B^0_{(i)} x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$B^p_{(i)} x_k = \sum_{\nu=0}^{p} \binom{p}{\nu} r^{p-\nu} s^\nu x_{k-i\nu}.$$

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = (a_{nk}), (n, k = 1, 2, 3, ...)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each n. If $x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y$ we say that A defines a (matrix) transformation from X to Y and we denote it by $A : X \to Y$.

Şengönül [33] defined the sequence $y = (y_k)$ which is frequently used as the Z-transformation of the sequence $x = (x_k)$ i.e.

$$y_k = \alpha x_k + (1 - \alpha)x_{k-1}$$

where $x_{-1} = 0, k \neq 0, 1 < k < \infty$ and Z denotes the matrix $Z = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} \alpha, & \text{if } n = k; \\ 1 - \alpha, & \text{if } n - 1 = k; \\ 0, & \text{otherwise.} \end{cases}$$

Şengönül [33] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in w : Z(x) \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w : Z(x) \in c_0\}.$$

For details on Zweier sequence spaces we refer to [7,15,16,18,19,21,22].

A function $M : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see [25]). An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x p(t)dt$$

where p is the known kernel of M, right differentiable for $t \geq 0, p(0) = 0, p(t) > 0$ for $t > 0$ and $p(t) \to \infty$ as $t \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Nakano [29], followed by Ruckle [32]. An Orlicz function M is said to satisfy $\Delta_2 - \mathit{condition}$ for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Two Orlicz functions M_1 and M_2 are said to be *equivalent* if there exist positive constants α, β and x_0 such that

$$M_1(\alpha) \leq M_2(x) \leq M_1(\beta)$$

for all x with $0 \leq x < x_0$.

Lindenstrauss and Tzafriri [27] studied some Orlicz type sequence spaces defined as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

A sequence $\mathbf{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (for details see [10,13,17,20]). Also a Musielak-Orlicz function $\phi = (\phi_k)$ is called a *complementary function* of a Musielak-Orlicz function $\mathbf{M}$ if

$$\phi_k(t) = \sup\{|\ t|s - M_k(s) : s \geq 0\}, \text{for } k = 1, 2, 3, ...$$

For a given Musielak-Orlicz function $\mathbf{M}$, the Musielak-Orlicz sequence space $l_{\mathbf{M}}$ and its subspace $h_{\mathbf{M}}$ are defined as follows:

$$l_{\mathbf{M}} = \{x = (x_k) \in w : I_{\mathbf{M}}(cx) < \infty, \text{for some } c > 0\};$$

$$h_{\mathbf{M}} = \{x = (x_k) \in w : I_{\mathbf{M}}(cx) < \infty, \text{for all } c > 0\},$$

where $I_{\mathbf{M}}$ is a convex modular defined by

$$I_{\mathbf{M}} = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in l_{\mathbf{M}}.$$

We consider $l_{\mathbf{M}}$ equipped with the Luxemburg norm

$$||\ x\ || = \inf\left\{k > 0 : I_{\mathbf{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$||\ x\ ||^0 = \inf\left\{\frac{1}{k}(1 + I_{\mathbf{M}}(kx)) : k > 0\right\}.$$

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = G$, $D = \max\{1, 2^{G-1}\}$ then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k})$$

for all $k \in \mathbb{N}$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max\{1, |a|^G\}$ for all $a \in \mathbb{C}$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [31] and many others (see [9,26,28,38]).

Remark 1.1. *It is well known if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.*

Throughout the paper X we denote a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous seminorms q. Also we denote I is an non-trivial admissible ideal of $\mathbb{N}$.

2. Ideal convergence in a locally convex space

In this section we define I-convergence in a locally convex space X and investigate some basic properties.

Definition 2.1. *A sequence $x = (x_k)$ in X is said to be I_q-convergent to $\ell \in X$ if for all $q \in Q$ and all $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : q(x_k - \ell) \geq \varepsilon\} \in I.$$

In this case we can write $I_q - \lim x_k = \ell$. We denote $I_q = \{\{k \in \mathbb{N} : q(x_k - \ell) \geq \varepsilon\} \in I\}$.

Further, since X is Hausdorff, the limit of ideal convergent sequence is unique.

Definition 2.2. *A sequence $x = (x_k)$ in X is said to be $B^p_{(i)}(I_q)$-convergent to $\ell \in X$ if for all $q \in Q$ and all $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : q(B^p_{(i)} x_k - \ell) \geq \varepsilon\} \in I.$$

In this case we can write $I_q - \lim B^p_{(i)}(x) = \ell$. We denote

$$B^p_{(i)}(I_q) = \{\{k \in \mathbb{N} : q(B^p_{(i)} x_k - \ell) \geq \varepsilon\} \in I\}.,$$

where

$$B^p_{(i)} x_k = \sum_{\nu=0}^{p} \binom{p}{\nu} \alpha^{p-\nu}(1-\alpha)^{\nu} x_{k-i\nu}.$$

Definition 2.3. *Let $\boldsymbol{M}$ be a Musielak-Orlicz function. We say that a sequence $x = (x_k)$ in $w^I(B^p_{(i)}, \boldsymbol{M})$ if and only if there exists $\ell \in X$ such that for all $q \in Q$ and for every $\varepsilon > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{\rho} \right) \right] \geq \varepsilon \right\} \in I \text{ for } \rho > 0. \tag{2.1}$$

When (2.1) holds we write

$$x_k \to \ell(w^I(B^p_{(i)}, \mathbf{M})).$$

The condition (2.1) provides a definition of ideal summability for a sequence in a locally convex space.

Theorem 2.1. *Let $A = (a_{nk})$ be a non-negative reguler matrix and $u = (u_k)$ be a bounded sequence of positive real numbers. Let $\boldsymbol{M}$ be a Musielak-Orlicz function. Then $x_k \to \ell(w(\boldsymbol{M}, A, u))$ implies that $x_k \to \ell(B^p_{(i)}(I_q)(A))$.*

Proof: Let $q \in Q$. Assume that $x_k \to \ell(w(\mathbf{M}, A, u))$, then for $\rho > 0$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{\rho} \right) \right]^{u_k} = 0 \text{ for } \ell \in \mathbb{C}.$$

Let $\varepsilon > 0$ be given. We define

$$K(\varepsilon) = \left\{ k \in \mathbb{N} : q(B^p_{(i)} x_k - \ell) \geq \varepsilon \right\}$$

and we write

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k}$$

$$= \sum_{k\in K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k} + \sum_{k\notin K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k}$$

$$\geq \left(\sum_{k\in K(\varepsilon)} a_{nk} \right) \left[M_k \left(\frac{\varepsilon}{r} \right) \right]^{u_k}.$$

Then we have $x_k \to \ell(B^p_{(i)}(I_q)(A))$. □

Theorem 2.2. *Let $A = (a_{nk})$ be a non-negative reguler matrix and $u = (u_k)$ be a bounded sequence of positive real numbers. Let $\boldsymbol{M}$ be a Musielak-Orlicz function. If $x = (x_k) \in \ell_\infty(B^p_{(i)})$ and $x_k \to \ell(B^p_{(i)}(I_q)(A))$, then $x_k \to \ell(w(\boldsymbol{M}, A, u))$.*

Proof: Suppose that $x = (x_k) \in \ell_\infty(B^p_{(i)})$ and $x_k \to \ell((B^p_{(i)}(I_q))(A))$. Then there is a set $K \in F(B^p_{(i)}(I_q))$ such that

$$\lim_{k\in K} q(B^p_{(i)} x_k - \ell) = 0.$$

Now

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k}$$

$$= \sum_{k\in K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k} + \sum_{k\notin K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k}$$

$$= \sum_{k=1}^{\infty} a_{nk} \chi_K(k) \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k} + \sum_{k=1}^{\infty} a_{nk} \chi_{K^c}(k) \left[M_k \left(\frac{q(B^p_{(i)} x_k - \ell)}{r} \right) \right]^{u_k}.$$

If we consider the regularity of A, $K^c \in B^p_{(i)}(I_q)$ and boundedness of (x_k) right side tends to zero. Hence $x_k \to \ell(w(\mathbf{M}, A, u))$. □

3. Difference Zweier ideal convergent sequences in a locally convex space

In this section we define some new classes of difference ideal convergent sequences by using infinite matrix and investigate their linear topological structures. Also we find out some relations related to these spaces.

Let I be an admissible ideal of $\mathbb{N}$, $u = (u_k)$ be a bounded sequence of positive real numbers and $A = (a_{nk})$ be an infinite matrix. Let $\mathbf{M}$ be a Musielak-Orlicz function. Further $w(X)$ denotes the space of all X-valued sequences. For each $\varepsilon > 0$, for all $q \in Q$ and for $\rho > 0$ we define the following sequence spaces.

$$[\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I =$$

$$\left\{ x = (x_k) \in w(X) : \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k - \ell\right)}{\rho} \right) \right]^{u_k} \geq \varepsilon \right\} \in I \text{ for } \ell \in X \right\},$$

$$[\mathcal{Z}_0(A, B^p_{(i)},\ \mathbf{M}, u, q)]^I =$$

$$\left\{ x = (x_k) \in w(X) : \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k\right)}{\rho} \right) \right]^{u_k} \geq \varepsilon \right\} \in I \right\},$$

$$[\mathcal{Z}_\infty(A, B^p_{(i)},\ \mathbf{M}, u, q)]^I =$$

$$\left\{ x = (x_k) \in w(X) : \exists K > 0 s.t. \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k\right)}{\rho} \right) \right]^{u_k} \geq K \right\} \in I \right\},$$

$$[\mathcal{Z}_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)] =$$

$$\left\{ x = (x_k) \in w(X) : \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k\right)}{\rho} \right) \right]^{u_k} < \infty \right\}.$$

Particular cases:

(i) If $p = 1$, then above spaces are denoted by $[\mathcal{Z}(A, B_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_0(A, B_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_\infty(A, B_{(i)}, \mathbf{M}, u, q)]^I$ and $[\mathcal{Z}_\infty(A, B_{(i)}, \mathbf{M}, u, q)]$.

(ii) If $i = 1$ then above spaces are denoted by $[\mathcal{Z}(A, B^p, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_0(A, B^p, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_\infty(A, B^p, \mathbf{M}, u, q)]^I$ and $[\mathcal{Z}_\infty(A, B^p, \mathbf{M}, u, q)]$.

(iii) If $M_k(x) = x$ for all $x \in [0, \infty), k \in \mathbb{N}$ then we obtain the above spaces as $[\mathcal{Z}(A, B^p_{(i)}, u, q)]^I$, $[\mathcal{Z}_0(A, B^p_{(i)}, u, q)]^I$, $[\mathcal{Z}_\infty(A, B^p_{(i)}, u, q)]^I$ and $[\mathcal{Z}_\infty(A, B^p_{(i)}, u, q)]$.

(iv) If $u = (u_k) = (1, 1, 1...)$, then above spaces are denoted by $[\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, q)]^I$, $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, q)]^I$, $[\mathcal{Z}_\infty(A, B^p_{(i)}, \mathbf{M}, q)]^I$ and $[\mathcal{Z}_\infty(A, B^p_{(i)}, \mathbf{M}, q)]$.

(v) If we take $A = (C, 1)$), i.e., the Cesàro matrix, then the above classes of sequences are denoted by $[\mathcal{Z}(B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_0(B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_\infty(B^p_{(i)}, \mathbf{M}, u, q)]^I$ and $[\mathcal{Z}_\infty(B^p_{(i)}, \mathbf{M}, u, q)]$.

(vi) If we take $A = (a_{nk})$ is a de la Vallée Poussin mean, i.e.,

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } k \in I_n = [n - \lambda_n + 1, n]; \\ 0, & \text{otherwise.} \end{cases}$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$, then the above classes of sequences are denoted by $[\mathcal{Z}(\lambda, B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_0(\lambda, B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_\infty(\lambda, B^p_{(i)}, \mathbf{M}, u, q)]^I$ and $[\mathcal{Z}_\infty(\lambda, B^p_{(i)}, \mathbf{M}, u, q)]$.

(vii) By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. As a final illustration let

$$a_{nk} = \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r = (k_{r-1}, k_r]; \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by $[\mathcal{Z}(\theta, B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_0(\theta, B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_\infty(\theta, B^p_{(i)}, \mathbf{M}, u, q)]^I$ and $[\mathcal{Z}_\infty(\theta, B^p_{(i)}, \mathbf{M}, u, q)]$.

Theorem 3.1. $[\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$, $[\mathcal{Z}_0(A, B^p_{(i)}, \boldsymbol{M}, u, q)]^I$ *and* $[\mathcal{Z}_\infty(A, B^p_{(i)}, \boldsymbol{M}, u, q)]^I$ *are linear spaces.*

Proof: We will proved the result for the space $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$ only and the others can be proved in similar way. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k\right)}{\rho_1} \right) \right]^{u_k} \ge \frac{\varepsilon}{2} \right\} \in I$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} y_k\right)}{\rho_2} \right) \right]^{u_k} \ge \frac{\varepsilon}{2} \right\} \in I.$$

Let α, β be two scalars in $\mathbb{R}$. Since $B^p_{(i)}$ is linear and the continuity of the Musielak-Orlicz function $\mathbf{M}$, the following inequality holds:

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} (\alpha x_k + \beta y_k)\right)}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{u_k}$$

$$\leq D\sum_{k=1}^{\infty}a_{nk}\left[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}M_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho_1}\right)\right]^{u_k}$$

$$+D\sum_{k=1}^{\infty}a_{nk}\left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}M_k\left(\frac{q\left(B^p_{(i)}y_k\right)}{\rho_2}\right)\right]^{u_k}$$

$$\leq DK\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho_1}\right)\right]^{p_k}+DK\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(B^p_{(i)}y_k\right)}{\rho_2}\right)\right]^{u_k},$$

where $K=\max\{1,\left(\frac{|\alpha|\rho_1}{|\alpha|\rho_1+|\beta|\rho_2}\right),\left(\frac{|\beta|\rho_2}{|\alpha|\rho_1+|\beta|\rho_2}\right)\}$.

From the above relation we get

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(B^p_{(i)}(\alpha x_k+\beta y_k)\right)}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)\right]^{u_k}\geq\varepsilon\right\}$$

$$\subseteq\left\{n\in\mathbb{N}:DK\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho_1}\right)\right]^{u_k}\geq\frac{\varepsilon}{2}\right\}$$

$$\cup\left\{n\in\mathbb{N}:DK\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(B^p_{(i)}y_k\right)}{\rho_2}\right)\right]^{u_k}\geq\frac{\varepsilon}{2}\right\}. \tag{3.1}$$

Since both of the sets on the right hand of (3.1) are belong to I, this completes the proof of the theorem. □

Remark 3.1. *It is easy to verify that the space $[\mathcal{Z}_\infty(A,B^p_{(i)},\boldsymbol{M},u,q)]$ is a linear space.*

Theorem 3.2. *Let $\mathbf{S}=(S_k)$ and $\mathbf{T}=(T_k)$ be Musielak-Orlicz functions. Then the following holds:*

$$[\mathcal{Z}_0(A,B^p_{(i)},\mathbf{S},u,q)]^I\cap[\mathcal{Z}_0(A,B^p_{(i)},\mathbf{T},u,q)]^I\subseteq[\mathcal{Z}_0(A,B^p_{(i)},\mathbf{S}+\mathbf{T},u,q)]^I.$$

Proof: Let $x=(x_k)\in[\mathcal{Z}_0(A,B^p_{(i)},\mathbf{S},u,q)]^I\cap[\mathcal{Z}_0(A,B^p_{(i)},\mathbf{T},u,q)]^I$. Then the result follows from the inequality

$$\sum_{k=1}^{\infty}a_{nk}\left[(S_k+T_k)\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right]^{u_k}$$

$$\leq D\sum_{k=1}^{\infty} a_{nk}\left[S_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right]^{u_k} + D\sum_{k=1}^{\infty} a_{nk}\left[T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right]^{p_k}.$$

□

Theorem 3.3. *Let* $\mathbf{S} = (S_k)$ *and* $\mathbf{T} = (T_k)$ *be Musielak-Orlicz functions. Then the following holds:*

$$[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{T}, u, q)]^I \subseteq [\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{ST}, u, q)]^I$$

provided $h = \inf u_k > 0$.

Proof: For a given $\varepsilon > 0$, we first choose $\varepsilon_0 > 0$ such that $\sup_n \left(\sum_{k=1}^n a_{nk}\right) \max\{\varepsilon_0^h, \varepsilon_0^H\} < \varepsilon$. Using the continuity of $\mathbf{M}$, choose $0 < \delta < 1$ such that $0 < \delta < t$ implies that $S_k(t) < \varepsilon_0$ for all $k \in \mathbb{N}$. Let $x = (x_k) \in [\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{T}, u, q)]^I$. For some $\rho > 0$ we denote

$$A_5 = \left\{ n \in \mathbb{N} : \sum_{k=1}^n a_{nk}\left[T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right]^{u_k} \geq \delta^H \right\} \in I.$$

If $n \notin A_5$, then we have

$$\sum_{k=1}^n a_{nk}\left[T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right]^{u_k} < \delta^H$$

$$i.e. \left[T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right]^{u_k} < \delta^H \text{ for all } k \in \mathbb{N}$$

$$i.e. T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right) < \delta \text{ for all } k \in \mathbb{N}$$

$$i.e. S_k\left(T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right) < \varepsilon_0 \text{ for all } k \in \mathbb{N}.$$

Consequently, we get

$$\sum_{k=1}^n a_{nk}\left[S_k\left(T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right)\right]^{u_k} < \sup_n\left(\sum_{k=1}^n a_{nk}\right)\max\{\varepsilon_0^h, \varepsilon_0^H\} < \varepsilon.$$

i.e.

$$\sum_{k=1}^n a_{nk}\left[S_k\left(T_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{\rho}\right)\right)\right]^{u_k} < \varepsilon.$$

This shows that

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk} \left[S_k\left(T_k\left(\frac{q\left(B^p_{(i)} x_k\right)}{\rho}\right)\right)\right]^{u_k} \geq \varepsilon\right\} \subset A_5 \in I.$$

This completes the proof. □

Theorem 3.4. *The inclusions* $[Y(A, B^{p-1}_{(i)}, \mathbf{M}, u, q)]^I \subset [Y(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$, *are strict for* $p \geq 1$. *In general* $[Y(A, B^j_{(i)}, \mathbf{M}, u, q)]^I \subset [Y(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$, *for* $j = 0, 1, 2, \ldots, p-1$ *and the inclusions are strict, where* $Y = \mathcal{Z}_0, \mathcal{Z}, \mathcal{Z}_\infty$.

Proof: We shall give the proof for $[\mathcal{Z}_0(A, B^{p-1}_{(i)}, \mathbf{M}, u, q)]^I$ only. The others can be proved by similar arguments. Let $x = (x_k)$ be any element in the space $[\mathcal{Z}_0(A, B^{p-1}_{(i)}, \mathbf{M}, u, q)]^I$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that the set

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^{p-1}_{(i)} x_k\right)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Since $\mathbf{M}$ is non-decreasing and convex, it follows that

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)} x_k\right)}{2\rho}\right)\right]^{p_k} = \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^{p-1}_{(i)} x_{k+1} - B^{p-1}_{(i)} x_k\right)}{2\rho}\right)\right]^{p_k}$$

$$\leq D \sum_{k=1}^{\infty} \left[\frac{1}{2} M_k\left(\frac{q\left(B^{p-1}_{(i)} x_{k+1}\right)}{\rho}\right)\right]^{p_k} + D \sum_{k=1}^{\infty} a_{nk} \left[\frac{1}{2} M_k\left(\frac{q\left(B^{p-1}_{(i)} x_k\right)}{\rho}\right)\right]^{p_k}$$

$$\leq DH \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^{p-1}_{(i)} x_{k+1}\right)}{\rho}\right)\right]^{p_k} + DH \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^{p-1}_{(i)} x_k\right)}{\rho}\right)\right]^{p_k},$$

where $H = \max\{1, (\frac{1}{2})^G\}$. Thus we have

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)} x_k\right)}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : DH \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^{p-1}_{(i)} x_{k+1}\right)}{\rho}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}$$

$$\cup \left\{n \in \mathbb{N} : DH \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^{p-1}_{(i)} x_k\right)}{\rho}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\} \tag{3.2}$$

Since both the sets in the right side of (3.2) belongs to I, we get

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k\right)}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

If follow from the following example that the inclusion is strict.

Example 3.1. *Let* $A = (C, 1)$, $M_k(x) = x$, *for all* $x \in [0, \infty), u_k = 1$ *for all* $k \in \mathbb{N}$ *and* $r = 1, s = -1$. *Consider a sequence* $x = (x_k) = (k^p)$. *Then* $x = (x_k)$ *belongs to* $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$ *but does not belong to* $[\mathcal{Z}_0(A, B^{p-1}_{(i)}, M, u, q)]^I$, *because* $B^p_{(i)}x_k = 0$ *and* $B^{p-1}_{(i)}x_k = (-1)^{p-1}(p-1)!$.

□

Theorem 3.5. *(a) Let* $0 < \inf u_k \leq u_k \leq 1$, *then* $[\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I \subset [\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, q)]^I$ *and* $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I \subset [\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, q)]^I$.

(b) If $1 < u_k \leq \sup u_k < \infty$, *then* $[\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, q)]^I \subset [\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$ *and* $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, q)]^I \subset [\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$.

Proof: (a) Let $x = (x_k) \in [\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$. Since $0 < \inf u_k \leq u_k \leq 1$, we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right] \leq \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right]^{p_k}$$

and therefore

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right] \geq \varepsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

(b) Let $1 < u_k \leq \sup u_k < \infty$ and let $x = (x_k) \in [\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, q)]^I$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right] \leq \varepsilon < 1$$

for all $n \geq N$. This implies that

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk} \left[M_k\left(\frac{q\left(B^p_{(i)}x_k - \ell\right)}{\rho}\right)\right].$$

Thus we have

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k - \ell\right)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k - \ell\right)}{\rho}\right)\right] \geq \varepsilon\right\} \in I.$$

This completes the proof of the theorem. □

Corollary 3.1. *Let $A = (C, 1)$ Cesáro matrix and let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function.*

- *(a) If $0 < \inf u_k \leq u_k \leq 1$, then*
  - *(i)* $[\mathcal{Z}(B^p_{(i)}, \mathbf{M}, u, q)]^I \subset [\mathcal{Z}(B^p_{(i)}, \mathbf{M}, q)]^I$;
  - *(ii)* $[\mathcal{Z}_0(B^p_{(i)}, \mathbf{M}, u, q)]^I \subset [\mathcal{Z}_0(B^p_{(i)}, \mathbf{M}, q)]^I$.
- *(b) If $1 < u_k \leq \sup u_k < \infty$, then*
  - *(i)* $[\mathcal{Z}(B^p_{(i)}, \mathbf{M}, q)]^I \subset [\mathcal{Z}(B^p_{(i)}, \mathbf{M}, u, q)]^I$;
  - *(ii)* $[\mathcal{Z}_0(B^p_{(i)}, \mathbf{M}, q)]^I \subset [\mathcal{Z}_0(B^p_{(i)}, \mathbf{M}, u, q)]^I$.

Theorem 3.6. *Let $0 < u_k \leq v_k$ for all $k \in \mathbb{N}$ and $\left(\frac{v_k}{u_k}\right)$ is bounded, then* $[\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, v, q)]^I \subseteq [\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$.

Proof: The proof of the theorem is straightforward, so we should omitted here. □

Theorem 3.7. *If $\lim_k u_k > 0$ and $x = (x_k) \to x_0([\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q))]^I$, then x_0 is unique.*

Proof: Let $\lim_k u_k = u_0 > 0$. Suppose that $(x_k) \to x_0([\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I)$ and $(x_k) \to y_0([\mathcal{Z}(A, B^p_{(i)}, \mathbf{M}, u, q)]^I)$.

Then there exist $\rho_1, \rho_2 > 0$ such that

$$B_1 = \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k - x_0\right)}{\rho_1}\right)\right]^{u_k} \geq \frac{\varepsilon}{2}\right\} \in I \tag{3.3}$$

and

$$B_2 = \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k - y_0\right)}{\rho_1}\right)\right]^{u_k} \geq \frac{\varepsilon}{2}\right\} \in I. \tag{3.4}$$

Let $\rho = \max\{2\rho_1, 2\rho_2\}$. Then we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(x_0 - y_0\right)}{\rho} \right) \right]^{u_k} \leq$$

$$D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B_{(i)}^p x_k - x_0\right)}{\rho_1} \right) \right]^{u_k} + D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B_{(i)}^p x_k - y_0\right)}{\rho_1} \right) \right]^{u_k}.$$

Thus from (3.3) and (3.4) we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(x_0 - y_0\right)}{\rho} \right) \right]^{u_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B_{(i)}^p x_k - x_0\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\}$$

$$\cup \left\{ n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(B_{(i)}^p x_k - y_0\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\} \subseteq B_1 \cup B_2 \in I.$$

Also we have

$$\left[M_k \left(\frac{q\left(x_0 - y_0\right)}{\rho} \right) \right]^{u_k} \to \left[M_k \left(\frac{q\left(x_0 - y_0\right)}{\rho} \right) \right]^{u_0} \text{ as } k \to \infty.$$

Therefore we have

$$\left[M_k \left(\frac{q\left(x_0 - y_0\right)}{\rho} \right) \right]^{u_k} \to \left[M_k \left(\frac{q\left(x_0 - y_0\right)}{\rho} \right) \right]^{u_0} = 0.$$

Hence $x_0 = y_0$. □

Theorem 3.8. *Let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $[\mathcal{Z}_\infty(A, B_{(i)}^p, u, q)]^I \subseteq [\mathcal{Z}_\infty(A, B_{(i)}^p, \mathbf{M}, u, q)]^I$

(ii) $[\mathcal{Z}_0(A, B_{(i)}^p, u, q)]^I \subseteq [\mathcal{Z}_\infty(A, B_{(i)}^p, \mathbf{M}, u, q)]^I$

(iii) $\sup_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} < \infty \ (t, \rho > 0)$.

Proof: (i)⇒(ii) is obvious, because $[\mathcal{Z}_0(A, B_{(i)}^p, u, q)]^I \subseteq [\mathcal{Z}_\infty(A, B_{(i)}^p, u, q)]^I$.

(ii)⇒(iii). Suppose $[\mathcal{Z}_0(A, B^p_{(i)}, u, q)]^I \subseteq [\mathcal{Z}_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$. We assume that (iii) is not satisfied. Then for some $t, \rho > 0$

$$\sup_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} = \infty,$$

and therefore there exists a sequence (n_j) of positive integers such that

$$\sum_{k=1}^{n_j} a_{n_j k} \left[M_k \left(\frac{j^{-1}}{\rho} \right) \right]^{u_k} > j, j = 1, 2, 3, ... \tag{3.5}$$

Define a sequence $x = (x_k)$ by

$$B^p_{(i)} x_k = \begin{cases} \frac{1}{j}, & \text{if } 1 \leq k \leq n_j, j = 1, 2, 3, ...; \\ 0, & \text{if } k > n_j \end{cases}$$

Then $x = (x_k) \in [\mathcal{Z}_0(A, B^p_{(i)}, u, q)]^I$ but by equation (3.5) we have $x = (x_k) \notin [\mathcal{Z}_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i) Suppose (iii) is satisfied and $x \in [\mathcal{Z}_\infty(A, B^p_{(i)}, u, q)]^I$. Suppose that $x \notin [\mathcal{Z}_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$. Then

$$\sup_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k\right)}{\rho} \right) \right]^{u_k} = \infty. \tag{3.6}$$

Put $t = q\left(Z(B^p_{(i)} x_k)\right)$ for all $k \in \mathbb{N}$. Then by the equation (3.6) we have

$$\sup_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} = \infty$$

which contradicts (iii). Hence (i) must hold. □

Theorem 3.9. *Let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function. Let $1 \leq u_k \leq sup_k u_k < \infty$. Then the following statements are equivalent:*

(i) $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, q)]^I \subseteq [\mathcal{Z}_0(A, B^p_{(i)}, u, q)]^I$

(ii) $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I \subseteq [\mathcal{Z}_\infty(A, B^p_{(i)}, u, q)]^I$

(iii) $\inf_n \sum_{k=1}^n a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} > 0 \ (t, \rho > 0)$.

Proof: (i)⇒(ii) is obvious.

(ii)⇒(iii). Suppose $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I \subseteq [\mathcal{Z}_\infty(A, B^p_{(i)}, u, q)]^I$. We assume that (iii) does not hold. Then for some $t, \rho > 0$

$$\inf_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} = 0.$$

We can choose an index sequence (n_j) of positive integers such that

$$\sum_{k=1}^{n_j} a_{n_j k} \left[M_k \left(\frac{i}{\rho} \right) \right]^{u_k} > \frac{1}{j}, j = 1, 2, 3, ... \tag{3.7}$$

Define a sequence $x = (x_k)$ by

$$B^p_{(i)} x_k = \begin{cases} j, & \text{if } 1 \leq k \leq n_j, j = 1, 2, 3, ...; \\ 0, & \text{if } k > n_j \end{cases}$$

Then by equation (3.7) we have $x = (x_k) \in [\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$ but $x = (x_k) \notin [\mathcal{Z}_\infty(A, B^p_{(i)}, u, q)]^I$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i) Let (iii) hold and $x \in [\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, u, q)]^I$. Then for every $\varepsilon > 0$, we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{q\left(B^p_{(i)} x_k \right)}{\rho} \right) \right]^{u_k} \geq \varepsilon \right\} \in I. \tag{3.8}$$

Suppose that $x \notin [\mathcal{Z}_0(A, B^p_{(i)}, u, q)]^I$. Then for some integer $\varepsilon_0 > 0$, we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk} \left[q\left(B^p_{(i)} x_k \right) \right]^{u_k} \geq \varepsilon_0 \right\} \notin I.$$

Therefore we have

$$\left[M_k \left(\frac{\varepsilon_0}{\rho} \right) \right]^{u_k} \leq \left[M_k \left(\frac{q\left(B^p_{(i)} x_k \right)}{\rho} \right) \right]^{u_k}$$

and consequently by the relation (3.8) we have

$$\inf_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{\varepsilon_0}{\rho} \right) \right]^{u_k} = 0$$

which contradicts (iii). Hence $[\mathcal{Z}_0(A, B^p_{(i)}, \mathbf{M}, q)]^I \subseteq [\mathcal{Z}_0(A, B^p_{(i)}, u, q)]^I$. □

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