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Bi Unique Range Sets -A Further Study

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abstract: The purpose of the paper is to obtain a new bi-unique range sets, as introduced in [4] with smallest cardinalities ever for derivative of meromorphic functions. Our results will improve all the results in connection to the bi-unique range sets to a large extent. Some examples have been exhibited to justify our certain claims. At last an open question have been posed for future investigations.

Key Words: Meromorphic function, Uniqueness, Shared Set, Weighted sharing.

Contents

1. Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r, h)$ any quantity satisfying

$$S(r, h) = o(T(r, h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$,

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we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

The uniqueness theory of meromorphic functions is a vast subject. Under the ambit of this theory several branches have been flourished. Among them set sharing problem exists as a distinguishable entity. We start the discussion with the question raised by Lin and Yi [17], in connection with the famous "Gross Question" {see [9]}.

Question A. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

To find the possible answer of the above question researchers have become more engaged to find explicitly a set S with minimum cardinalities such that any two meromorphic functions f and g having common poles sharing the set S become identical {cf. [1]-[3], [5]-[8], [11], [15]-[17], [22]-[23]}. The advent of the new notion of gradation of sharing of values and sets in [13,14] further add essence to-wards the investigations. This notion is a scaling between CM and IM and measures how close a shared value is to being shared IM or to being shared CM. In the following we recall the definition.

Definition 1.1. *[13,14] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. *[13] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a; f)$.*

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Recently to study the possible answer of *Question A* the present first author [4] have introduced the notion of bi unique range sets for entire or meromorphic function with weight p, m as follows :

Definition 1.3. *[4] A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is called bi unique range sets for meromorphic (entire) functions with weights p, m if for any two non-constant meromorphic (entire) functions f and g, $E_f(S_1, p) = E_g(S_1, p)$, $E_f(S_2, m) = E_g(S_2, m)$ implies $f \equiv g$. We write S_i's $i = 1, 2$ as BURSMp, m (BURSEp, m) in short. As usual if both $p = m = \infty$, we say S_i's $i = 1, 2$ as BURSM (BURSE).*

In [4] the present first author manipulated the above definition in order to get the possible answer of *Question A* for two finite sets in $\mathbb{C}$, which significantly improved the results obtained in [20] and [19]. Below we are recalling the result in [4]. The purpose of the paper is to investigate this fact.

Theorem A. *[4] Let* $S_1 = \{0, 1\}$*,* $S_2 = \left\{z : \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c = 0\right\}$*, where* $n (\geq 5)$ *is an integer and* $c \neq 0,\ 1,\ \frac{1}{2}$ *is a complex number such that* $c^2 - c + 1 \neq 0$*. Then* S_i*'s* $i = 1,\ 2$ *are* BURSM1, 3.

Theorem B. *[4] Let* S_i*,* $i = 1,\ 2$ *be given as in* Theorem A*. Then* S_i*'s* $i = 1,\ 2$ *are* BURSM3, 2.

It is to be observed that in [4] we were unable to diminish the cardinalities of the range sets as mentioned in [19]. So it is natural to ask the following question. *Question 1: Can there exists any pair of range sets in the sense of* Definition 1.3 *whose cardinalities(s) are less than that given in* Theorems A-B ?

Possible answer of the above question is the motivation of the paper. We shall show that if we take the set sharing problem of derivatives of meromorphic functions, in stead of the original functions, a pair of range sets with cardinalities 2 and 3 different from those used in *Theorems A-B* provide the answer of *Question 1.* Till date this is the best result obtained in terms of bi-unique range sets.

Throughout the paper for an integer n and a nonzero constant a we shall denote $-a\frac{n-1}{n} = c_1$ and $\beta = -c_1^n - ac_1^{n-1}$. Below we are giving our main theorem.

Theorem 1.4. *Let* $S_1 = \{0, c_1\}$*,* $S_2 = \{z : z^n + az^{n-1} + b = 0\}$*, where* $n (\geq 3)$ *be an integer and* a *and* b *be two nonzero constants such that* $b \neq \beta, \frac{\beta}{2}$*. Then* S_i*'s* $i = 1,\ 2$ *are bi-unique range sets with weights* 1 *and* 3 *for* $f^{(k)}$ *and* $g^{(k)}$*.*

The following example shows that in *Theorem 1.4* $a \neq 0$ is necessary.

Example 1.5. *Let* $f(z) = \sqrt[3]{-b}\ e^z$ *and* $g(z) = (-1)^k \sqrt[3]{-b}\ e^{-z}$ *and* $S_1 = \{0\}$*,* $S_2 = \{z : z^3 + b = 0\}$*. Then* $f^{(k)}$*,* $g^{(k)}$ *share* (S_i, ∞)*,* $i = 1, 2$ *but* $f^{(k)} \not\equiv g^{(k)}$*.*

From the following example we see that if in our main result we discard $-a\frac{n-1}{n}$ in S_1 and replace $f^{(k)}$ and $g^{(k)}$ simply by f and g then the conclusion ceases to hold. In other words, the presence of the element $-a\frac{n-1}{n}$ in S_1 is essential in that case.

Example 1.6. *Let* $S_1 = \{0\}$*,* $S_2 = \{z : z^3 + az^2 + b = 0\}$ *where* $a \neq 0$*,* b *be so chosen that* S_2 *has distinct elements. Let* f *and* g *be two non-constant meromorphic functions such that* $f(z) = -a\frac{e^z + e^{2z}}{1 + e^z + e^{2z}}$*,* $g(z) = -a\frac{1 + e^z}{1 + e^z + e^{2z}}$*. Then they share* (S_i, ∞)*,* $i = 1, 2$ *but* $f \not\equiv g$*.*

So natural question would be whether the cardinality of the set S_1 in *Theorem 1.4* can further be diminished ?

It is seen from the next example that the sets S_i, $(i = 1, 2)$ in *Theorem 1.4* can not be replaced by two arbitrary sets.

Example 1.7. *Let* $f(z) = e^z$ *and* $g(z) = (-1)^k \alpha e^{-z}$ *and for a constant* $\alpha \neq 0, \frac{1}{2}, 1$ *we take* $S_1 = \{1, \alpha\}$*,* $S_2 = \{0, \frac{1}{2}, 2\alpha\}$*. Then* $f^{(k)}$*,* $g^{(k)}$ *share* (S_i, ∞)*,* $i = 1, 2$ *but* $f^{(k)} \not\equiv g^{(k)}$*.*

Though for the standard definitions and notations of the value distribution theory we refer to [10], we now explain some notations which are used in the paper.

Definition 1.8. *[12] For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting function of simple* a *points of* f*. For a positive integer* m *we denote by* $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ *the counting function of those* a *points of* f *whose multiplicities are not greater(less) than* m *where each* a *point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ *are defined similarly, where in counting the* a*-points of* f *we ignore the multiplicities.*

Also $N(r, a; f \mid< m)$*,* $N(r, a; f \mid> m)$*,* $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ *are defined analogously.*

Definition 1.9. *[14] We denote by* $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$*.*

Definition 1.10. *[13,14] Let* f*,* g *share a value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g*. Clearly* $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ *and in particular if* f *and* g *share* (a, p) *then* $\overline{N}_*(r, a; f, g) \leq \overline{N}(r, a; f \mid\geq p + 1) = \overline{N}(r, a; g \mid\geq p + 1)$*.*

Definition 1.11. *Let* $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$*. We denote by* $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ *the counting function of those* a*-points of* f*, counted according to multiplicity, which are not the* b_i*-points of* g *for* $i = 1, 2, \ldots, q$*.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$ as follows

$$F = \frac{P(f^{(k)})}{-b} = \frac{\left(f^{(k)}\right)^{n-1}\left(f^{(k)} + a\right)}{-b}, \qquad G = \frac{P(g^{(k)})}{-b} = \frac{\left(g^{(k)}\right)^{n-1}\left(g^{(k)} + a\right)}{-b}, \tag{2.1}$$

where $n(\geq 2)$ and k are two positive integers and for a meromorphic function h we put
$P(h) = (h)^n + a(h)^{n-1}$. Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right) \tag{2.2}$$

and

$$\Phi = \frac{F'}{F - 1} - \frac{G'}{G - 1}. \tag{2.3}$$

Lemma 2.1. *([14], Lemma 1) Let* F*,* G *be two non-constant meromorphic functions sharing* $(1, 1)$ *and* $H \not\equiv 0$*. Then*

$$N(r, 1; F \mid= 1) = N(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2. *Let S_1 and S_2 be defined as in* Theorem 1.4 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,p)=E_{g^{(k)}}(S_1,p)$, $E_{f^{(k)}}(S_2,0)=E_{g^{(k)}}(S_2,0)$, where $0\leq p<\infty$ and $H\not\equiv 0$ then*

$$\begin{aligned}N(r,H) &\leq \overline{N}(r,0;f^{(k)}\mid\geq p+1)+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\mid\geq p+1\right)+\overline{N}_*(r,1;F,G)\\ &\quad+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f^{(k+1)})+\overline{N}_0(r,0;g^{(k+1)}),\end{aligned}$$

where $\overline{N}_0(r,0;f^{(k+1)})$ is the reduced counting function of those zeros of $f^{(k+1)}$ which are not the zeros of $f^{(k)}\left(f^{(k)}-a\frac{n-1}{n}\right)(F-1)$ and $\overline{N}_0(r,0;g^{(k+1)})$ is similarly defined.

Proof: We note that $F^{'}=\frac{(f^{(k)})^{n-2}(nf^{(k)}+a(n-1))f^{(k+1)}}{-b}$, $G^{'}=\frac{(g^{(k)})^{n-2}(ng^{(k)}+a(n-1))g^{(k+1)}}{-b}$ and

$$F^{''}=\frac{(f^{(k)})^{n-2}(nf^{(k)}+a(n-1))f^{(k+2)}+(f^{(k)})^{n-3}(n(n-1)f^{(k)}+a(n-1)(n-2))(f^{(k+1)})^2}{-b},$$

$$G^{''}=\frac{(g^{(k)})^{n-2}(ng^{(k)}+a(n-1))g^{(k+2)}+(g^{(k)})^{n-3}(n(n-1)g^{(k)}+a(n-1)(n-2))(g^{(k+1)})^2}{-b}.$$

So

$$\begin{aligned}H = &\frac{(n-1)(nf^{(k)}+a(n-2))f^{(k+1)}}{f^{(k)}(nf^{(k)}+a(n-1))}-\frac{(n-1)(ng^{(k)}+a(n-2))g^{(k+1)}}{g^{(k)}(ng^{(k+1)}+a(n-1))}\\ &+\frac{f^{(k+2)}}{f^{(k+1)}}-\frac{g^{(k+2)}}{g^{(k+1)}}-\left(\frac{2F^{'}}{F-1}-\frac{2G^{'}}{G-1}\right).\end{aligned}$$

Since $E_{f^{(k)}}(S_2,0)=E_{g^{(k)}}(S_2,0)$ it follows that if z_0 is a 0-point of $f^{(k)}$ $(g^{(k)})$ then either $g^{(k)}(z_0)=0$ $(f^{(k)}(z_0)=0)$ or $g^{(k)}(z_0)=-a\frac{n-1}{n}$ $(f^{(k)}(z_0)=-a\frac{n-1}{n})$. Clearly F and G share $(1,0)$. Since H has only simple poles, the lemma can easily be proved by simple calculation. □

Lemma 2.3. *[6] Let f and g be two meromorphic functions sharing $(1,m)$, where $1\leq m<\infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\leq\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]$$

Lemma 2.4. *[18] Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\neq 0$ and $b_m\neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *Let S_1 and S_2 be defined as in* Theorem 1.4 *with $n \geq 3$ and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,p) = E_{g^{(k)}}(S_1,p)$, $E_{f^{(k)}}(S_2,m) = E_{g^{(k)}}(S_2,m)$, $0 \leq p < \infty$ and $\Phi \not\equiv 0$ then*

$$\begin{aligned} & (2p+1)\left\{\overline{N}\left(r,0;f^{(k)} \mid\geq p+1\right) + \overline{N}\left(r,c_1;f^{(k)} \mid\geq p+1\right)\right\} \\ \leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + S(r,f^{(k)}) + S(r,g^{(k)}). \end{aligned}$$

Proof: By the given condition clearly F and G share $(1,m)$. Also we see that

$$\Phi = \frac{(f^{(k)})^{n-2}\left(nf^{(k)} + a(n-1)\right)f^{(k+1)}}{-b(F-1)} - \frac{(g^{(k)})^{n-2}\left(ng^{(k)} + a(n-1)\right)g^{(k+1)}}{-b(G-1)}.$$

Let z_0 be a zero or a c_1- point of $f^{(k)}$ with multiplicity r. Since $E_{f^{(k)}}(S_1,p) = E_{g^{(k)}}(S_1,p)$ then that would be a zero of Φ of multiplicity min $\{(n-2)r + r - 1,\ r + r - 1\}$ i.e., of multiplicity min $\{(n-1)r - 1,\ 2r - 1\}$ if $r \leq p$ and a zero of multiplicity at least $min\{(n-2)(p+1) + p,\ p+1+p\}$ i.e., a zero of multiplicity at least $min\{(n-1)p + (n-2),\ 2p+1\}$ if $r > p$. So using *Lemma 2.4* by a simple calculation we can write

$$\begin{aligned} & \min\{(n-1)p + (n-2), (2p+1)\}\left\{\overline{N}(r,0;f^{(k)} \mid\geq p+1) + \overline{N}(r,c_1;f^{(k)} \mid\geq p+1)\right\} \\ \leq\ & N(r,0;\Phi) \\ \leq\ & T(r,\Phi) \\ \leq\ & N(r,\infty;\Phi) + S(r,F) + S(r,G) \\ \leq\ & \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \end{aligned}$$

□

Lemma 2.6. *Let S_1, S_2 be defined as in* Theorem 1.4 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,p) = E_{g^{(k)}}(S_1,p)$, $E_{f^{(k)}}(S_2,m) = E_{g^{(k)}}(S_2,m)$, where $0 \leq p < \infty$, $2 \leq m < \infty$ and $H \not\equiv 0$, then*

$$\begin{aligned} & (n+1)\left\{T(r,f^{(k)}) + T(r,g^{(k)}\right\} \\ \leq\ & 2\left\{\overline{N}(r,0;f^{(k)}) + \overline{N}\left(r,c_1;f^{(k)}\right)\right\} + \overline{N}\left(r,0;f^{(k)} \mid\geq p+1\right) \\ & + \overline{N}\left(r,c_1;f^{(k)} \mid\geq p+1\right) + 2\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} \\ & + \frac{1}{2}\left[N(r,1;F) + N(r,1;G)\right] - \left(m - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) \\ & + S(r,f^{(k)}) + S(r,g^{(k)}). \end{aligned}$$

Proof: By the second fundamental theorem we get

$$\begin{aligned} & (n+1)\{T(r,f^{(k)})+T(r,g^{(k)})\} \qquad (2.4)\\ \leq\ & \overline{N}(r,1;F)+\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,c_1;f^{(k)}\right)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)\\ & +\overline{N}(r,0;g^{(k)})+\overline{N}\left(r,c_1;g^{(k)}\right)+\overline{N}(r,\infty;g)-N_0(r,0;f^{(k+1)})\\ & -N_0(r,0;g^{(k+1)})+S(r,f^{(k)})+S(r,g^{(k)}). \end{aligned}$$

Using *Lemmas 2.1, 2.2, 2.3* and *2.4* we note that

$$\begin{aligned} & \overline{N}(r,1;F)+\overline{N}(r,1;G) \qquad (2.5)\\ \leq\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\ \leq\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+\overline{N}(r,0;f^{(k)}\mid\geq p+1)\\ & +\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\mid\geq p+1\right)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ & -\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f^{(k+1)})+\overline{N}_0(r,0;g^{(k+1)})\\ & +S(r,f^{(k)})+S(r,g^{(k)}). \end{aligned}$$

Using (2.5) in (2.4) and noting that

$$\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,c_1;f^{(k)}\right)=\overline{N}(r,0;g^{(k)})+\overline{N}\left(r,c_1;g^{(k)}\right)$$

the lemma follows. □

Lemma 2.7. *Let $f^{(k)}$, $g^{(k)}$ be two non-constant meromorphic functions such that $E_{f^{(k)}}(\{0,c_1\},0)=E_{g^{(k)}}(\{0,c_1\},0)$. Then $\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right)\equiv\left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right)$ implies $f^{(k)}\equiv g^{(k)}$, where n (≥ 2) is an integer, k is a positive integer and a is a nonzero finite constant.*

Proof: Let z_0 be a zero of $f^{(k)}$ $(g^{(k)})$. Then z_0 must be either a 0-point or a c_1-point of $g^{(k)}$ $(f^{(k)})$. But from the given condition if z_0 is not a zero of $g^{(k)}$, then it must be a zero of $g^{(k)}+a$, which is impossible. So we conclude that here $f^{(k)}$ and $g^{(k)}$ share $(0,\infty)$ and f, g share (∞,∞). We also note that $\Theta\left(\infty;f^{(k)}\right)+\Theta\left(\infty;g^{(k)}\right)\geq 2-\frac{2}{k+1}=\frac{2k}{k+1}>0$. Now the lemma can be proved in the line of proof of *Lemma 3* [16]. □

Lemma 2.8. *Let F, G be given by (2.1) and they share $(1,m)$. Also let $\omega_1,\omega_2\ldots\omega_n$ are the members of the set S_2 as defined in Theorem 1.4. Then*

$$\overline{N}_*(r,1;F,G)\leq\frac{1}{m}\left[\overline{N}(r,0;f^{(k)})+\overline{N}(r,c_1;f^{(k)})\right]+S(r,f^{(k)}).$$

Proof: First we note that since S_1 has distinct elements, c_1 can not be a member of S_2. So

$$\begin{aligned}
& \overline{N}_*(r,1;F,G) \\
\leq\ & \overline{N}(r,1;F\mid\geq m+1) \\
\leq\ & \frac{1}{m}\left(N(r,1;F)-\overline{N}(r,1;F)\right) \\
\leq\ & \frac{1}{m}\left[\sum_{j=1}^{n}\left(N(r,\omega_j;f^{(k)})-\overline{N}(r,\omega_j;f^{(k)})\right)\right] \\
\leq\ & \frac{1}{m}\left[N\left(r,0;f^{(k+1)}\mid f^{(k)}\neq 0,\ c_1\right)\right] \\
\leq\ & \frac{1}{m}\left[\overline{N}\left(r,\infty;\frac{f^{(k)}(f^{(k)}-c_1)}{f^{(k+1)}}\right)\right] \\
\leq\ & \frac{1}{m}\left[N\left(r,\infty;\frac{f^{(k+1)}}{f^{(k)}(f^{(k)}-c_1)}\right)\right]+S(r,f^{(k)}) \\
\leq\ & \frac{1}{m}\left[\overline{N}(r,0;f^{(k)})+\overline{N}(r,c_1;f^{(k)})\right]+S(r,f^{(k)})
\end{aligned}$$

□

Lemma 2.9. *[21] If $H\equiv 0$, then F, G share $(1,\infty)$.*

Lemma 2.10. *Let S_1, S_2 be defined as in Theorem 1.4 with $n\geq 3$ an integer. If for two non-constant meromorphic function f and g, $E_{f^{(k)}}(S_1,p)=E_{g^{(k)}}(S_1,p)$, $E_{f^{(k)}}(S_2,m)=E_{g^{(k)}}(S_2,m)$, where $0\leq p<\infty$, $2\leq m<\infty$ and $\Phi\not\equiv 0$ then*

$$\begin{aligned}
& \left\{\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}\left(r,c_1;f^{(k)}\right)\right\} \\
\leq\ & \left(\frac{m}{m-1}\right)[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}$$

Proof: Using *Lemma 2.5* for $p=0$ and *Lemma 2.8* we get

$$\begin{aligned}
& \overline{N}\left(r,0;f^{(k)}\right)+\overline{N}\left(r,c_1;f^{(k)}\right) \\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)}) \\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\frac{1}{m}\left[\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}\left(r,c_1;f^{(k)}\right)\right] \\
& +S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}$$

From above the lemma follows. □

3. Proof of the theorem

Proof: [Proof of Theorem 1.4] Let F, G be given by (2.1). Then F and G share $(1,3)$. We consider the following cases.
Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1. Let $H \not\equiv 0$. Then using *Lemma 2.6* for $m=3$, $p=1$, *Lemma 2.5* for $p=0$ and $p=1$, *Lemma 2.8* for $m=3$, *Lemma 2.10* and *Lemma 2.4* we obtain

$$\begin{aligned}
& (n+1)\left\{T(r,f^{(k)})+T(r,g^{(k)})\right\} \qquad (3.1)\\
\leq\; & 2\left\{\overline{N}(r,0;f^{(k)})+\overline{N}\left(r,c_1;f^{(k)}\right)\right\}+\overline{N}(r,0;f^{(k)}\mid\geq 2)\\
& +\overline{N}\left(r,c_1;f^{(k)}\mid\geq 2\right)+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\
& +\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\; & \left\{4+\frac{1}{3}\right\}\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
& +\left\{\frac{1}{2}+\frac{1}{3}\right\}\overline{N}_*(r,1;F,G)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\; & \frac{13}{3}\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
& +\frac{5}{12}\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\; & \frac{57}{12}\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+\frac{n}{2}[T(r,f^{(k)}+T(r,g^{(k)}]\\
& +S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\; & \left\{\frac{n}{2}+\frac{57}{24}\right\}[T(r,f^{(k)}+T(r,g^{(k)}]+S(r,f^{(k)})+S(r,g^{(k)}).
\end{aligned}$$

(3.1) gives a contradiction for $n \geq 3$.
Subcase 1.2 Let $H \equiv 0$. Then

$$\frac{1}{F-1} \equiv \frac{A}{G-1}+B, \qquad (3.2)$$

where $A \neq 0$, B are constants. Also $T(r,F)=T(r,G)+O(1)$. i.e.,

$$nT(r,f^{(k)}) = nT(r,g^{(k)})+O(1). \qquad (3.3)$$

In view of *Lemma 2.9* it follows that F and G share $(1,\infty)$. We now consider the following cases.
Subcase 1.2.1.
Let $B=0$. From (3.2) we get

$$\frac{1}{F-1} \equiv \frac{A}{G-1}.$$

i.e.,

$$G^{'} \equiv AF^{'}.$$

i.e.,

$$\Phi \equiv 0,$$

a contradiction.

Subcase 1.2.2.

If $B \neq 0$, then

$$F - 1 \equiv \frac{G-1}{BG + A - B}. \tag{3.4}$$

Subcase 1.2.2.1.

If $A - B \neq 0$, then from (3.4) we get

$$F - 1 \equiv \frac{G-1}{B\left(G - (\frac{B-A}{B})\right)}. \tag{3.5}$$

So

$$\overline{N}(r, \frac{B-A}{B}; G) = \overline{N}(r, \infty; F).$$

Subcase 1.2.2.1.1.

If $g^{(k)} - c_1$ is a repeated factor of $G - \frac{B-A}{B}$, then

$$(g^{(k)} - c_1)^2 \prod_{i=1}^{n-2} (g^{(k)} - \alpha_i) \equiv \frac{1}{B}\frac{G-1}{F-1},$$

where $g^{(k)} - \alpha_i$'s $(i = 1, 2, \ldots, n-2)$ are the distinct simple factors of $G - \frac{B-A}{B}$. Since $f^{(k)}$, $g^{(k)}$ share $\{0, c_1\}$ and F, G share $(1, \infty)$ it follows that c_1 points of $g^{(k)}$ can not be a pole of f and so it must be an e.v.P. of $g^{(k)}$. Therefore α_i's are neutralised by the poles of f. Now if z_0 is a zero of $g^{(k)} - c_1$ of order p, then it would be pole of $f^{(k)}$ of order q such that $p = nq \geq n(k+1)$. So in view of the second fundamental theorem and (3.3) we get

$$(n-2)T(r, g^{(k)}) \leq \sum_{i=1}^{n-2} \overline{N}(r, \alpha_i; g^{(k)}) + \overline{N}(r, c_1; g^{(k)}) + \overline{N}(r, \infty; g) + S(r, g^{(k)})$$

i.e.,

$$(n-2)T(r, g^{(k)}) \leq \frac{(n-2)}{n(k+1)} T(r, g^{(k)}) + \frac{1}{k+1} T(r, g^{(k)}) + S(r, g^{(k)}),$$

which gives a contradiction for $n \geq 3$.

Subcase 1.2.2.1.2. If $(g^{(k)} - c_1)$ is not a factor of $G - \frac{B-A}{B}$, then

$$\prod_{i=1}^{n} (g^{(k)} - \beta_i) \equiv \frac{1}{B}\frac{G-1}{F-1},$$

where $g^{(k)}-\beta_i$'s $(i=1,2,\ldots,n)$ are the distinct simple factors of $G-\frac{B-A}{B}$. Clearly from above we get

$$\sum_{i=1}^{n}\overline{N}(r,\beta_i;g^{(k)})=\overline{N}(r,\infty;f).$$

Again by the second fundamental theorem we get

$$\begin{aligned}(n-1)T(r,g^{(k)}) &\leq \sum_{i=1}^{n}\overline{N}(r,\beta_i;g^{(k)})+\overline{N}(r,\infty;g)+S(r,g^{(k)})\\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,g^{(k)}),\end{aligned}$$

i.e., in view of (3.3)

$$\left(n-1-\frac{2}{k+1}\right)T(r,g^{(k)}) \leq S(r,g^{(k)}),$$

which is a contradiction for $n\geq 3$.
Subcase 1.2.2.2.
If $A-B=0$, then from (3.4) we get

$$\frac{B}{-b}\left(g^{(k)}\right)^{n-1}(g^{(k)}+a)\equiv\frac{G-1}{F-1}.$$

Using the same argument as in *Subcase 1.2.2.1.1.* we get that 0 is an e.v.P. of g and

$$\overline{N}(r,-a;g^{(k)})\leq\frac{1}{n(k+1)}T(r,f^{(k)}).$$

So by the second fundamental theorem and (3.3) we get

$$\begin{aligned}T(r,g^{(k)}) &\leq \overline{N}(r,-a;g^{(k)})+\overline{N}(r,0;g^{(k)})+\overline{N}(r,\infty;g)+S(r,g^{(k)})\\ &\leq \left\{\frac{1}{n(k+1)}+\frac{1}{k+1}\right\}T(r,g^{(k)})+S(r,g^{(k)}),\end{aligned}$$

a contradiction for $n\geq 3$.
Case 2. Suppose that $\Phi\equiv 0$. On integration we get

$$(F-1)\equiv A(G-1) \tag{3.6}$$

for some nonzero constant A. Here also in view of *Lemma 2.4*, (3.3) holds. Since by the given condition of the theorem $E_f(S_1,0)=E_g(S_1,0)$, we consider the following cases.
Subcase 2.1. Let us first assume $f^{(k)}$ and $g^{(k)}$ share $(0,0)$ and $(c_1,0)$. If one of 0 or c_1 is an e.v.P. of both $f^{(k)}$ and $g^{(k)}$, then we get $A=1$ and we have $F\equiv G$, which in view of *Lemma 2.7* implies $f^{(k)}\equiv g^{(k)}$. If both 0 and c_1 are e.v.P. of $f^{(k)}$

as well as of $g^{(k)}$ then noting that here $F \equiv AG + (1-A)$, suppose $A \neq 1$. Using *Lemma 2.4*, (3.3) and the second fundamental theorem we get

$$\begin{aligned} & nT(r, f^{(k)}) \\ \leq\; & \overline{N}(r, 0; F) + \overline{N}(r, 1-A; F) + \overline{N}(r, \infty; F) + S(r, F) \\ \leq\; & \overline{N}(r, 0; f^{(k)}) + \overline{N}(r, -a; f^{(k)}) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; f) + S(r, f^{(k)}) \\ \leq\; & (1 + \frac{1}{k+1})T(r, f^{(k)}) + T(r, g^{(k)}) + S(r, f^{(k)}) \\ \leq\; & (2 + \frac{1}{k+1})T(r, f^{(k)}) + S(r, f^{(k)}), \end{aligned}$$

which implies a contradiction since $n \geq 3$.
Subcase 2.2. Next suppose that $f^{(k)}$ and $g^{(k)}$ do not share $(0, 0)$ and $(c_1, 0)$. We now consider the following subcases.
Subcase 2.2.1.
Suppose none of 0, c_1 is e.v.P. of $f^{(k)}$ i.e., none of c_1, 0 is e.v.P. of $g^{(k)}$. Also from (3.6) we get

$$P(f^{(k)}) + b(1-A) \equiv AP(g^{(k)}).$$

Since at least one c_1-point of $f^{(k)}$ corresponds to at least one 0-point of $g^{(k)}$, from above we have

$$b(1-A) = \beta. \tag{3.7}$$

Again form (3.6) we get

$$\frac{P(f^{(k)})}{A} \equiv P(g^{(k)}) - \frac{b(1-A)}{A}. \tag{3.8}$$

We claim that $-\frac{b(1-A)}{A} \neq \beta$. For if $-\frac{b(1-A)}{A} = \beta$, then in view of (3.7) we have $A = -1$, which again in view of (3.7) implies $b = \frac{\beta}{2}$, a contradiction. So $P(g^{(k)}) - \frac{b(1-A)}{A}$ has n distinct factors. Let them be γ_i, $(i = 1, 2, \ldots, n)$. Hence from (3.8) we have

$$\prod_{i=1}^{n}(g^{(k)} - \gamma_i) \equiv \frac{1}{A}\,(f^{(k)})^{n-1}(f^{(k)} + a). \tag{3.9}$$

Since none of γ_i, $(i = 1, 2, \ldots, n)$ coincides with 0 or c_1, from (3.9) it follows that 0 is an e.v.P. of $f^{(k)}$, a contradiction to the initial assumption of this subcase.
Subcase 2.2.2.
Let one of 0 or c_1 is an e.v.P. of $f^{(k)}$.
Subcase 2.2.2.1.
Suppose first 0 is an e.v.P. of $f^{(k)}$. If c_1 is not an e.v.P. of $g^{(k)}$, then there would be at least one z_0 such that $g(z_0) = f(z_0) = c_1$ and then from (3.6) we get $A = 1$, which in view of *Lemma 2.6* yields $f^{(k)} \equiv g^{(k)}$ and we are done. So c_1 must be an e.v.P. of $g^{(k)}$. Now using the similar argument as used in *Subcase 2.2.1.*, from (3.9)

and the second fundamental theorem we get

$$\begin{aligned} nT(r,g^{(k)}) &\le \sum_{i=1}^{n}\overline{N}(r,\gamma_i;g^{(k)})+\overline{N}(r,c_1;g^{(k)})+\overline{N}(r,\infty;g)+S(r,g^{(k)})\\ &\le \overline{N}(r,-a;f^{(k)})+\frac{1}{k+1}T(r,g^{(k)})+S(r,g^{(k)}), \end{aligned}$$

which in view of (3.3) again gives a contradiction for $n\ge 3$.

Subcase 2.2.2.2.

Suppose next c_1 is an e.v.P. of $f^{(k)}$, i.e., 0 is an e.v.P. of $g^{(k)}$. Here noticing the fact that in (3.6) F and G are interchangeable, using the same argument as in *Subcase 2.2.2* this subcase can be disposed off. So we omit the details.

Subcase 2.2.3.

Let 0, c_1 are both e.v.P. of $f^{(k)}$, i.e., c_1, 0 are both e.v.P. of $g^{(k)}$, then again in view of (3.6) we consider the following subcases.

Subcase 2.2.3.1.

Suppose $F+A-1$ has n distinct zeros, ξ_i, $i=1,2,\ldots,n$. Then (3.6) takes the form

$$A(g^{(k)})^{n-1}(g^{(k)}+a)\equiv(f^{(k)}-\xi_1)(f^{(k)}-\xi_2)\ldots(f^{(k)}-\xi_n).$$

Then from the second fundamental theorem we get

$$\begin{aligned} &(n+1)T(r,f^{(k)})\\ \le\ &\sum_{i=1}^{n}\overline{N}(r,\xi_i;f^{(k)})+\overline{N}(r,0;f^{(k)})+\overline{N}(r,c_1;f^{(k)})+S(r,f^{(k)})\\ \le\ &\overline{N}(r,-a;g^{(k)})+\frac{1}{k+1}T(r,f^{(k)})+S(r,f^{(k)}), \end{aligned}$$

which in view of (3.3) gives a contradiction for $n\ge 3$.

Subcase 2.2.3.2.

Suppose $F+A-1$ has $n-2$ distinct zeros, η_i, $i=1,2,\ldots,n-2$ and a double zero at c_1. Then (3.6) changes to the form

$$A(g^{(k)})^{n-1}(g^{(k)}+a)\equiv\left(f^{(k)}-c_1\right)^2(f^{(k)}-\eta_1)(f^{(k)}-\eta_2)\ldots(f^{(k)}-\eta_{n-2}).$$

So again from the second fundamental theorem we get

$$\begin{aligned} &(n-1)T(r,f^{(k)})\\ \le\ &\sum_{i=1}^{n-2}\overline{N}(r,\eta_i;f^{(k)})+\overline{N}(r,0;f^{(k)})+\overline{N}(r,c_1;f^{(k)})+\overline{N}(r,\infty;f)+S(r,f^{(k)})\\ \le\ &\overline{N}(r,-a;g^{(k)})+\frac{1}{k+1}T(r,f^{(k)})+S(r,f^{(k)}), \end{aligned}$$

which in view of (3.3) gives a contradiction for $n\ge 3$. □

4. Concluding Remark and an Open Question

We see from the statement of *Example 1.7* that conclusion of *Theorem 1.4* does not hold for any arbitrary sets different from that used in *Theorem 1.4*. So natural question would be
i) Whether the sets S_i used in *Theorem 1.4* are the only bi-unique range sets for the derivatives of two meromorphic functions for the case $n = 3$?

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