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Upper Bound of Second Hankel determinant for generalized Sakaguchi type spiral-like functions

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abstract: In this paper, the authors introduce a generalized Sakaguchi type spiral-like function class $S(\lambda, \beta, s, t)$ and obtain sharp upper bound to the second Hankel determinant $|H_2(1)|$ for the function f in the above class. Relevances of the main result are also briefly indicated.

Key Words: Analytic functions, Starlike functions, Sakaguchi type functions, λ-spiral-like functions, Second Hankel determinant, Toeplitz determinants

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1. Introduction and Motivation

Let

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

be the unit disk in the complex z-plane. Let $\mathcal{A}$ be the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in $\mathbb{U}$ and satisfy the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions f in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be λ-spiral starlike function of order β, denoted by $SP(\lambda, \beta)$ if and only if the following inequality holds true:

$$\Re\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] > \beta \quad (0 \le \beta < 1, \ |\lambda| \le \frac{\pi}{2}; \ z \in \mathbb{U}). \tag{1.2}$$

For $\beta = 0$, the class $SP(\lambda, 0)$ reduces to $S_p(\lambda)$ which has been studied by Spacek [22]. Observed that for $\lambda = 0$, $S_p(0) = S^*$, the familiar class of starlike functions

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in $\mathbb{U}$.
Recently, Frasin [9] introduced and studied a generalized Sakaguchi type function class $S(\alpha, s, t)$ as follows. A function $f(z) \in \mathcal{A}$ is said to be in the class $S(\alpha, s, t)$ if it satisfies

$$\Re\left[\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right] > \alpha \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$, $s,\ t \in \mathbb{C}$, $s \neq t$ and for all $z \in \mathbb{U}$.

Motivated by work of Frasin [9], we introduce here a new subclass of $\mathcal{A}$ as follows:

Definition 1.1. *A function $f(z) \in \mathcal{A}$ is said to be in the generalized Sakaguchi type spiral-like class $S(\lambda, \beta, s, t)$ if it satisfies*

$$\Re\left[e^{i\lambda}\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}\right] > \beta cos\lambda \quad (z \in \mathbb{U}), \tag{1.4}$$

for some β $(0 \leq \beta < 1)$, s and t are real parameters , $s > t$ and λ is real with $|\lambda| < \frac{\pi}{2}$.

It may be noted that for $s = 1,\ \lambda = 0$, the class $S(0, \beta, 1, t) = S(\beta, t)$ has been studied by Owa et al. [23,24], Goyal and Goswami [10] and Cho et al. [4]; while for $s = 1,\ \lambda = 0,\ \beta = 0,\ t = -1$, the class $S(0, 0, 1, -1) = S(0, -1)$ has introduced and studied by Sakaguchi [21]. Further, for $\lambda = t = 0,\ s = 1$, the above class reduces to the well-known subclass of $\mathcal{A}$ consisting of univalent starlike functions of order β (see [6]).

The qth Hankel determinant for $q \geq 1$ and $n \geq 1$ is stated by Noonan and Thomas [19] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot & \cdot \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2q-2} \end{vmatrix}.$$

A good amount of literature is available about the importance of Hankel determinant. It is useful in the study of power series with integral coefficients (see [3]), meromorphic functions (see [25]) and also singularities (see [5]). Noor (see [20]) determined the rate of growth of $H_q(n)$ as $n \longrightarrow \infty$ for the functions in $\mathcal{S}$ with a bounded boundary while Ehrenborg (see [7]) studied the Hankel determinant of exponential polynomials.
For $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

It is well-known [6] that for $f \in \mathcal{S}$ and given by (1.1), the sharp inequality $|a_3 - a_2^2|$ holds. Fekete-Szegö (see [8]) then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in \mathcal{S}$. For a given family $\mathcal{F}$ of the functions in $\mathcal{A}$, the sharp upper bound for the nonlinear functional $|a_2a_4 - a_3^2| = |H_2(1)|$ is popularly known as the second Hankel determinant. Second Hankel determinant for various subclasses of analytic functions were obtained by various authors. For details, (see [1,2,11,12, 13,14,15,18]).
Following the techniques devised by Libera and Zlotkiewicz (see [16,17]), in the present paper, the authors determine a sharp upper bound of the second Hankel determinant $|H_2(1)|$ for the function f belonging to the class $S(\lambda, \beta, s, t)$.

2. Preliminaries

Let $\mathcal{P}$ denote the class of functions normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

which are regular in $\mathbb{U}$ and satisfying $\Re\{p(z)\} > 0$ for every $z \in \mathbb{U}$. Here $p(z)$ is called caratheòdory function (see [6]).

To investigate the main result, we need the following lemmas.

Lemma 2.1. *(see [6]) If $p \in \mathcal{P}$, then $|c_n| \leq 2$, for each $n \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. *([16], also see [17, p. 254]) Let the function $p \in \mathcal{P}$ be given by the power series (2.1). Then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.2}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - (4 - c_1^2)c_1x^2 + 2(4 - c_1^2)(1 - |x|^2)y \tag{2.3}$$

for some complex numbers x, y satisfying $|x| \leq 1$ and $|y| \leq 1$.

3. Main Result

Theorem 3.1. *Let the function f given by (1.1) be in the class $S(\lambda, \beta, s, t)$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4(1-\beta)^2 cos^2\lambda}{(2s^2 - st - t^2)^2}. \tag{3.1}$$

The estimate in (3.1) is sharp.

Proof: Let the function $f(z)$ given by (1.1) be in the class $S(\lambda, \beta, s, t)$. Then from the Definition 1.1, there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{U}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ such that

$$\begin{aligned} e^{i\lambda}\frac{(s-t)zf'(sz)}{f(sz) - f(tz)} &= [(1-\beta)p(z) + \beta]cos\lambda + isin\lambda \\ \Longrightarrow e^{i\lambda}(s-t)zf'(sz) &- e^{i\lambda}(f(sz) - f(tz)) \\ &= (f(sz) - f(tz))[(1-\beta)p(z) + \beta - 1]cos\lambda. \end{aligned} \tag{3.2}$$

Replacing $f(tz)$, $f(sz)$, $f'(sz)$ and $p(z)$ by their equivalent series in (3.2), after simplification, we obtain

$$\begin{aligned}
&e^{i\lambda}[a_2(s-t)z + a_3(2s^2-st-t^2)z^2 + a_4(3s^3-s^2t-st^2-t^3)z^3+\cdots] \\
&= [c_1z + \{a_2(s+t)c_1+c_2\}z^2 \\
&+\{a_3(s^2+st+t^2)c_1 + a_2(s+t)c_2+c_3\}z^3+\cdots](1-\beta)cos\lambda.
\end{aligned} \tag{3.3}$$

Equating the coefficients of z, z^2 and z^3 in (3.3), we get

$$\begin{aligned}
a_2 &= \frac{e^{-i\lambda}(1-\beta)cos\lambda}{s-t}c_1 \\
a_3 &= \frac{e^{-i\lambda}(1-\beta)cos\lambda}{2s^2-st-t^2}\left[\frac{e^{-i\lambda}(1-\beta)(s+t)cos\lambda c_1^2+(s-t)c_2}{s-t}\right], \\
a_4 &= \frac{e^{-i\lambda}(1-\beta)cos\lambda}{3s^3-st^2-s^2t-t^3}\left[\frac{(3s^3-2t^3+s^2t-2st^2)e^{-i\lambda}(1-\beta)cos\lambda}{(s-t)(2s^2-st-t^2)}c_1c_2\right. \\
&\left. +\frac{e^{-2i\lambda}(1-\beta)^2cos^2\lambda(s^3+2s^2t+2st^2+t^3)}{(s-t)(2s^2-st-t^2)}c_1^3 + c_3\right].
\end{aligned} \tag{3.4}$$

Substituting the values of a_2, a_3 and a_4 from (3.4) in the second Hankel functional $|a_2a_4-a_3^2|$ for the function $f \in S(\lambda,\beta,s,t)$, we obtain

$$\begin{aligned}
|a_2a_4-a_3^2| &= \left|\frac{e^{-2i\lambda}(1-\beta)^2cos^2\lambda}{(s-t)(3s^3-st^2-s^2t-t^3)}\right. \\
&+\left[\frac{3s^3-2t^3+s^2t-2st^2}{(s-t)(2s^2-st-t^2)}e^{-i\lambda}(1-\beta)cos\lambda c_1^2c_2\right. \\
&+ e^{-2i\lambda}(1-\beta)^2cos^2\lambda\frac{s^3+2s^2t+2st^2+t^3}{(s-t)(2s^2-st-t^2)}c_1^4 \\
&+ c_1c_3] - \frac{e^{-2i\lambda}(1-\beta)^2cos^2\lambda}{(s-t)^2(2s^2-st-t^2)^2}\left[(s-t)^2c_2^2\right. \\
&\left.\left.+e^{-2i\lambda}(1-\beta)^2(s+t)^2cos^2\lambda c_1^4 + 2e^{-i\lambda}(1-\beta)(s^2-t^2)cos\lambda c_1^2c_2\right]\right| \\
&= \left|\frac{e^{-2i\lambda}(1-\beta)^2cos^2\lambda}{(s-t)^2(3s^3-st^2-s^2t-t^3)(2s^2-st-t^2)^2}\left[(s-t)(2s^2-st-t^2)^2c_1c_3\right.\right. \\
&+ e^{-i\lambda}(1-\beta)cos\lambda(s^4t-3s^2t^3+2st^4)c_1^2c_2 + e^{-2i\lambda}(1-\beta)^2cos^2\lambda \\
&\left.\left.(-s^5-2s^4t+s^3t^2+2s^2t^3)c_1^4-(s-t)^2(3s^3-st^2-s^2t-t^3)c_2^2\right]\right|.
\end{aligned} \tag{3.5}$$

Making use of the result $|xa+yb| \le |x||a|+|y||b|$, where x, y, a and b are real numbers and $|e^{-in\lambda}| = 1$, where n is a real number, after simplification, we obtain

$$\begin{aligned}
|a_2a_4-a_3^2| &\le \frac{(1-\beta)^2cos^2\lambda}{(s-t)^2(3s^3-st^2-s^2t-t^3)(2s^2-st-t^2)^2} \\
&\quad |d_1c_1c_3+d_2cos\lambda c_1^2c_2+d_3c_2^2+d_4cos^2\lambda c_1^4|,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} d_1 &= (s-t)(2s^2-st-t^2)^2, \\ d_2 &= (s^4t-3s^2t^3+2st^4)(1-\beta), \\ d_3 &= -(s-t)^2(3s^3-st^2-s^2t-t^3), \\ d_4 &= -(s^5+2s^4t-s^3t^2-2s^2t^3)(1-\beta)^2. \end{aligned} \tag{3.7}$$

Substituting the values of c_2 and c_3 from Lemma 2.2 in the right hand side of (3.6), we have

$$\begin{aligned} |d_1c_1c_3+d_2cos\lambda c_1^2c_2+d_3c_2^2+d_4cos^2\lambda c_1^4| = \Big| &\frac{d_1c_1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2 \\ &+2(4-c_1^2)(1-|x|^2)z\}+\frac{d_2cos\lambda}{2}c_1^2\{c_1^2+x(4-c_1^2)\} \\ &+\frac{d_3}{4}\{c_1^2+x(4-c_1^2)\}^2+d_4cos^2\lambda c_1^4\Big|. \end{aligned} \tag{3.8}$$

Making use of well-known fact that $|z|<1$ in (3.8), upon simplification gives

$$\begin{aligned} 4|d_1c_1c_3+d_2cos\lambda c_1^2c_2+d_3c_2^2+d_4cos^2\lambda c_1^4| = \Big|&(d_1+2d_2cos\lambda+d_3+4d_4cos^2\lambda)c_1^4 \\ &+2d_1c_1(4-c_1^2)+2(d_1+d_2cos\lambda+d_3)c_1^2(4-c_1^2)|x| \\ &-\left((d_1+d_3)c_1^2+2d_1c_1-4d_3\right)(4-c_1^2)|x|^2\Big|. \end{aligned} \tag{3.9}$$

Using the values of $d_1,\ d_2,\ d_3$ and d_4 given in (3.7), after simplification, we obtain

$$\begin{aligned} d_1+2d_2cos\lambda+d_3+4d_4cos^2\lambda &= (s-t)(s^4-3s^2t^2+2st^3)+2(1-\beta) \\ &(s^4t-3s^2t^3+2st^4)cos\lambda-4(s^5+2s^4t-s^3t^2-2s^2t^3)(1-\beta)^2cos^2\lambda, \end{aligned} \tag{3.10}$$

$$2(d_1+d_2cos\lambda+d_3) = 2[(s-t)(s^4-3s^2t^2+2st^3)+(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda], \tag{3.11}$$

and

$$\begin{aligned} (d_1+d_3)c_1^2+2d_1c_1-4d_3 &= [(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)]c_1^2 \\ &+2(s-t)(2s^2-st-t^2)^2c_1+4(s-t)^2(3s^3-st^2-s^2t-t^3). \end{aligned} \tag{3.12}$$

Consider

$$\begin{aligned}
&\left[(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^2-st^2-s^2t-t^3)\right] c_1^2\\
&+2(s-t)(2s^2-st-t^2)^2c_1+4(s-t)^2(3s^3-st^2-s^2t-t^3)=(s-t)(s^4-3s^2t^2+2st^3)\\
&\left[c_1^2+\frac{2(2s^2-st-t^2)^2}{s^4-3s^2t^2+2st^3}c_1+4\frac{(s-t)(3s^3-st^2-s^2t-t^3)}{s^4-3s^2t^2+2st^3}\right]\\
&=(s-t)(s^4-3s^2t^2+2st^3)\left[\left\{c_1+\frac{(2s^2-st-t^2)^2}{s^4-3s^2t^2+2st^3}\right\}^2\right.\\
&\left.-\left\{\frac{(2s^2-st-t^2)^4}{(s^4-3s^2t^2+2st^3)^2}-\frac{4(s-t)(3s^3-st^2-s^2t-t^3)}{s^4-3s^2t^2+2st^3}\right\}\right]\\
&=(s-t)(s^4-3s^2t^2+2st^3)\left[\left(c_1+\frac{(2s^2-st-t^2)^2}{s^4-3s^2t^2+2st^3}\right)^2\right.\\
&\left.-\left\{\frac{\sqrt{4s^8+28s^6t^2-16s^7t+29s^4t^4-32s^5t^3+10s^2t^6-20s^3t^5-4st^7+t^8}}{s^4-3s^2t^2+2st^3}\right\}^2\right]\\
&=(s-t)(s^4-3s^2t^2+2st^3)\left[c_1+\left\{\frac{2s^2-st-t^2}{s^4-3s^2t^2+2st^3}\right.\right.\\
&\left.\left.+\frac{\sqrt{4s^8+28s^6t^2-16s^7t+29s^4t^4-32s^5t^3+10s^2t^6-20s^3t^5-4st^7+t^8}}{s^4-3s^2t^2+2st^3}\right\}\right]\\
&\times\left[c_1+\left\{\frac{2s^2-st-t^2}{s^4-3s^2t^2+2st^3}\right.\right.\\
&\left.\left.-\frac{\sqrt{4s^8+28s^6t^2-16s^7t+29s^4t^4-32s^5t^3+10s^2t^6-20s^3t^5-4st^7+t^8}}{s^4-3s^2t^2+2st^3}\right\}\right].
\end{aligned} \tag{3.13}$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_2+b) \geq (c_1-a)(c_1-b)$, where $a,\ b \geq 0$ in the right hand side of (3.13), upon simplification, we obtain

$$\begin{aligned}
&[(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)]c_1^2\\
&+2(s-t)(2s^2-st-t^2)^2c_1+4(s-t)^2(3s^3-st^2-s^2t-t^3)\\
&\geq [(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)]c_1^2\\
&-2(s-t)(2s^2-st-t^2)^2c_1+4(s-t)^2(3s^3-st^2-s^2t-t^3)
\end{aligned} \tag{3.14}$$

From (3.12) and (3.14), it follows that

$$-\left[(d_1+d_3)c_1^2+2d_1c_1-4d_3\right] \leq -\left[\left\{(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)\right\}c_1^2 \right.$$
$$\left. -2(s-t)(2s^2-st-t^2)^2c_1+4(s-t)^2(3s^3-st^2-s^2t-t^3)\right]. \tag{3.15}$$

Substituting the values from the relation (3.10), (3.11) and (3.15) in the right hand side of (3.9), we get

$$\begin{aligned} 4|d_1c_1c_3+d_2cos\lambda c_1^2c_2+d_3c_2^2+d_4cos^2\lambda c_1^4| \leq |\{(s-t)(s^4-3s^2t^2+2st^3)+2(1-\beta)\\ (s^4t-3s^2t^3+2st^4)cos\lambda-4(s^5+2s^4t-s^3t^2-2s^2t^3)(1-\beta)^2cos^2\lambda\}c_1^4\\ +2(s-t)(2s^2-st-t^2)^2c_1(4-c_1^2)+\{2(s-t)(2s^2-st-t^2)^2\\ +2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda-2(s-t)^2(3s^3-st^2-s^2t-t^3)\}\\ c_1(4-c_1^2)|x|-[\{(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)\}c_1^2-\\ 2(s-t)(2s^2-st-t^2)^2c_1+4(s-t)^2(3s^3-st^2-s^2t-t^3)](4-c_1^2)|x|^2| \end{aligned} \tag{3.16}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right hand sides of (3.16), we obtain

$$\begin{aligned} 4|d_1c_1c_3+d_2cos\lambda c_1^2c_2+d_3c_2^2+d_4cos^2\lambda c_1^4| \leq [4cos^2\lambda(1-\beta)^2(s^5+2s^4t-s^3t^2-2s^2t^3)\\ -(s-t)(s^4-3s^2t^2+2st^3)-2(1-\beta)(s^4t-3s^2t^3+2st^4)cos\lambda]c^4\\ +2(s-t)(2s^2-st-t^2)^2c(4-c^2)+\{2(s-t)(2s^2-st-t^2)^2\\ +2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda-2(s-t)^2(3s^3-st^2-s^2t-t^3)\}\\ c^2(4-c^2)\mu+[\{(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)\}c^2\\ -2(s-t)(2s^2-st-t^2)^2c+4(s-t)^2(3s^3-st^2-s^2t-t^3)](4-c^2)\mu^2\\ = H(c,\mu)\ (say)\ (0 \leq \mu = |x| \leq 1,\ 0 \leq c \leq 2), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} H(c,\mu) = [4cos^2\lambda(1-\beta)^2(s^5+2s^4t-s^3t^2-2s^2t^3)\\ -(s-t)(s^4-3s^2t^2+2st^3)-2(1-\beta)(s^4t-3s^2t^3+2st^4)cos\lambda]c^4\\ +2(s-t)(2s^2-st-t^2)^2c(4-c^2)+\{2(s-t)(2s^2-st-t^2)^2\\ +2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda-2(s-t)^2(3s^3-st^2-s^2t-t^3)\}\\ c^2(4-c^2)\mu+[\{(s-t)(2s^2-st-t^2)^2-(s-t)^2(3s^3-st^2-s^2t-t^3)\}c^2\\ -2(s-t)(2s^2-st-t^2)^2c+4(s-t)^2(3s^3-st^2-s^2t-t^3)](4-c^2)\mu^2. \end{aligned} \tag{3.18}$$

Now we maximize the function $H(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating on both sides of (3.18) partially with respect to μ, we get

$$\frac{\partial H}{\partial \mu}=\big[2(s-t)(2s^2-st-t^2)^2+2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda -2(s-t)^2(3s^3-st^2-s^2t-t^3)\big]c^2(4-c^2)+2\big[\{(s-t)(2s^2-st-t^2)^2 -(s-t)^2(3s^3-st^2-s^2t-t^3)\}c^2-2(s-t)(2s^2-st-t^2)^2c +4(s-t)^2(3s^3-st^2-s^2t-t^3)\big](4-c^2)\mu \quad (3.19)$$

For $0<\mu<1$, for fixed c, $0<c<2$, we observe from (3.19) that $\frac{\partial H}{\partial \mu}>0$. Therefore, $H(c,\mu)$ is an increasing function of μ and hence cannot have a maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for a fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le 1} H(c,\mu)=H(c,1)=T(c) \quad (3.20)$$

Upon simplification, it follows from (3.18) and (3.20) that

$$T(c)=[4cos^2\lambda(1-\beta)^2(s^5+2s^4t-s^3t^2-2s^2t^3)-(s-t)(s^4-3s^2t^2+2st^3) 2(1-\beta)(s^4t-3s^2t^3+2st^4)cos\lambda]c^4+[2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda+3(s-t) (s^4-3s^2t^2+2st^3)]c^2(4-c^2)+4(s-t)^2(3s^3-st^2-s^2t-t^3)(4-c^2). \quad (3.21)$$

Therefore,

$$T'(c)=4[4cos^2\lambda(1-\beta)^2(s^5+2s^4t-s^3t^2-2s^2t^3)-(s-t)(s^4-3s^2t^2+2st^3) 2(1-\beta)(s^4t-3s^2t^3+2st^4)cos\lambda]c^3+[2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda+3(s-t) (s^4-3s^2t^2+2st^3)](8c-4c^3)-8(s-t)^2(3s^3-st^2-s^2t-t^3)c. \quad (3.22)$$

and

$$T''(c)=12[4cos^2\lambda(1-\beta)^2(s^5+2s^4t-s^3t^2-2s^2t^3)-(s-t)(s^4-3s^2t^2+2st^3) 2(1-\beta)(s^4t-3s^2t^3+2st^4)cos\lambda]c^2+[2(s^4t-3s^2t^3+2st^4)(1-\beta)cos\lambda+3(s-t) (s^4-3s^2t^2+2st^3)](8-12c^2)-8(s-t)^2(3s^3-st^2-s^2t-t^3). \quad (3.23)$$

For extreme values of $T(c)$, consider $T'(c)=0$. From (3.22) we have $c=0$. Putting the values of $c=0$ in (3.23) and simplify, we get

$$T''(c)=-8(s-t)[(9s^2t^2-6st^3-4s^3t+t^4)-2(s^3t-2st^3+s^2t^2)(1-\beta)cos\lambda] \le 0 \quad \left(0\le\beta<1, |\lambda|<\frac{\pi}{2}\right). \quad (3.24)$$

By second derivative test, $T(c)$ has maximum values at $c = 0$ and for a fixed value of λ ($|\lambda| < \frac{\pi}{2}$), we obtain

$$\max_{0\leq c\leq 2} T(c) = T(0) = 16(s-t)^2(3s^3 - st^2 - s^2t - t^3). \tag{3.25}$$

Consider the maximum value of $T(c)$ only at $c = 0$, simplifying the relation (3.17) and (3.25), we obtain

$$|d_1c_1c_3 + d_2cos\lambda c_1^2c_2 + d_3c_2^2 + d_1cos^2\lambda c_1^4| \leq 4(s-t)^2(3s^3 - st^2 - s^2t - t^3). \tag{3.26}$$

From (3.6) and (3.26), after simplifying, we get

$$|a_2a_4 - a_3^2| \leq \frac{4(1-\beta)^2cos^2\lambda}{(2s^2 - st - t^2)^2}. \tag{3.27}$$

By choosing $c_1 = c = 0$ and selecting $x = -1$ in (2.2) and (2.3), we find that $c_2 = -2$ and $c_3 = 0$. Under such case it follows from (3.4) that $a_2 = 0$, $a_3 = -\frac{2e^{-i\lambda}(1-\beta)cos\lambda}{2s^2-st-t^2}$ and $a_4 = 0$. Substituting these values in the functional $|a_2a_4 - a_3^2|$, we observed that the equality is attained which shows our result is sharp. This completes the proof of Theorem 3.1. □

Concluding Remark: In this paper, we have determined the sharp upper bound for the functional $|a_2a_4 - a_3^2|$ for the functions $f \in \mathcal{A}$ belonging to the class $S(\lambda, \beta, s, t)$. We conclude this paper by remarking that the above theorem include several previously established results for particular values of the parameters λ, β, s, t. For example, taking $s = 1, t = 0$ and $\beta = 0$ in Theorem 3.1 we get the result due to Krishna and Reddy (see [15]). Further, by letting $s = 1$, $t = 0$, $\beta = 0$ and $\lambda = 0$ in Theorem 3.1, we obtain the result $|a_2a_4 - a_3^2| \leq 1$. This result is sharp and coincides with that of Janteng et al. (see [13]). Now we are working on to find the sharp upper bound for the above function class using third Hankel determinant.

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