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Spectrum and fine spectrum of the Zweier matrix over the sequence space cs

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ABSTRACT: In this article we have determined the spectrum and fine spectrum of the Zweier matrix Z_s on the sequence space cs. In a further development, we have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator Z_s on the sequence space cs.

Key Words: Spectrum of an operator; matrix mapping; sequence space.

Contents

1. Introduction

By w, we denote the space of all real or complex valued sequences. Throughout the paper c, c_0, bv, cs, bs, ℓ_1, ℓ_∞ represent the spaces of all convergent, null, bounded variation, convergent series, bounded series, absolutely summable and bounded sequences respectively. Also bv_0 denotes the sequence space $bv \cap c_0$.

Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. Fine spectrum of the operator $\Delta_{a,b}$ on the sequence space c was determined by Akhmedov and El-Shabrawy [1]. The fine spectra of the Cesàro operator C_1 over the sequence space bv_p, $(1 \leq p < \infty)$ was determined by Akhmedov and Başar [2]. Altay and Başar [3,4] determined the fine spectrum of the difference operator Δ and the generalized difference operator $B(r,s)$ on the sequence spaces c_0 and c. The spectrum and fine spectrum of the Zweier Matrix on the sequence spaces ℓ_1 and bv were studied by Altay and Karakuş [5]. Altun [6,7] determined the fine spectra of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. Furkan, Bilgiç and Kayaduman [14] have determined the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv. Fine spectra of operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv and generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p, $(1 \leq p < \infty)$ were studied by Bilgiç and Furkan [11,12]. Fine spectrum of the generalized difference operator Δ_v on the sequence space ℓ_1

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was investigated by Srivastava and Kumar [28]. Panigrahi and Srivastava [24,25] studied the spectrum and fine spectrum of the second order difference operator Δ^2_{uv} on the sequence space c_0 and generalized second order forward difference operator Δ^2_{uvw} on the sequence space ℓ_1. Fine spectra of upper triangular double-band matrix $U(r,s)$ over the sequence spaces c_0 and c were studied by Karakaya and Altun [20]. Karaisa and Başar [19] have determined the spectrum and fine spectrum of the upper traiangular matrix $A(r,s,t)$ over the sequence space ℓ_p, $(0 < p < \infty)$. In a further development, they have also determined the approximate point spectrum, defect spectrum and compression spectrum of the operator $A(r,s,t)$ on the sequence space ℓ_p, $(0 < p < \infty)$.

In this paper, we shall determine the spectrum and fine spectrum of the lower triangular matrix Z_s on the sequence space cs . Also,we determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator Z_s on the sequence space cs. Clearly, $cs = \{x = (x_n) \in w : \lim_{n\to\infty} \sum_{i=0}^{n} x_i \quad exists\}$ is a Banach space with respect to the norm $||x||_{cs} = \sup_n |\sum_{i=0}^{n} x_i|$.

2. Preliminaries and Background

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$,we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ be a complex normed linear space, where θ is the zero element and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1},$$

and call it the *resolvent* operator of T.

A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded

(R3) T_λ^{-1} is defined on a set which is dense in X i.e. $\overline{R(T_\lambda)} = X$.

The *resolvent set* of T, denoted by $\rho(T,X)$, is the set of all regular values λ of T. Its complement $\sigma(T,X) = \mathbb{C} - \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point(discrete) spectrum* $\sigma_p(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an eigenvalue of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies $(R3)$, but not $(R2)$, that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists (and may be bounded or not), but does not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X.

From Goldberg [17], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1} :

(I) $R(T) = X$,

(II) $R(T) \neq \overline{R(T)} = X$

(III) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in the Table 1.

	I	II	III
1	$\rho(T, X)$		$\sigma_r(T, X)$
2	$\sigma_c(T, X)$	$\sigma_c(T, X)$	$\sigma_r(T, X)$
3	$\sigma_p(T, X)$	$\sigma_p(T, X)$	$\sigma_p(T, X)$

Table 1: Subdivisions of spectrum of a linear operator

These are labeled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2 and III_3 . If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in I_2$,then λ is in the resolvent set $\rho(T, X)$ of T . The further classification gives rise to the fine spectrum of T . If an operator is in state II_2 , then $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ^{-1} exists but is discontinuous and we write $\lambda \in II_2\sigma(T, X)$. The state II_1 is impossible as if T_λ is injective, then from Kreyszig $\left[[22], \text{Problem 6, p.290}\right]$ T_λ^{-1} is bounded and hence continuous if and only if $R(T_\lambda)$ is closed.

Again, following Appell et al. [8], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $||x_k|| = 1$ and $||Tx_k|| \to 0$ as $k \to \infty$.

The *approximate point spectrum* of T , denoted by $\sigma_{ap}(T,X)$, is defined as the set

$$\sigma_{ap}(T,X)=\{\lambda\in\mathbb{C}: \textit{there exists a Weyl sequence for} \quad T-\lambda I\} \tag{2.1}$$

The *defect spectrum* of T, denoted by $\sigma_{\delta}(T,X)$, is defined as the set

$$\sigma_{\delta}(T,X)=\{\lambda\in\mathbb{C}: T-\lambda I \quad \textit{is not surjective}\} \tag{2.2}$$

The two subspectra given by equations (2.1) and (2.2) form a (not necessarily disjoint) subdivisions

$$\sigma(T,X)=\sigma_{ap}(T,X)\cup\sigma_{\delta}(T,X) \tag{2.3}$$

of the spectrum. There is another subspectrum

$$\sigma_{co}(T,X)=\{\lambda\in\mathbb{C}: \overline{R(T-\lambda I)}\neq X\}$$

which is often called the *compression spectrum* of T. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X)=\sigma_{ap}(T,X)\cup\sigma_{co}(T,X) \tag{2.4}$$

Clearly, $\sigma_p(T,X)\subseteq\sigma_{ap}(T,X)$ and $\sigma_{co}(T,X)\subseteq\sigma_{\delta}(T,X)$. Moreover, it is easy to verify that $\sigma_r(T,X)=\sigma_{co}(T,X)\setminus\sigma_p(T,X)$ and $\sigma_c(T,X)=\sigma(T,X)\setminus[\sigma_p(T,X)\cup\sigma_{co}(T,X)]$.

By the definitions given above, we can illustrate the subdivisions of spectrum of a bounded linear operator in the Table 2.

<table>
<tr><td></td><td></td><td>1</td><td>2</td><td>3</td></tr>
<tr><td></td><td></td><td>T_λ^{-1} exists and is bounded</td><td>T_λ^{-1} exists and is not bounded</td><td>T_λ^{-1} does not exist</td></tr>
<tr><td>I</td><td>$R(T-\lambda I)=X$</td><td>$\lambda\in\rho(T,X)$</td><td>$\cdots$</td><td>$\lambda\in\sigma_p(T,X)$
$\lambda\in\sigma_{ap}(T,X)$</td></tr>
<tr><td>II</td><td>$\overline{R(T-\lambda I)}=X$</td><td>$\lambda\in\rho(T,X)$</td><td>$\lambda\in\sigma_c(T,X)$
$\lambda\in\sigma_{ap}(T,X)$
$\lambda\in\sigma_{\delta}(T,X)$</td><td>$\lambda\in\sigma_p(T,X)$
$\lambda\in\sigma_{ap}(T,X)$
$\lambda\in\sigma_{\delta}(T,X)$</td></tr>
<tr><td>III</td><td>$\overline{R(T-\lambda I)}\neq X$</td><td>$\lambda\in\sigma_r(T,X)$
$\lambda\in\sigma_{\delta}(T,X)$
$\lambda\in\sigma_{co}(T,X)$</td><td>$\lambda\in\sigma_r(T,X)$
$\lambda\in\sigma_{ap}(T,X)$
$\lambda\in\sigma_{\delta}(T,X)$
$\lambda\in\sigma_{co}(T,X)$</td><td>$\lambda\in\sigma_p(T,X)$
$\lambda\in\sigma_{ap}(T,X)$
$\lambda\in\sigma_{\delta}(T,X)$
$\lambda\in\sigma_{co}(T,X)$</td></tr>
</table>

Table 2: Subdivisions of spectrum of a linear operator

Proposition 2.1. [Appell et al. [8], Proposition 1.3, p. 28] *Spectra and subspectra of an operator $T\in B(X)$ and its adjoint $T^*\in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T, X)$.

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.

(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.

(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.

(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.

(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell et al. [8]).

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n \in \mathbb{N}_0. \tag{2.5}$$

By $(\lambda : \mu)$, we denote the class of all matrices such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right hand side of equation (2.5) converges for each $n \in \mathbb{N}_0$ and every $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}_0} \in \mu$ for all $x \in \lambda$.

The Zweier matrix Z_s is an infinite lower triangular matrix of the form

$$Z_s = \begin{pmatrix} s & 0 & 0 & 0 & \cdots \\ 1-s & s & 0 & 0 & \cdots \\ 0 & 1-s & s & 0 & \cdots \\ 0 & 0 & 1-s & s & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $s \neq 0, 1$.

The following results will be used in order to establish the results of this article.

Lemma 2.1. [Wilansky [35] Example 6B, Page 130] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(cs)$ from cs to itself if and only if:*

(i) $\sup_m \sum_k |\sum_{n=1}^{m}(a_{nk} - a_{n,k-1})| < \infty$.

(ii) $\sum_n a_{nk}$ *is convergent for each* k.

Lemma 2.2. [Goldberg [17], Page 59] *T has a dense range if and only if T^* is one to one.*

Lemma 2.3. [Goldberg [17], Page 60] *T has a bounded inverse if and only if T^* is onto.*

3. Spectrum and fine spectrum of the operator Z_s over the sequence space cs

Theorem 3.1. *$Z_s : cs \to cs$ is a bounded linear operator and*

$$\| Z_s \|_{(cs:cs)} \leq |s| + |1-s|.$$

Proof: From Lemma 2.1, it is easy to show that $Z_s : cs \to cs$ is a bounded linear operator.
Now,

$$\begin{aligned} |Z_s(x)| &= \left| \sum_{i=0}^{n} s x_i + \sum_{i=0}^{n-1} (1-s) x_i \right| \\ &\leq |s| \left| \sum_{i=0}^{n} x_i \right| + |1-s| \left| \sum_{i=0}^{n-1} x_i \right| \\ &\leq (|s| + |1-s|) \| x \|_{cs} \end{aligned}$$

and hence,$\| Z_s \|_{(cs:cs)} \leq |s| + |1-s|$. Hence the result. □

From Theorem 2.1 in [4], we get the spectrum of the operator $B(r,s)$ on the sequence space c_0 is $\sigma(B(r,s), c_0) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$, where the operator $B(r,s)$ is given by the lower triangular matrix

$$B(r,s) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ 0 & s & r & 0 & \cdots \\ 0 & 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $s \neq 0$. The lower triangular matrix Z_s is a special case of $B(r,s)$.Also the sequence space cs is a subspace of c_0. Therefore we can expect that

$$\sigma(Z_s, cs) \subseteq \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1-s|\}.$$

In the following theorem we give an independent proof of our expectation.

Theorem 3.2. *The spectrum of the operator Z_s over cs is given by*

$$\sigma(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1-s|\}.$$

Proof: We prove this theorem by showing that $(Z_s - \alpha I)^{-1}$ exists and is in $(cs : cs)$ for $|\alpha - s| > |1 - s|$, and then show that the operator $Z_s - \alpha I$ is not invertible for $|\alpha - s| \leq |1 - s|$.

Let α be such that $|\alpha - s| > |1 - s|$. Since $s \neq 1$ we have $\alpha \neq s$ and so $Z_s - \alpha I$ is a triangle, therefore $(Z_s - \alpha I)^{-1}$ exists. Let $y = (y_n) \in cs$. Solving $(Z_s - \alpha I)x = y$ for x in terms of y we get

$$\begin{aligned}(Z_s - \alpha I)^{-1} &= (a_{nk}) \\ &= \begin{pmatrix} \frac{1}{s-\alpha} & 0 & 0 & 0 & \cdots \\ \frac{s-1}{(s-\alpha)^2} & \frac{1}{s-\alpha} & 0 & 0 & \cdots \\ \frac{(s-1)^2}{(s-\alpha)^3} & \frac{s-1}{(s-\alpha)^2} & \frac{1}{s-\alpha} & 0 & \cdots \\ \frac{(s-1)^3}{(s-\alpha)^4} & \frac{(s-1)^2}{(s-\alpha)^3} & \frac{s-1}{(s-\alpha)^2} & \frac{1}{s-\alpha} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\end{aligned}$$

It is easy to show that for all m,

$$\sum_k \left| \sum_{n=1}^{m} (a_{nk} - a_{n,k-1}) \right| \leq \frac{1}{|s-\alpha|} + \frac{|s-1|}{|s-\alpha|^2} + \frac{|s-1|^2}{|s-\alpha|^3} + \cdots + \frac{|s-1|^m}{|s-\alpha|^{m+1}}$$

and hence, $\sup_m \sum_k \left| \sum_{n=1}^{m} (a_{nk} - a_{n,k-1}) \right| < \infty$, as $|\alpha - s| > |1 - s|$.

Since $|\alpha - s| > |1 - s|$, so for all k, the series

$$\sum_n a_{nk} = \frac{1}{r-\alpha} - \frac{s}{(r-\alpha)^2} + \frac{s^2}{(r-\alpha)^3} - \cdots \tag{3.1}$$

is also convergent. So, by Lemma 2.1, $(Z_s - \alpha I)^{-1}$ is in $(cs : cs)$.
This shows that $\sigma(Z_s, cs) \subseteq \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\}$.

Now, let $\alpha \in \mathbb{C}$ be such that $|\alpha - s| \leq |1 - s|$. If $\alpha \neq s$, then $Z_s - \alpha I$ is a triangle and hence, $(Z_s - \alpha I)^{-1}$ exists.
Let $y = (1, 0, 0, 0, ...)$. Then $y \in cs$.
Now, $(Z_s - \alpha I)^{-1} y = x$ gives

$$x_n = \frac{(s-1)^n}{(s-\alpha)^{n+1}}.$$

Since $|\alpha - s| \leq |1 - s|$, so the series

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} \frac{(s-1)^n}{(s-\alpha)^{n+1}} = \frac{1}{s-\alpha} \sum_{n=0}^{\infty} \left(-\frac{s-1}{s-\alpha} \right)^n$$

is not convergent and hence, $x = (x_n) \notin cs$. Therefore, $(Z_s - \alpha I)^{-1}$ is not in $(cs : cs)$ and so $\alpha \in \sigma(Z_s, cs)$.

If $\alpha = s$, then the operator $Z_s - \alpha I$ is represented by the matrix

$$Z_s - sI = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1-s & 0 & 0 & 0 & \cdots \\ 0 & 1-s & 0 & 0 & \cdots \\ 0 & 0 & 1-s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Since, the range of $Z_s - \alpha I$ is not dense, so $\alpha \in \sigma(Z_s, cs)$. Hence,

$$\{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\} \subseteq \sigma(Z_s, cs).$$

This completes the proof. □

Theorem 3.3. *The point spectrum of the operator Z_s over cs is given by*

$$\sigma_p(Z_s, cs) = \phi.$$

Proof: Let α be an eigenvalue of the operator Z_s . Then there exists $x \neq \theta = (0, 0, 0, ...)$ in cs such that $Z_s x = \alpha x$. Then, we have

$$\begin{aligned} sx_0 &= \alpha x_0 \\ (1-s)x_0 + sx_1 &= \alpha x_1 \\ (1-s)x_1 + sx_2 &= \alpha x_2 \\ &\cdots \\ (1-s)x_n + sx_{n+1} &= \alpha x_{n+1}, \quad n \geq 0 \end{aligned}$$

If x_k is the first non-zero entry of the sequence (x_n) , then $\alpha = s$. Then from the relation
$(1-s)x_k + sx_{k+1} = \alpha x_{k+1}$, we have $(1-s)x_k = 0$. But $s \neq 1$ and hence, $x_k = 0$, a contradiction. Hence,$\sigma_p(Z_s, cs) = \phi$. □

If $T : cs \to cs$ is a bounded linear operator represented by a matrix A, then it is known that the adjoint operator $T^* : cs^* \to cs^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space cs^* of cs is isometrically isomorphic to the Banach space bv of all bounded variation sequences normed by $\| x \|_{bv} = \sum_{n=0}^{\infty} |x_{n+1} - x_n| + \lim_{n\to\infty} |x_n|$.

Theorem 3.4. *The point spectrum of the operator Z_s^* over cs^* is given by*

$$\sigma_p(Z_s^*, cs^* \cong bv) = \{\alpha \in \mathbb{C} : |\alpha - s| < |1 - s|\}.$$

Proof: Let α be an eigenvalue of the operator Z_s^*. Then there exists $x \neq \theta = (0, 0, 0, ...)$ in bv such that $Z_s^* x = \alpha x$. Then, we have

$$\begin{aligned} Z_s^t x &= \alpha x \\ \Rightarrow s x_0 + (1-s) x_1 &= \alpha x_0 \\ s x_1 + (1-s) x_2 &= \alpha x_1 \\ s x_2 + (1-s) x_3 &= \alpha x_2 \\ &\cdots \\ s x_n + (1-s) x_{n+1} &= \alpha x_n, \quad n \geq 0 \end{aligned}$$

Then, we have

$$x_n = \left(\frac{\alpha - s}{1 - s} \right)^n x_0.$$

Since $x = (x_n) \in bv$, so $x = (x_n) \in c$. The sequence (x_n) is convergent if and only if $|\alpha - s| < |1 - s|$. Hence, $\sigma_p(Z_s^*, cs^* \cong bv) = \{\alpha \in \mathbb{C} : |\alpha - s| < |1 - s|\}$. □

Theorem 3.5. *The residual spectrum of the operator Z_s over cs is given by*

$$\sigma_r(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| < |1 - s|\}.$$

Proof: Since,

$$\sigma_r(Z_s, cs) = \sigma_p(Z_s^*, cs^*) \setminus \sigma_p(Z_s, cs),$$

so we get the required result by using Theorem 3.3 and Theorem 3.4. □

Theorem 3.6. *The continuous spectrum of the operator Z_s over cs is given by*

$$\sigma_c(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| = |1 - s|\}.$$

Proof: Since, $\sigma(Z_s, cs)$ is the disjoint union of $\sigma_p(Z_s, cs)$, $\sigma_r(Z_s, cs)$ and $\sigma_c(Z_s, cs)$, therefore, by Theorem 3.2, Theorem 3.3 and Theorem 3.5, we get

$$\sigma_c(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| = |1 - s|\}.$$

□

Theorem 3.7. *If $\alpha = s$, then $\alpha \in III_1\sigma(Z_s, cs)$.*

Proof: If $\alpha = s$, the range of $Z_s - \alpha I$ is not dense. So, from Table 2 and Theorem 3.3, we have $\alpha \in \sigma_r(Z_s, cs)$.
From Table 2,

$$\sigma_r(Z_s, cs) = III_1\sigma(Z_s, cs) \cup III_2\sigma(Z_s, cs).$$

Therefore, $\alpha \in III_1\sigma(Z_s, cs)$ or $\alpha \in III_2\sigma(Z_s, cs)$.
Also for $\alpha = s$,

$$Z_s - \alpha I = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1-s & 0 & 0 & 0 & \cdots \\ 0 & 1-s & 0 & 0 & \cdots \\ 0 & 0 & 1-s & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

To prove the result, it is enough to show that the operator $Z_s - \alpha I$ is bounded below. It is easy to verify that for all $x \in cs$, we have

$$\| (Z_s - \alpha I)x \| \geq \frac{|1-s|}{2} \| x \|$$

which shows that the operator $Z_s - \alpha I$ is bounded below and so $Z_s - \alpha I$ has a bounded inverse. This completes the theorem. □

Theorem 3.8. *If $\alpha \neq s$ and $\alpha \in \sigma_r(Z_s, cs)$, then $\alpha \in III_2\sigma(Z_s, cs)$.*

Proof: Since,$\alpha \in \sigma_r(Z_s, cs)$, therefore,from Table 2,

$$\alpha \in III_1\sigma(Z_s, cs) \quad \text{or} \quad \alpha \in III_2\sigma(Z_s, cs).$$

Now, $\alpha \in \sigma_r(Z_s, cs)$ implies that $|\alpha - s| < |1 - s|$. Therefore, the series (3.1) in Theorem 3.2 is not convergent and hence, the operator $Z_s - \alpha I$ has no bounded inverse.
Therefore, $\alpha \in III_2\sigma(Z_s, cs)$. □

Theorem 3.9. *If $\alpha \in \sigma_c(Z_s, cs)$, then $\alpha \in II_2\sigma(Z_s, cs)$.*

Proof: If $\alpha \in \sigma_c(Z_s, cs)$, then $|\alpha - s| = |1 - s|$. Therefore, the series (3.1) in Theorem 3.2 is not convergent and hence, the operator $Z_s - \alpha I$ has no bounded inverse. So, α satisfies Goldberg's condition 2.
Now we shall show that the operator $Z_s - \alpha I$ is not onto.
Let $y = (y_n) = (1, 0, 0, 0, ...)$. Clearly,$(y_n) \in cs$.
Let $x = (x_n)$ be a sequence such that $(Z_s - \alpha I)x = y$.
On solving, we get

$$x_n = \frac{(s-1)^n}{(s-\alpha)^{n+1}}.$$

Now,the series

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} \frac{(s-1)^n}{(s-\alpha)^{n+1}} = \frac{1}{s-\alpha} \sum_{n=0}^{\infty} \left(\frac{s-1}{s-\alpha}\right)^n$$

is not convergent as $|\alpha - s| = |1 - s|$ and hence the operator $Z_s - \alpha I$ is not onto.
So, α satisfies Goldberg's condition II.
This completes the proof. □

Theorem 3.10. *The approximate point spectrum of the operator Z_s over cs is given by*

$$\sigma_{ap}(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\} \setminus \{s\}.$$

Proof: From Table 2,

$$\sigma_{ap}(Z_s, cs) = \sigma(Z_s, cs) \setminus III_1\sigma(Z_s, cs).$$

By Theorem 3.7, $III_1\sigma(Z_s, cs) = \{s\}$ .This completes the proof. □

Theorem 3.11. *The compression spectrum of the operator Z_s over cs is given by*

$$\sigma_{co}(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| < |1 - s|\}.$$

Proof: By Proposition 2.1(e), we get

$$\sigma_p(Z_s^*, cs^*) = \sigma_{co}(Z_s, cs).$$

Using Theorem 3.4, we get the required result. □

Theorem 3.12. *The defect spectrum of the operator Z_s over cs is given by*

$$\sigma_\delta(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\}.$$

Proof: From Table 2, we have

$$\sigma_\delta(Z_s, cs) = \sigma(Z_s, cs) \setminus I_3\sigma(Z_s, cs).$$

Also,

$$\sigma_p(Z_s, cs) = I_3\sigma(Z_s, cs) \cup II_3\sigma(Z_s, cs) \cup III_3\sigma(Z_s, cs).$$

By Theorem 3.3, we have $\sigma_p(Z_s, cs) = \phi$ and so $I_3\sigma(Z_s, cs) = \phi$.
Hence, $\sigma_\delta(Z_s, cs) = \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\}$. □

Corollary 3.1. *The following statements hold:*

(i) $\sigma_{ap}(Z_s^*, cs^* \cong bv) = \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\}$.

(ii) $\sigma_\delta(Z_s^*, cs^* \cong bv) = \{\alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s|\} \setminus \{s\}$.

Proof: Using Proposition 2.1 (c) and (d), we get

$$\sigma_{ap}(Z_s^*, cs^* \cong bv) = \sigma_\delta(Z_s, cs)$$

and

$$\sigma_\delta(Z_s^*, cs^* \cong bv) = \sigma_{ap}(Z_s, cs).$$

Using Theorem 3.10 and Theorem 3.12, we get the required results. □

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