p-$\mathcal{I}$-GENERATOR AND p_1-$\mathcal{I}$-GENERATOR IN BITOPOLOGY

Santanu Acharjee[1] and Binod Chandra Tripathy[2]

[1,2]Mathematical Sciences Division
Institute of Advanced Study in Science and Technology
Paschim Boragaon, Garchuck, Guwahati-781035, India.
E-mail: [1]santanuacharjee@rediffmail.com
[2]tripathybc@rediffmail.com, tripathybc@yahoo.com

Abstract. In this article we have investigated the relations of p-$\mathcal{I}$-generator, p_1-$\mathcal{I}$-generator with p-Lindelöf and p_1-Lindelöf using τ_i-codense, (i, j)-meager, (i, j)-nowhere dense and perfect mapping of bitopological space. The relations between p-compactness, p-Lindelöfness, p_1-Lindelöfness and topological ideal, (i, j)-meager, (i, j)-Baire space in bitopological space are investigated. Some properties are studied on product bitopology using perfect mapping. It can be found that bitopological space has many applications in real life problems. Hence, we hope that this theory will help to fulfill some interlinks which may have applications in near future.

Keywords : Topological ideal, p-Lindelöf, p_1-Lindelöf, pairwise weakly Lindelöf, pairwise almost Lindelöf.

AMS Classification no. 54E55

1. Introduction, motivation and scopes of bitopological space in other areas of mathematics and natural sciences

Kelly [1] introduced bitopological space via quasi-pseudo metric and systematically investigated its various important properties. It has drawn direct and indirect attentions of many point set topologists, fuzzy topologists, engineers and researchers of medical sciences, computer scientists etc. for its applications.

Definition of topological ideal is a very old concept. Topological ideal $\mathcal{I}$ and σ-ideal can be found in Dontchev etal [2]. Ideal of all nowhere dense sets and ideal of all meager sets of a ideal topological space $(X, \tau, \mathcal{I})$ are denoted by $\mathcal{N}$ and $\mathcal{M}$ respectively. Throughout no separation properties are considered unless it is stated clearly.

Kuratowski [3] introduced the notion of local function of $A \subseteq X$ in (X, τ) with respect to $\mathcal{I}$ and τ (briefly A^*). $A^*(\mathcal{I})$ or $A^* = \{x \in X | U \cap A \notin \mathcal{I}, x \in U$ for all $U \in \tau\}$.

It is well known that $cl^*(A) = A^* \cup A$, defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ finer than τ.

Throughout this paper "bitopological space" will be denoted by BS.

A cover $\mathcal{U}$ of a BS (X, τ_1, τ_2) is called $\tau_1\tau_2$-open (Swart[4], Definition 4.1) if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If in addition $\mathcal{U}$ contains atleast one non-empty member of τ_1 and atleast one nonempty member of τ_2; it is called pairwise open(see for instance Fletcher et al. [5]). Pairwise compactness was defined by Fletcher et al. [5]. p-compact, p_1- compact, p-Lindelöf and p_1-Lindelöf are defined by Kilićkman and Salleh[6]. According to Reilly [7]; (X, τ_1, τ_2) is pairwise Lindelöf (pairwise compact)if each pairwise open cover has a countable (finite) subcover. Cooke et al. [8] investigated relation between semi-compactness and pairwise compactness in bitopological space.

Kilićkman and Salleh [9-11] also investigated various properties of pairwise Lindelöfness. Cocompactness, cotopology, (i, j)-baire space etc. were studied by Dvalishvili [12].

Frolik [13] introduced weakly Lindelöf space and Willard and Dissanayake [14] introduced almost Lindelöf space in a topological space and their bitopological version are studied by Kilićkman and Salleh [9]. In the last two decades various developments have taken place in bitopological space. Still a little progress has been observed in case of generalized closed sets of bitopological space and related areas. Fuzzy version of some generalized closed sets and related structures of both topology and bitopology has been investigated (one may refer to [15-17]). Fuzzy version of topological ideal was introduced in [18].

Bitopological space and their properties have many useful applications in real world. In 2010, Salama [19] used lower and upper approximations of Pawlak's rough sets by using a class of generalized closed set of bitopological space for data reduction of rheumatic fever data sets. Fuzzy topology integrated support vector machine (FTSVM)-classification method for remotely sensed images based on standard support vector machine (SVM) were introduced by using fuzzy topology by Zhang etal [20]. For some of recent indirect applications of topological or bitopological space as fuzzy version, one may refer to [19-21]. Topological ideal has also huge applications in real world. Recently Tripathy and Acharjee [22] have introduced a class of generalized closed set in bitopological space using topological ideal, two expansion operators and local functions. The application of this set can be found in market price equilibrium [23]. There are the maximum nine out eleven strategies under which expected value of daily used items decided by a consumer and value decided by govt.is equal. Other two strategies are special cases. These are useful from the view point that no one will have to face poverty in year 2015 if she have price table of items of year 2014 and 2013. She is free to choose daily used items according to her preferences.

One may refer to [41] for inter related research works on topology, order of mathematics and utility theory of mathematical economics. In this paper one may find that how concepts of countability, compactness, normality, Lindelöfness etc. of general topology and order (i.e. LOTS etc) have been used to represent countable representation of utility function.For reference; Prof. G. Bosi's [42,43] extensive publications and vast expertise lies on mathematical economics using general topology, bitopology, order topology.

Hence there is need and scope of investigation considering different types of pairwise compactness, pairwise Lindelöf from the point of view of topological ideal, (i, j)-meager and (i, j)-Baire space so that these results may contribute to both theory and application in various areas of sciences.

In this paper we are trying to give some possible answers of the following questions. They are as follows:

(i) Is there any relation between different forms of pairwise Lindelöfness and (i, j)-meager and pairwise Baire space in bitopological space?

(ii) Is there any relation between different forms of pairwise Lindelöfness and topological ideal in bitopological space?

(iii) What are the results related to pairwise Lindelöfness in product bitopology using Dutta's perfect mapping?

In this paper we particularly consider only two pairwise Lindelöfnss. They are p-Lindelöf due to Kilićkman and Salleh [2] and p_1-Lindelöf due to Birsan [24] as defined by Kilićkman and Salleh [2]. Dvalishvili [25] defined (i, j)-nowhere dense set. Dontchev et al. [26] studied ideal irresoluteness in topology. Dutta [27] defined perfect map from bitopological view point. Researchers have investigated Khalimsky digital line considering generalized closed sets in topological space (one may refer to [28-30]). Many topologists are now focusing on ideal and its various consequences. Systematic study on pairwise Lindelöfness also can be found in Salleh and Kilićkman[31]. Recently Acharjee and Tripathy [32] studied pairwise compactness on (γ, δ)-BSC set in bitopological spaces. Throughout this paper we will consider $i, j \in \{1, 2\}, i \neq j$

From above it can be considered that bitopology is also gaining speed now a days as an applied branches as many research areas are using bitopological properties as their tools to solve mechanical, medical, economical problems etc. Hence these above questions might play significant roles in applied sectors in near future. Often it seems easy to assume that bitopological results are extensions of results from general topology; but actually is it not as it seems. One may simply say that bitopology has more than two definitions of Lindelöfness using only pairwise open sets etc.

Variation of i and j between only 1 and 2 in a bitopological space often signifies different properties which general topology never follows. In [44], Acharjee and Papadopoulos, gave some answers to some open questions and one suitable counterexample.

2. Some preliminary definitions

Definition 2.1 ([9], Definition 2.7). A BS (X, τ_1, τ_2) is said to be (i, j)-nearly Lindelöf (resp. (i, j)-almostLindelöf, (i, j)-weakly Lindelöf) if every τ_i-open cover $\{U_\alpha | \alpha \in \Delta\}$ of X, there exists a countable subcollection $\{U_{\alpha_n} | n \in N\}$ such that $X = \bigcup_{n\in N} \tau_i int \tau_j cl(U_{\alpha_n})$ (resp. $X = \bigcup_{n\in N} \tau_j cl(U_{\alpha_n})$, $X = \tau_j cl(\bigcup_{n\in N} U_{\alpha_n})$).

X is said to be pairwise nearly Lindelöf if it is both (i, j)-nearly Lindelöf and (j, i)-nearly Lindelöf. Similarly we can define pairwise almost Lindelöf, pairwise weakly Lindelöf.

Definition 2.2 ([25], Definition 1.1). A subset A of a BS (X, τ_1, τ_2) is termed as

(i,j)-nowhere dense if $\tau_i int \tau_j cl(A) = \emptyset$. The families of all (i,j)-nowhere dense subsets of X are denoted by (i,j)-$\mathcal{ND}(X)$.

Let $\mathcal{I}$ be a topological ideal then $\mathcal{I} \neq \emptyset$ and $\mathcal{I}$ is said to be codense [2] for a topological space (X,τ) if and only if $\mathcal{I} \cap \tau = \{\emptyset\}$. Keeping same meaning in our mind we may define τ_i-codense; $i = 1,2$ for a BS (X,τ_1,τ_2). An ideal $\mathcal{I}$ is said to be pairwise codense if it is both τ_1-codense and τ_2-codense. We denote ideal of (i,j)-nowhere dense subsets of BS (X,τ_1,τ_2) by $\mathcal{I}_i\mathcal{N}_j(X)$

Definition 2.3 ([12], Definition 1.6). A subset A of a BS (X,τ_1,τ_2) is termed as (i,j)-first category (or (i,j)-meager) if $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \in (i,j)$-$\mathcal{ND}(X)$; for every $n \in N$ and A is of (i,j)-second category (or (i,j)-non meager) if it is not of (i,j)-first category. The families of all sets of (i,j)-first categories (or (i,j)-second categories) in X are denoted by (i,j)-$Catg_I(X)((i,j)$-$Catg_{II}(X))$.

If $X \in (i,j)$-$Catg_I(X)(X \in (i,j)$-$Catg_{II}(X))$ is abbreviated to X is of (i,j)-$Catg_I((i,j)$-$Catg_{II})$.

We denote σ-ideal of (i,j)-meager subsets of a BS (x,τ_1,τ_2) by $\sigma_i\mathcal{M}_j(X)$ (see [2]).

Now We define following definition.

Definition 2.4. A BS (X,τ_1,τ_2) is said to be (i,j)-non-nearly Lindelöf (resp. (i,j)-non-almost Lindelöf, (i,j)-non-weakly Lindelöf) if every τ_i-open cover $\{U_\alpha | \alpha \in \Delta\}$ of X, there exists a τ_j-open countable sub-collection $\{U_{\alpha_n} | n \in N\}$ such that $X = \bigcup_{n\in N} \tau_j int \tau_i cl(U_{\alpha_n})$ (resp. $X = \bigcup_{n\in N} \tau_i cl(U_{\alpha_n})$, $X = \tau_i cl(\bigcup_{n\in N} U_{\alpha_n})$).

X is said to be pairwise non-nearly Lindelöf if it is both (i,j)-non-nearly Lindelöf and (j,i)-non-nearly Lindelöf. Similarly we have pairwise non-almost Lindelöf, pairwise non-weakly Lindelöf.

Kilićkman and Salleh defined p-Lindelöf [6, Definition 6]. It was stated that Birsan defined p_1-Lindelöf [6, Definition 1].

Definition 2.5. ([44], Definition 3.1) Let (X,τ_1,τ_2) be a bitopological space, then:

(i) (X,τ_1,τ_2) is said to be an (i,j)-second countable bitopological space if (X,τ_i) is second countable with respect to τ_j.

(ii) (X,τ_1,τ_2) is said to be a contra second countable bitopological space if it is both $(1,2)$-second countable bitopological space and $(2,1)$-second countable bitopological space.

We procure the following results those will be used in this paper.

Lemma 2.1([6], Theorem 6) If (X,τ_1,τ_2) is second countable space, then (X,τ_1,τ_2)

is p-Lindelöf.

Definition 2.6. ([7]) A bitopological space (X, τ_1, τ_2) is *pairwise compact* (resp. *pairwise Londelöf*) if each pairwise open cover of (X, τ_1, τ_2) has a finite (resp. countable) subcover.

Definition 2.7. ([46]) (X, τ_1, τ_2) is said to be *pairwise countably compact* if every countable pairwise open cover of (X, τ_1, τ_2) has a finite subcover.

Proposition 2.1. ([7]) In a pairwise Lindelöf space pairwise countable compactness is equivalent to pairwise compactness.

Proposition 2.2. ([7]) Any second countable bitopological space is pairwise Lindelöf.

Proposition 2.3. ([7]) If (X, τ_1, τ_2) is pairwise Lindelöf and A is a proper subset of X which is τ_1-closed then A is pairwise Lindelöf and τ_2-Lindelöf.

Proposition 2.4. ([7]) If (X, τ_1, τ_2) is pairwise Lindelöf and pairwise regular then it is pairwise normal.

3. Main results

In this section we have defined a new class in a bitopological space which will generate p-Lindelöf space and p_1-Lindelöf space.

Definition 3.1. A BS $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be τ_i-$\mathcal{I}$-generator (τ_i^c-$\mathcal{I}$-generator) if every τ_i-open cover $\{U_\alpha | \alpha \in \Delta\}$ of X, there exists a (τ_j-open) countable sub collection $\{U_{\alpha_n} | n \in N\}$ such that $X \setminus \bigcup_{n\in N} U_{\alpha_n} \in \mathcal{I}$.

X is said to be p-$\mathcal{I}$-generator (p_1-$\mathcal{I}$-generator) if it is both τ_i-$\mathcal{I}$-generator (τ_i^c-$\mathcal{I}$-generator) and τ_j-$\mathcal{I}$-generator (τ_j^c-$\mathcal{I}$-generator).

Remark 3.1. From definition of ideal it is clear that $\mathcal{I} \neq \emptyset$. If $\mathcal{I} = \{\emptyset\}$ then Definition 3.1 reduces to p-Lindelöf (p_1-Lindelöf) i.e. p-$\{\emptyset\}$-generator$\Leftrightarrow$ p-Lindelöf and p_1-$\{\emptyset\}$-generator$\Leftrightarrow$ p_1-Lindelöf.

From [2,26] we know that a subset S of $(X, \tau, \mathcal{I})$ will be a topological space with ideal $\mathcal{I}_S$=$\{I \cap S : I \in \mathcal{I}\}$.

A subset A of X of (X, τ_1, τ_2) is said to be pairwise clopen if it is both τ_1-clopen and τ_2-clopen.

Theorem 3.1. (i) Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a p-$\mathcal{I}$-generator. If A be a pairwise closed subset of X then $(A, \tau_1|_A, \tau_2|_A, \mathcal{I}_A)$ is also p-$\mathcal{I}_A$-generator.

(ii) Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a p_1-$\mathcal{I}$-generator. If A be a pairwise clopen subset of X then $(A, \tau_1|_A, \tau_2|_A, \mathcal{I}_A)$ is also p_1-$\mathcal{I}_A$-generator.

Proof.(i) Let $\mathcal{U}_A = \{U_\alpha \cap A : U_\alpha \in \tau_i, \alpha \in \Delta\}$ be a $\tau_i|_A$-open cover of A. Thus $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\} \cup (X \setminus A)$ is τ_i open cover of X. Thus X has a countable subcollection $\mathcal{V}$=$\{U_{\alpha_n} : U_{\alpha_n} \in \tau_i, n \in N\} \cup (X \setminus A)$ such that $X \setminus \{\bigcup_{n\in N} U_{\alpha_n} \cup (X \setminus A)\} = R(\text{say}) \in \mathcal{I}$. Then $A \subseteq \bigcup_{n\in N}\{U_{\alpha_n} : n \in N\} \cup R$. Thus $A = \bigcup_{n\in N}(U_{\alpha_n} \cap A) \cup (R \cap A)$. So, clearly we have $A \setminus \{\bigcup_{n\in N}(U_{\alpha_n} \cap A)\} \subseteq (R \cap A) \in \mathcal{I}_A$. Thus $\mathcal{V}_A = \{U_{\alpha_n} \cap A : n \in N\}$ is satisfying required condition for p-$\mathcal{I}_A$-generator. Hence proof.

(ii) It can be established following the technique of (i).

Remark 3.2. If $\mathcal{I} = \{\emptyset\}$ then $\mathcal{I}_A = \{\emptyset\}$, then by Theorem 3.1, A is p-$\{\emptyset\}$-generator. Which implies Lemma 1. of Kilićkman and Salleh [6] and vice-versa. Similarly If A is p_1-$\{\emptyset\}$-generator, then it implies Lemma 4. of Kilićkman and Salleh [6]

In view of Lemma 2.1 and Remark 3.1 we have the following result.

Corollary 3.1. Every second countable space is p-$\{\emptyset\}$-generator.

Theorem 3.2. (i) Let (X, τ_1, τ_2) be a BS, then X is pairwise weakly Lindelöf if and only if X is both τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator and τ_j-$\mathcal{I}_i\mathcal{N}_j$-generator.

(ii) Let (X, τ_1, τ_2) be a BS, then X is pairwise non-weakly Lindelöf if and only if X is both τ_i-$\mathcal{I}_i\mathcal{N}_j$-generator and τ_j-$\mathcal{I}_j\mathcal{N}_i$-generator.

Proof. (i) Necessity.
we have only to show that if X is (i, j)-weakly Lindelöf then it is τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator.

Let us assume that X is (i, j)-weakly Lindelöf and let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be an τ_i-open cover of X. Then by Definition 2.1, there exists a countable subcollection $\{U_{\alpha_n} | n \in N\}$ such that $X = \tau_j cl(\bigcup_{n\in N} U_{\alpha_n})$. Then $X \setminus \bigcup_{n\in N} U_{\alpha_n} \in \mathcal{I}_j\mathcal{N}_i(X)$. Similarly established for (j, i)-weakly Lindelöf case.

Sufficiency

We proof that if X is τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator then X is (i, j)-weakly Lindelöf.

Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be an τ_i-open cover of X, then by Definition 3.1; there exists a countable subcollection $\{U_{\alpha_n} | n \in N\}$ such that $X \setminus \bigcup_{n\in N} U_{\alpha_n} \in \mathcal{I}_j\mathcal{N}_i(X)$. Then $X = \tau_j cl(\bigcup_{n\in N} U_{\alpha_n})$. Thus X is (i, j)-weakly Lindelöf. Similarly we can prove for τ_j-$\mathcal{I}_i\mathcal{N}_j$-generator case.

(ii) It can be established by following the technique of proof of (i).

Theorem 3.3.(i) A BS (X, τ_1, τ_2) is pairwise weakly Lindelöf if and only if it is both

τ_i-R-generator and τ_j-S-generator for some τ_j-codense ideal R and τ_i-codense ideal S.

(ii) A BS (X, τ_1, τ_2) is pairwise non-weakly Lindelöf if and only if it is both τ_i-R-generator and τ_j-S-generator for some τ_i-codense ideal R and τ_j-codense ideal S.

Proof. (i) Necessity.

If (X, τ_1, τ_2) is pairwise weakly Lindelöf, then by Theorem 3.2(i), X is both τ_i-$\mathcal{I}_j\mathcal{N}_i$-generator and τ_j-$\mathcal{I}_i\mathcal{N}_j$-generator. It can be checked that $\mathcal{I}_j\mathcal{N}_i(X) \cap \tau_j = \{\emptyset\}$. So, $\mathcal{I}_j\mathcal{N}_i(X)$ is τ_j-codense. Similarly it can be shown for other case.

Sufficiency.

Let R be any τ_j-codense ideal and X is τ_i-R-generator. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be any τ_i-open cover of X. Then there is a countable subcover $U_{\alpha_n} | n \in N\}$ such that $X \setminus \bigcup_{n \in N} U_{\alpha_n} \in R$. As R is τ_j-codense ideal, so $X = \tau_j cl(\bigcup_{n \in N} U_{\alpha_n})$. Thus X is (i, j)-weakly Lindelöf. Similarly we can prove for the other case. Thus X is pairwise weakly Lindelöf. Hence the proof.

Dvalishvili ([12, Definition 1.7],[25]) cited (i, j)-Baire space and pairwise Baire space.

In next theorem we establish the relation between pairwise weakly Lindelöf space and pairwise σ-ideal generator under certain condition.

Theorem 3.4. Let (X, τ_1, τ_2) is a pairwise Baire space. Then (i) (X, τ_1, τ_2) is pairwise weakly Lindelöf if and only if (X, τ_1, τ_2) is both τ_i-$\sigma_j\mathcal{M}_i$-generator and τ_j-$\sigma_i\mathcal{M}_j$-generator.

(ii) (X, τ_1, τ_2) is pairwise non-weakly Lindelöf if and only if (X, τ_1, τ_2) is both τ_i-$\sigma_i\mathcal{M}_j$-generator and τ_j-$\sigma_j\mathcal{M}_i$-generator.

Proof. (i) (X, τ_1, τ_2) is (i, j)-Baire space and (j, i)-Baire space $\Rightarrow$ X is (i, j)-$Catg_{II}$ and (j, i)-$Catg_{II}$.

(X, τ_1, τ_2) is (i, j)-Baire space and (j, i)-Baire space $\Leftrightarrow$ $\sigma_i\mathcal{M}_j(X)$ is τ_i-codense and $\sigma_j\mathcal{M}_i(X)$ is τ_j-codense. Then from Theorem 3.3(i) proof follows.

A BS (X, τ_1, τ_2) is said to have property * if $\tau_i cl(\tau_j cl(U)) = \tau_j cl(U)$ whenever $U \subseteq X$ and $i, j \in \{1, 2\}, i \neq j$.

We state the following result without proof.

Theorem 3.5.(i) If (X, τ_1, τ_2) is pairwise almost Lindelöf with property * then it is both τ_i-$\sigma_j\mathcal{M}_i$-generator and τ_j-$\sigma_i\mathcal{M}_j$-generator.

(ii) If (X, τ_1, τ_2) is pairwise non-almost Lindelöf with property * then it is both τ_i-$\sigma_i\mathcal{M}_j$-generator and τ_j-$\sigma_j\mathcal{M}_i$-generator.

In view of Theorem 3.4 and Theorem 3.5 we state the following result.

Corollary 3.2. (i) If a BS (X, τ_1, τ_2) is pairwise almost Lindelöf with property * and pairwise Baire space then it is pairwise weakly Lindelöf.

(ii) If a BS (X, τ_1, τ_2) is pairwise non-almost Lindelöf with property * and pairwise Baire space then it is pairwise non weakly Lindelöf.

The following result is a consequence of Theorem 3.1 and Theorem 3.2.

Corollary 3.3. (i) If A be a pairwise clopen subset of a pairwise weakly Lindelöf space (X, τ_1, τ_2) then $(A, \tau_1|_A, \tau_2|_A)$ is pairwise weakly Lindelöf.

(ii) If A be a pairwise clopen subset of a pairwise non-weakly Lindelöf space (X, τ_1, τ_2) then $(A, \tau_1|_A, \tau_2|_A)$ is pairwise non-weakly Lindelöf.

During the preparation of this present article with refer to Kilicman and Salleh's article [6], some open questions were aroused; some of whose answers were positive and counter example was provided by Acharjee and Papadopoulos [44] using interlocking and nest in a bitopological space. These questions were as follows:

"What type of a countable space in a bitopological space is a p_1-Lindelöf space?". "Is every p_1-Lindelöf space implies that same type of countable space?" The positive answer of first question and counter example of non-existence of second questions using nest and interlocking were provided in [44]. Thus we have one theorem.

Theorem 3.6. Let (X, τ_1, τ_2) be a contra second countable bitopological space, then it is p_1-$\{\emptyset\}$-generator.

Proof. Remark 3.1. and Theorem 3.1. of [44] give the proof.

Theorem 3.7. Every pairwise closed subset of a contra second countable bitopological space is p_1-$\{\emptyset\}$-generator.

Proof. By corollary 3.1. of [44] and Remark 3.1. we have the proof.

4. Relation with perfect mapping

The following definition on perfect mapping is due to Datta [27].

Definition 4.1([27], Definition 2.1) A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2)$ is said to be perfect if

(i) f is continuous; that is f is τ_1-ψ_1-continuous and τ_2-ψ_2-continuous.

(ii) f is compact, that is, the inverse image of every point of Y is τ_1-compact,τ_2-

compact and pairwise compact.

(iii) f is closed, that is, the image of every τ_1-closed (τ_2-closed) subset of X is ψ_1-closed (ψ_2-closed) subset of Y.

Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \psi_1, \psi_2, \mathcal{J})$ be a function, then we denote $f(\mathcal{I}) = \{f(I)|I \in \mathcal{I}\}$ and $f^{-1}(\mathcal{J}) = \{f^{-1}(J)|J \in \mathcal{J}\}$. In this case $f(\mathcal{I})$ and $f^{-1}(\mathcal{J})$ are ideal of Y and X respectively.

Theorem 4.1 (i) Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \psi_1, \psi_2)$ be a continuous function and surjection. If $(X, \tau_1, \tau_2, \mathcal{I})$ is p-$\mathcal{I}$-generator, then (Y, ψ_1, ψ_2) is also p-$f(\mathcal{I})$-generator.

(ii) Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \longrightarrow (Y, \psi_1, \psi_2)$ be a continuous function and surjection. If $(X, \tau_1, \tau_2, \mathcal{I})$ is p_1-$\mathcal{I}$-generator then (Y, ψ_1, ψ_2) is also p_1-$f(\mathcal{I})$-generator.

Proof.(i) We only prove that if $(X, \tau_1, \tau_2, \mathcal{I})$ is τ_i-$\mathcal{I}$-generator then (Y, ψ_1, ψ_2) is also ψ_i-$f(\mathcal{I})$-generator.

Let $\mathcal{U} = \{U_\alpha|\alpha \in \Delta\}$ be any ψ_i-open cover of Y, Then by Definition 4.1, $\mathcal{V} = \{f^{-1}(U_\alpha)|\alpha \in \Delta\}$ is τ_i-open cover of X. So by definition we have a subcollection $\{f^{-1}(U_{\alpha_n})|n \in N\}$ such that $X \setminus \bigcup_{n\in N} f^{-1}(U_{\alpha_n}) \in \mathcal{I}$. Suppose $f^{-1}(Y \setminus \bigcup_{n\in N} U_{\alpha_n}) = I$. So,$(Y \setminus \bigcup_{n\in N} U_{\alpha_n}) = f(I) \in f(\mathcal{I})$ as $I \in \mathcal{I}$. Thus we have the proof

(ii) It can be established following the technique used in establishing part(i).

We state the following result without proof.

Theorem 4.2. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2, \mathcal{J})$ be a perfect, open and surjective. Then

(i) If $(Y, \psi_1, \psi_2, \mathcal{J})$ is p-$\mathcal{J}$-generator then (X, τ_1, τ_2) is p-$f^{-1}(\mathcal{J})$-generator.

(ii) If $(Y, \psi_1, \psi_2, \mathcal{J})$ is p_1-$\mathcal{J}$-generator then (X, τ_1, τ_2) is p_1-$f^{-1}(\mathcal{J})$-generator.

Lemma 4.1. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2, \mathcal{J})$ be an open function and surjective. If $\mathcal{J}$ is ψ_i-codense then $f^{-1}(\mathcal{J})$ is τ_i-codense.

Proof. Let $f^{-1}(\mathcal{J})$ is not τ_i-codense. Let $f^{-1}(J) \in f^{-1}(\mathcal{J}) \cap \tau_i \neq \{\emptyset\}$. Then $f^{-1}(J) \in \tau_i \setminus \{\emptyset\}$. Due to surjective and open $f(f^{-1}(J)) = J \in \psi_i \setminus \{\emptyset\}$. This contradicts the fact that $\mathcal{J}$ is ψ_i-codense. Hence the proof.

Corollary 4.1. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2)$ be a perfect,open and surjective. Then

(i)If (Y, ψ_1, ψ_2) is p-Lindelöf then (X, τ_1, τ_2) is p-Lindelöf .

(ii)If (Y, ψ_1, ψ_2) is p_1-Lindelöf then (X, τ_1, τ_2) is p_1-Lindelöf .

Proof. (i) (Y, ψ_1, ψ_2) is p-Lindelöf implies it is p-$\{\emptyset\}$-generator.Then the proof follows from Theorem 4.2(i) and Remark 3.1.

(ii) Proof follows similar to the case (i)

Applying Theorem 3.3 and Lemma 4.1 one can get the following result.

Corollary 4.2. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \psi_1, \psi_2)$ be a open and surjective. Then

(i) If (Y, ψ_1, ψ_2) is pairwise weakly Lindelöf then (X, τ_1, τ_2) is pairwise weakly Lindelöf .

(ii) If (Y, ψ_1, ψ_2) is pairwise non-weakly Lindelöf then (X, τ_1, τ_2) is pairwise non-weakly Lindelöf .

5. On product bitopology

It is well known that every continuous mapping between p-compact spaces is p-compact in bitopological space. One may refer to Dutta ([27], pg no- 124)

Theorem 5.1. (i) If $(X, \tau_1, \tau_2, \mathcal{I})$ is p-$\mathcal{I}$-generator and (Y, ψ_1, ψ_2) is p-compact then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p-$\pi^{-1}(\mathcal{I})$-generator where $\pi : X \times Y \longrightarrow X$ is a projection map.

(ii) If $(X, \tau_1, \tau_2, \mathcal{I})$ is p_1-$\mathcal{I}$-generator and (Y, ψ_1, ψ_2) is p-compact then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p_1-$\pi^{-1}(\mathcal{I})$-generator where $\pi : X \times Y \longrightarrow X$ is a projection map.

Proof. The projection map is perfect. Hence the rest follows from Theorem 4.2.

The following result is a consequence of Theorem 3.3, Lemma 4.1 and Theorem 5.1.

Corollary 5.1. (i) If (X, τ_1, τ_2) is pairwise weakly Lindelöf and (Y, ψ_1, ψ_2) is p-compact then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is pairwise weakly Lindelöf.

(ii) If (X, τ_1, τ_2) is pairwise non-weakly Lindelöf and (Y, ψ_1, ψ_2) is p-compact then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is pairwise non-weakly Lindelöf .

Corollary 5.2. (i) If (X, τ_1, τ_2) is p-Lindelöf and (Y, ψ_1, ψ_2) is p-compact then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p-Lindelöf .

(ii) If (X, τ_1, τ_2) is p_1-Lindelöf and (Y, ψ_1, ψ_2) is p-compact then $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p_1-Lindelöf .

Proof. (i) By Remark 3.1, (X, τ_1, τ_2) is p-Lindelöf $\Leftrightarrow$ (X, τ_1, τ_1) is p-$\{\emptyset\}$-generator. By Theorem 5.1(i), $(X \times Y, \tau_1 \times \psi_1, \tau_2 \times \psi_2)$ is p-$\{\emptyset\}$-generator. Hence the proof.

Conclusion

In this paper we investigated that p-Lindelöf ness and p_1-Lindelöf ness can be derived by defining new classes in bitopological space. We also proved results related to perfect mapping of bitopological space and used them in the area of product bitopology. We have used perfect mapping to prove various results. One may understand from classical literatures of bitopology that various types of pairwise mappings play crucial role in contradiction of results related to various pairwise concept. This idea may extend on other types of Lindelöf ness in bitopological space.These methods give a short and concrete way to prove various results in product of Lindelöf spaces. We are hoping that this paper will attract attentions of both topologists, economists and researchers of other branches. The connection between countability and p_1-Lindelöfness or p_1-$\{\emptyset\}$-generator will help economists to use bitopological space, Lindelöfness etc in their respective research areas as possible idea can be gained from [41-43] where authors studied utility functions and various results based on compactness, Lindelöfness and other properties of general topology and order.

References

[1] J.C. Kelly, Bitopological spaces, Proc. Lond. Math. Soc., 13 (1963), 71-89.

[2] J. Dontchev, M. Ganster and D. Rose, Ideal resolvability, Topology Appl., 93 (1999), 1-16.

[3] K. Kuratowski, *Topology I*, Warsaw, (1933)

[4] J. Swart, Total disconnectedness in bitopological spaces and product bitopological spaces, Indag. math.,33 (1971), 135-145.

[5] P. Fletcher, H.B. Hoyle III and C.W. Patty, The comparison of topologies, Duke Math. Jour., 36 (1969), 325-332.

[6] A. Kilićkman and Z. Salleh, On pairwise Lindelöf bitopological spaces, Topology Appl., 154 (2007), 1600-1607.

[7] I.L. Reilly, Pairwise Lindelöf bitopological spaces, Kyungpook. Math. Jour,13(1) (1973), 1-4.

[8] I.E. Cooke and I.L. Reilly, On bitopological compactness, J. Lond. Math. Soc.,9(2) (1975), 518-522.

[9] A. Kilićkman and Z. Salleh, Pairwise weakly regular Lindelöf spaces, Abstarct and App. Anal., (2008), 1-13.

[10] A. Kilićkman and Z. Salleh, Pairwise almost Lindelöf bitopological spaces II, Malaysian Jour. Math.Sc., 1(2) (2007), 227-238.

[11] A. Kilićkman and Z. Salleh, Mappings and pairwise continuity on pairwise Lindelöf bitopological spaces, Albanian J. Math., 1(2) (2007), 115-120.

[12] B. Dvalishvili, Relative compactness, cotopology and some other notions from the bitopological point of view, Topology Appl., 140 (2004), 37-55.

[13] Z. Frolik, Generalizations of compact and Lindelöf spaces, Czech. Math. Jour., 9(8) (1959), 172-217.

[14] S. Willard and U.N.B. Dissanayake, The almost Lindelöf degree, Canad. Math. Bull., 27(4) (1984), 452-455.

[15] B.C. Tripathy and G.C.Roy, On mixed fuzzy topological spaces and countability, Soft Comput., 16(10) (2012), 1691-1695.

[16] B.C. Tripathy and G.C.Roy, Mixed ideal topological spaces, Appl. Math. Comput., 220 (2013), 602-607.

[17] S. Sampath Kumar, Semi-open sets, semi-continuity and semi-open mappings in fuzzy bitopological spaces, Fuzzy sets and systems, 64(3) (1994), (421-426).

[18] D. Sarkar, Fuzzy ideal theory,fuzzy local function and generated fuzzy topology,Fuzzy sets and systems, 87(1) (1997), 117-123.

[19] A.S. Salama, Bitopological rough approximations with medical applications, Jour.King Saud Univ.(Sc), 22 (2010), 117-183.

[20] H. Zhang, W.Shi and K. Liu, Fuzzy topology integrated support vector machine for remote sensed image classification, IEEE trans. on Geoscience and Remote sensing, 50(3) (2012), 850-862.

[21] W. Shi,K. Liu and C. Huang, Fuzzy topology based area object extraction method, IEEE trans. on Geoscience and Remote sensing, 48(1) (2012), 147-154.

[22] B.C. Tripathy and S. Acharjee, On (γ, δ)-Bitopological semi closed set via topological ideal, Proyecciones Jour. Math., 33(3) (2014), 245-257.

[23] S. Acharjee and B.C. Tripathy, Strategies in mixed budget-a bitopological approach, (submitted after minor revision to Comptes Rend. Math, September 2015).

[24] T. Birsan, Compacité dens les espaces bitopologiques, An.St.Univ. Iasi.s.I.a mathematica, 15 (1969), 317-328.

[25] B. Dvalishvili, Bitopologiacal spaces: Theory,relations with generalized algebraic structures and applications, 1st edition, Elsevier, North Holland(2005)

[26] J. Dontchev, On Hausdroff spaces via topological ideal and $\mathcal{I}$-irresolute functions.

Papers on Gen.Top.and Appl, Ann. New york Acad. Sci. ,767 (1995), 28-38.

[27] M.C. Datta, Projective bitopological spaces II, J. Aus. Math. Soc., 14 (1972), 119-128.

[28] J. Dontchev, M.Ganster and T.Noiri, Unified operation approach of generalized closed sets via topological ideals, Math. Japon, 49 (1999), 395-401.

[29] J.Dontchev and H. Maki, Groups of θ-generalized homeomorphisms and digital line, Topo. Appl., 95(2) (1999), 113-128.

[30] J. Slapál, A Jordan curve theorem with respect to a pretopology on Z^2, Int.J.Comp. Maths., 90(8) (2013), 1618-1628.

[31] Z. Salleh and A. Kilićkman, Generalizations of Lindelöf properties in bitopological spaces, Lap Lambert Publication, Germany (2012).

[32] S. Acharjee and B.C.Tripathy, Pairwise compactness on (γ, δ)-BSC sets of bitopological spaces,(under communication), 2014.

[33] A.S. Davis, Indexed systems of neighborhoods of general topological spaces, Amer. Math. Monthly, 68(9) (1961), 886-893.

[34] Y.K. Song and Y.Y. Zhang, Some remarks on almost Lindelöf spaces and weakly Lindelöf spaces, Math. Vesnik, 62(1) (2010), 77-83.

[35] W.J. Pervin, Connectedness in bitopological spaces, Indag. Math., 29 (1967), 369-372.

[36] M.C. Datta, Projective bitopological spaces, J.Aus. Math. Soc., 13 (1972), 327-334.

[37] S. Romaguera, On bitopological quasi-pseudometrization, J. Aus. Math. Soc (series A),36 (1984), 126-129.

[38] W. Wu and L.Zhou, On intuitionistic fuzzy topologies based on intuitionistic fuzzy reflexive and transitive relation, Soft Comput., 15 (2011), 1183-1194.

[39] A. Kilićkman and Z.Salleh, Pairwise almost Lindelöf bitopological space, Paper in Proc. 2nd IMT-GT regional conf. on Mathematics, Statistics and their applications, Universiti Scins Malayesia, Penag, Malayesia, June 2006.

[40] B.C.Tripathy, M.Sen and S. Nath, Lacunary-I-convergence in probabilistic n-normed space, Journal of Egyptian Math. Soc., 23(1) (2015), 9094

[41] E. Minguzzi, Normally preordered spaces and utilities, *Order*, 30 (2013), 137150.

[42] G. Bosi, http://www.deams.units.it/en/department/people/bosi-gianni/1167 (personal web page of University of Tristie)

[43] G. Bosi and G.B. Mehta, Existence of a semicontinuous or continuous utility function: a unified approach and an elementary proof, *Jour. Math.Eco*, 38(3) (2002), 311-328.

[44] S. Acharjee and K. Papadopoulos, On relation between countable space and p_1-Lindelöfness via nest in a bitopological space (submitted to BSPM, 2015)

[45] C. Good and K. Papadopoulos, A topological characterization of ordinals: van Dalen and Wattel revisited. Top. Appl., 159 (2012), 1565-1572.

[46] D. H. Pahk and B. D. Choi, Notes on pairwise compactness, Kyungpook Math. Jour., 11(1) (1971), 45-52.