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Some results on common best proximity point in fuzzy metric spaces*

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abstract: In this paper, we define the concepts of commute proximally, dominate proximally, weakly dominate proximally and common best proximity point in fuzzy metric space (abbreviated, FM-space). We prove some common best proximity point and common fixed point theorems for dominate proximally and weakly dominate proximally mappings in FM-space under certain conditions. Our results generalize many known results in metric space.

Key Words: Commute proximally, Dominate proximally, Weakly dominate proximally, Common best proximity point, Fuzzy metric space.

Contents

1. Introduction and preliminaries

The notion of fuzzy sets introduced by Zadeh [25], proved a turning point in the development of mathematics. This notion laid the foundation of fuzzy mathematics. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. Kramosil and Michalek [15] introduced the notion of a fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [12], modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [15].
Fixed point theory in fuzzy metric spaces was initiated by Grabiec [13]. Subrahmanyam [22] gave a generalization of Jungck [14] common fixed point theorem for commuting mappings in the setting of fuzzy metric spaces, whereas Vasuki [23] gave a fuzzy version of a result contained in Pant [18]. Thereafter, many authors established fuzzy versions of a host of classical metrical common fixed point theorems (e.g. [1,20,23]).

In nonlinear analysis, the theory of fixed points is an essential instrument to solve the equation $Tx = x$ for a self mapping T defined on a subset of an abstract space such as a metric space, a normed linear space or a topological vector space. If T is a non-self mapping from A to B, then the aforementioned equation does

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not necessarily admit a solution. However, in such circumstances, it may be speculated to determine an element x for which the error $d(x, Tx)$ is minimum, where d is the distance function, in which case x and Tx are in close proximity to each other. In fact, best approximation theorems and best proximity point theorems are applicable for solving such problems. In view of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T. Further, it is interesting to observe that best proximity theorems also emerge as a natural generalization of fixed point theorems. A best proximity point reduces to a fixed point if the mapping under consideration is a self mapping. Investigation of several variants of contractions for the existence of a best proximity point can be found in [2,3,4,5,7,8,9,10,17,19,21,24]. Eldred et al. [11] have established a best proximity point theorem for relatively non-expansive mappings. Further, Anuradha and Veeramani have focussed on best proximity point theorems for proximal pointwise contraction mappings [6].

In this paper, we establish some definitions and basic concepts of the common best proximity point in the framework of fuzzy metric spaces.

We first bring notation, definitions and known results, which are related to our work.

Definition 1.1. *A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:*

(i) $a * b = b * a$,

(ii) $a * (b * c) = (a * b) * c$,

(iii) $a * b \geq c * d, \qquad \text{whenever } \ a \geq c \ \text{ and } \ b \geq d$,

(iv) $a * 1 = a$,

for every $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a *_M b = \min\{a, b\}$ and $a *_P b = ab$.

Lemma 1.2. *If $*$ is a t-norm, then $a * a \geq a$, for all $a \in [0,1]$, if and only if $* = *_M$.*

Proof: For an arbitrary t-norm $*$ we get $* \leq *_M$. Let $a, b \in [0,1]$ and $a \leq b \leq 1$, so

$$a \leq a * a \leq a * b \leq a *_M b = a,$$

then $a * b = a *_M b$. □

Definition 1.3. *(George and Veeramani [12]) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space (abbreviated, FM-space) if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:*

(FM1) $M(x,y,t) > 0,\ M(x,y,0) = 0,$

(FM2) $M(x,y,t) = 1$ *iff* $x = y,$

(FM3) $M(x,y,t) = M(y,x,t),$

(FM4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s),$

(FM5) $M(x,y,.) : (0,\infty) \to [0,1]$ *is continuous.*

for any $x,y,z \in X$ *and* $t,s > 0$.

Example 1.4. *[12] Let* (X,d) *be a metric space. Define*

$$M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}, \qquad k,m,n \in \mathbb{R}^+,$$

then $(X,M,*_M)$ *is a fuzzy metric space.*

Lemma 1.5. *[16] Let* $(X,M,*)$ *be a FM-space. If there exists* $q \in (0,1)$ *such that for all* $t \geq 0$, $M(x,y,qt) \geq M(z,w,t) \geq M(x,y,t)$ *where* $x,y,z,w \in X$, *then* $x = y$ *and* $z = w$.

Proposition 1.6. *[13] Let* $(X,M,*)$ *be a fuzzy metric space. Then for all* $x,y \in X$, $M(x,y,\cdot)$ *is non-decreasing.*

Definition 1.7. *Let* $(X,M,*)$ *be a FM-space. An open ball with center* x *and radius* λ $(0 < \lambda < 1)$ *in* X *is the set* $U_x(\varepsilon,\lambda) = \{y \in X : M(x,y,\epsilon) > 1-\lambda\}$, *for all* $\varepsilon > 0$. *It is easy to see that* $\mathfrak{U} = \{U_x(\varepsilon,\lambda) :\ x \in X,\ \varepsilon > 0,\ \lambda \in (0,1)\}$ *determines a Hausdorff topology for* X *[12].*

Definition 1.8. *Let* $(X,M,*)$ *be a FM-space. For each* $\lambda \in (0,1)$, *define the function*

$$d_\lambda : X \times X \to \mathbb{R},$$

by

$$d_\lambda(x,y) = \sup_{t\in\mathbb{R}}\{t \in \mathbb{R}\ :\ M(x,y,t) \leq 1-\lambda\}.$$

Since $M(x,y,t)$ *is non-decreasing, continuous with* $\inf_{t\in\mathbb{R}} M(x,y,t) = 0$ *and* $\sup_{t\in\mathbb{R}}$ $M(x,y,t) = 1$, *then* $d_\lambda(x,y)$ *is finite.*

Proposition 1.9. *Let* $(X,M,*_M)$ *be a FM-space. The function* d_λ *is a pseudo-metric for each* $\lambda \in (0,1)$. *Furthermore* $d_\lambda(x,y) = 0$ *for all* $\lambda \in (0,1)$ *if and only if* $x = y$.

Proof: *Since* $M(x,y,0) = 0$ *and* $M(x,y,.)$ *is non-decreasing, it is clear that* $d_\lambda(x,y) \geq 0$. *Obviously* $d_\lambda(x,y) = d_\lambda(y,x)$, *since* $M(x,y,t) = M(y,x,t)$. *Furthermore* $d_\lambda(x,x) = 0$ *from* $(FM2)$. *It remains to verify the triangle inequality. To this end, assume towards a contradiction that*

$$d_\lambda(x,y) + d_\lambda(y,z) < d_\lambda(x,z),$$

for some x, y and z in X. We can choose $t > d_\lambda(x, y)$ and $s > d_\lambda(y, z)$ so that

$$t + s < d_\lambda(x, z).$$

Hence

$$M(x, y, t) = 1 - \lambda_1 > 1 - \lambda,$$

and

$$M(y, z, s) = 1 - \lambda_2 > 1 - \lambda.$$

Choose $\tilde{\lambda} = \max\{\lambda_1, \lambda_2\}$, then

$$1 - \tilde{\lambda} \leq 1 - \tilde{\lambda} *_M 1 - \tilde{\lambda} \leq M(x, y, t) *_M M(y, z, s) \leq M(x, z, t + s),$$

which is a contradiction, since $1 - \tilde{\lambda} > 1 - \lambda$. □

Proposition 1.10. *Let $(X, M, *_M)$ be a FM-space. For each $(x, y) \in X \times X$, $d_\lambda(x, y)$ is a left continuous non-increasing function of λ, such that $M(x, y, d_\lambda(x, y)) \leq 1 - \lambda$.*

Proof: *Fix (x, y). If $\lambda_1, \lambda_2 \in (0, 1), \lambda_1 \leq \lambda_2$ and $t_0 \in \{t \in \mathbb{R} : M(x, y, t) \leq 1 - \lambda_2\}$ then $M(x, y, t_0) \leq 1 - \lambda_2 \leq 1 - \lambda_1$ so $t_0 \in \{t \in \mathbb{R} : M(x, y, t) \leq 1 - \lambda_1\}$, therefore $d_\lambda(x, y)$ is non-increasing. Let $\{t_n\}$ be a sequence that $t_n \in \{t \in \mathbb{R} : M(x, y, t) \leq 1 - \lambda\}$ and $t_n \to d_\lambda(x, y)$. Since $M(x, y, .)$ is continuous, we have $M(x, y, t_n) \to M(x, y, d_\lambda(x, y))$. Because $M(x, y, t_n) \leq 1 - \lambda$, then $M(x, y, d_\lambda(x, y)) \leq 1 - \lambda$. To see that $d_\lambda(x, y)$ is left continuous, let, $\lambda \in (0, 1)$ and let $\{\lambda_i\}$ be a non-decreasing sequence converging to λ, Then $\{d_{\lambda_i}(x, y)\}$ is a non-increasing sequence bounded below by zero and hence converges to a number p; since $d_{\lambda_i}(x, y) \geq d_\lambda(x, y)$, $p \geq d_\lambda(x, y)$. We complete the proof by showing that $p = d_\lambda(x, y)$. If $p > d_\lambda(x, y)$, then from the definition of $d_\lambda(x, y)$, $M(x, y, p) > 1 - \lambda$. Choose an n large enough so that $M(x, y, p) > 1 - \lambda_n \geq 1 - \lambda$. However, since $p \leq d_{\lambda_n}(x, y)$, $M(x, y, p) \leq M(x, y, d_{\lambda_n}(x, y)) \leq 1 - \lambda_n$ and we have a contradiction.* □

The family $\{d_\lambda : \lambda \in (0, 1)\}$ of all such pseudometrics will be called the family of pseudometrics associated with the fuzzy metric M.

Lemma 1.11. *$M(x, y, \epsilon) > 1 - \lambda$, if and only if $d_\lambda(x, y) < \epsilon$.*

Proof: Let $d_\lambda(x, y) < \epsilon$, then from the definition of $d_\lambda(x, y)$ we get $M(x, y, \epsilon) > 1 - \lambda$. If $M(x, y, \epsilon) > 1 - \lambda$, then $d_\lambda(x, y) < \epsilon$, since if $d_\lambda(x, y) \geq \epsilon$, from the fact that $M(x, y, d_\lambda(x, y)) \leq 1 - \lambda$ and $M(x, y, .)$ is non-decreasing we have

$$1 - \lambda \geq M(x, y, d_\lambda(x, y)) \geq M(x, y, \epsilon) > 1 - \lambda,$$

and we have a contradiction. Therefore $d_\lambda(x, y) < \epsilon$. □

From this lemma it is clear that the neighborhood $U_x(\varepsilon, \lambda)$ of x in the M topology is the d_λ-spherical neighborhood of x of radius ϵ. We thus have immediately the following basic result.

Theorem 1.12. *Let $(X, M, *_M)$ be a FM-space. The topology on X generated by the family of pseudometrics associated with the fuzzy metric M is the same as the topology induced by M.*

Definition 1.13. *A sequence (x_n) in a FM-space $(X, M, *)$ is said to be convergent to a point $x \in X$ if and only if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $M(x_n, x, \varepsilon) > 1 - \lambda$ for all $n \geq n_0(\varepsilon, \lambda)$ or for every $\lambda \in (0,1)$, $d_\lambda(x_n, x) \to 0$ or $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, in this case we say that limit of the sequence (x_n) is x.*

Definition 1.14. *A sequence (x_n) in a FM-space $(X, M, *)$ is said to be Cauchy sequence if and only if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $M(x_{n+p}, x_n, \varepsilon) > 1 - \lambda$ for all $n \geq n_0(\varepsilon, \lambda)$ and every $p \in \mathbb{N}$ or for every $\lambda \in (0,1)$, $d_\lambda(x_{n+p}, x_n) \to 0$ for all $p \in \mathbb{N}$, or $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and $p \in \mathbb{N}$.*

Also, a FM-space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X, is convergent.

Proposition 1.15. *The limit of a convergent sequence in a FM-space $(X, M, *)$ is unique.*

Proof: It is obvious. □

Proposition 1.16. *Let $(X, M, *)$ be a FM-space and (x_n) be a sequence in X. If sequence (x_n) converges to $x \in X$, then $M(x, x, t) = 1$ for all $t > 0$.*

Proof: It is obvious. □

Definition 1.17. *Let A and B be nonempty subsets of a FM-space $(X, M, *)$. Let*

$$M(A, B, t) = \sup_{x \in A, y \in B} M(x, y, t), \quad t \geq 0,$$

which is said to be the fuzzy distance of A, B.

Definition 1.18. *Let A and B be nonempty subsets of a FM-space $(X, M, *)$. We define the following sets:*

$$A_0 = \{x \in A : \ \exists\, y \in B \text{ s.t. } \forall\, t \geq 0,\ M(x, y, t) = M(A, B, t)\},$$
$$B_0 = \{y \in B : \ \exists\, x \in A \text{ s.t. } \forall\, t \geq 0,\ M(x, y, t) = M(A, B, t)\}.$$

Definition 1.19. *Let A and B be nonempty subsets of a FM-space $(X, M, *)$ and $T, S : A \to B$ be two mappings. We say that an element $x \in A$ is a common best proximity point of the mappings S and T, if*

$$M(x, Sx, t) = M(A, B, t) = M(x, Tx, t),$$

for all $t \geq 0$.

It is clear that the notion of a common fixed point coincided with the notion of a common best proximity point when the underlying mapping is a self mapping. Also, it can be noticed that common best proximity point is an element at which both function $x \to M(x, Sx, t)$ and $x \to M(x, Tx, t)$ for all $t \geq 0$, attain global supremum.

Definition 1.20. *Let A and B be nonempty subsets of a FM-space $(X, M, *)$ and $T, S : A \to B$ be two mappings. We say that T, S are commute proximally if*

$$M(u, Sx, t) = M(A, B, t) = M(v, Tx, t),$$

for all $t \geq 0$, then $Sv = Tu$, where $x, y, u, v \in A$.

Example 1.21. *Let $(X, M, *)$ be a FM-space and $T, S : X \to X$ be two mappings such that $TS = ST$. Clearly $M(X, X, t) = 1$ for all $t \geq 0$ and so if*

$$M(u, Sx, t) = M(X, X, t) = M(v, Tx, t), \qquad (x, u, v \in X, \ t \geq 0),$$

then by the hypothesis, $u = Sx$ and $v = Tx$. Therefore $Sv = STx = TSx = Tu$, hence T, S are commute proximally.

Definition 1.22. *Let A and B be nonempty subsets of a FM-space $(X, M, *)$ and $T, S : A \to B$ be two mappings. We say that the mapping T is to dominate the mapping S proximally if*

$$M(u_1, Sx_1, t) = M(u_2, Sx_2, t) = M(A, B, t) = M(v_1, Tx_1, t) = M(v_2, Tx_2, t),$$

for all $t \geq 0$, then there exists a $\alpha \in (0, 1)$ such that for all $t \geq 0$,

$$M(u_1, u_2, \alpha t) \geq M(v_1, v_2, t),$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$.

Example 1.23. *Let $X = [-2, 2]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$, it is easy to see that $(X, M, *_M)$ is a FM-space. Define non-self mappings $S : X \to [-1, 1]$ and $T : X \to [-1, 1]$ as*

$$Sx = \frac{1}{8}x, \qquad Tx = -\frac{1}{2}x, \qquad (x \in X).$$

It is easy to see that $M(X, [-1, 1], t) = 1$. If for all $t \geq 0$

$$M(u_1, Sx_1, t) = M(u_2, Sx_2, t) = M(X, [-1, 1], t) = 1 = M(v_1, Tx_1, t) = M(v_2, Tx_2, t),$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$. Then $u_i = Sx_i$ and $v_i = Tx_i$ $(i = 1, 2)$ and so for $\alpha = 1/4$ we have $M(u_1, u_2, \alpha t) = M(v_1, v_2, t)$, hence T dominates S proximally for $\alpha = 1/4$.

Definition 1.24. *Let A and B be nonempty subsets of a FM-space $(X, M, *)$ and $T, S : A \to B$ be two mappings. We say that the mapping T is to weakly dominate the mapping S proximally if*

$$M(u_1, Sx_1, t) = M(u_2, Sx_2, t) = M(A, B, t) = M(v_1, Tx_1, t) = M(v_2, Tx_2, t),$$

for all $t \geq 0$, then there exists a $\alpha \in (0, 1)$ such that for all $t \geq 0$,

$$M(u_1, u_2, \alpha t) \geq \min\{M(v_1, v_2, t), M(v_1, u_1, t), M(v_1, u_2, t), M(v_2, u_1, t)\},$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$.

Obviously, if T dominates S proximally, then T weakly dominates S proximally. The following example shows that the converse is not true, in general.

Example 1.25. *Let $X = [0, 1] \times [0, 1]$ and $d : X \times X \to [0, \infty)$ be given by $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and define*

$$M((x_1, x_2), (y_1, y_2), t) = \frac{t}{t + d((x_1, x_2), (y_1, y_2))}.$$

Clearly, $(X, M, *_M)$ is a FM-space. Let $A = \{(0, x) : x \in [0, 1]\}$, $B = \{(1, x) : x \in [0, 1]\}$ and $S, T : A \to B$ be defined as $T(0, x) = (1, x)$ for all $x \in [0, 1]$ and

$$S(0, x) = \begin{cases} (1, \frac{1}{3})\ ; & x < 1, \\ (1, \frac{1}{2})\ ; & x = 1, \end{cases} \qquad (\forall\ x \in [0, 1]).$$

It is easy to see that $M(A, B, t) = \frac{t}{t+1}$. We show that T does not dominate S proximally. To show the claim, suppose that there exists $\alpha \in (0, 1)$ such that for all $t \geq 0$,

$$M(U_1, U_2, \alpha t) \geq M(V_1, V_2, t),$$

where $U_1 = (0, u_1)$, $U_2 = (0, u_2)$, $V_1 = (0, v_1)$, $V_2 = (0, v_2)$, $X_1 = (0, x_1)$ and $X_2 = (0, x_2)$ be elements in A satisfying

$$\begin{aligned} M(U_1, SX_1, t) =& M(U_2, SX_2, t) = M(A, B, t) = M(V_1, TX_1, t) \\ =& M(V_2, TX_2, t), \end{aligned} \tag{1.1}$$

for all $t \geq 0$. Let $U_1 = (0, \frac{1}{3})$, $U_2 = (0, \frac{1}{2})$, $V_1 = (0, x)$, $V_2 = (0, 1)$, $X_1 = (0, x)$ and $X_2 = (0, 1)$ where $0 \leq x < 1$. Then U_1, U_2, V_1, V_2, X_1 and X_2 satisfy (1.1) and then, we have

$$M(U_1, U_2, \alpha t) = \frac{t}{t + \frac{1}{\alpha 6}} \geq M(V_1, V_2, t) = \frac{t}{t + (1 - x)}, \qquad (\forall\ x \in [0, 1)),$$

a contradiction. Then we show that T weakly dominates S proximally for $\alpha = 1/4$, to verify this, let $x_1, x_2, u_1, u_2, v_1, v_2 \in [0, 1]$ and

$$\begin{aligned} M((0, u_1), S(0, x_1), t) =& M((0, u_2), S(0, x_2), t) = M(A, B, t) \\ =& M((0, v_1), T(0, x_1), t) = M((0, v_2), T(0, x_2), t). \end{aligned}$$

Now we need to consider several possible cases.

Case 1. Let $x_1, x_2 \in [0,1)$. *Then* $u_1 = u_2 = \frac{1}{3}$ *and*

$$M((0,u_1),(0,u_2),\frac{1}{4}t) = 1 \geq \min\{M((0,v_1),(0,v_2),t), M((0,v_1),(0,u_1),t), M((0,v_1),(0,u_2),t), M((0,v_2),(0,u_1),t)\}.$$

Case 2. Let $x_1 = 1 = x_2$. *Then* $u_1 = u_2 = \frac{1}{2}$ *and*

$$M((0,u_1),(0,u_2),\frac{1}{4}t) = 1 \geq \min\{M((0,v_1),(0,v_2),t), M((0,v_1),(0,u_1),t), M((0,v_1),(0,u_2),t), M((0,v_2),(0,u_1),t)\}.$$

Case 3. Let $x_1 \in [0,1)$ *and* $x_2 = 1$. *Then* $u_1 = \frac{1}{3}$, $u_2 = \frac{1}{2}$, $v_2 = 1$ *and* $M((0,u_1),(0,u_2),\frac{1}{4}t) = \frac{t}{t+\frac{2}{3}} = M((0,v_2),(0,u_1),t)$, *so*

$$M((0,u_1),(0,u_2),\frac{1}{4}t) \geq \min\{M((0,v_1),(0,v_2),t), M((0,v_1),(0,u_1),t), M((0,v_1),(0,u_2),t), M((0,v_2),(0,u_1),t)\}.$$

Case 4. Let $x_1 = 1$ *and* $x_2 \in [0,1)$. *Then* $u_1 = \frac{1}{2}$, $u_2 = \frac{1}{3}$, $v_1 = 1$ *and* $M((0,u_1),(0,u_2),\frac{1}{4}t) = \frac{t}{t+\frac{2}{3}} = M((0,v_1),(0,u_2),t)$, *so*

$$M((0,u_1),(0,u_2),\frac{1}{4}t) \geq \min\{M((0,v_1),(0,v_2),t), M((0,v_1),(0,u_1),t), M((0,v_1),(0,u_2),t), M((0,v_2),(0,u_1),t)\}.$$

In this article, we introduce two new classes of mappings, called dominate proximally and weakly dominate proximally in fuzzy metric space. We provide sufficient conditions for the existence and uniqueness of common best proximity points and common fixed points for weakly dominate proximally mappings in FM-space. Our results generalize many known results in metric space. Examples are given to support our main results.

2. Main Results

Now we state and prove our main theorem about existence and uniqueness of a common best proximity point for dominate proximally and weakly dominate proximally mappings in FM-space under certain conditions.

Theorem 2.1. *Let* A *and* B *be nonempty subsets of a complete fuzzy metric space* $(X, M, *_M)$ *such that* A_0 *and* B_0 *are nonempty and* A_0 *is closed. If the mappings* $T, S : A \to B$ *satisfy the following conditions:*

(i) T *weakly dominates* S *proximally,*

(ii) *S and T commute proximally,*

(iii) *S and T are continuous,*

(iv) *$S(A_0) \subset B_0$ and $S(A_0) \subset T(A_0)$.*

Then, there exists a unique element $x \in A$ such that

$$M(x, Sx, t) = M(A, B, t) = M(x, Tx, t),$$

for all $t \geq 0$.

Proof: First, suppose that there exists an element $u \in A_0$ such that $Su = Tu$. By the hypothesis, there exists an element $x \in A_0$ such that

$$M(x, Su, t) = M(A, B, t) = M(x, Tu, t), \qquad \forall t \geq 0, \tag{2.1}$$

so, $Sx = Tx$. Once again, by the hypothesis, there exists an element $v \in A_0$ such that

$$M(v, Sx, t) = M(A, B, t) = M(v, Tx, t), \qquad \forall t \geq 0. \tag{2.2}$$

Since T weakly dominates S, then from (2.1) and (2.2), for all $t \geq 0$, we get

$$M(x, v, \alpha t) \geq \min\{M(x, v, t), M(x, x, t), M(x, v, t), M(v, v, t)\} = M(x, v, t),$$

which implies $x = v$, by Lemma 1.5. Therefore, it follows that

$$M(x, Sx, t) = M(v, Sx, t) = M(A, B, t) = M(v, Tx, t) = M(x, Tx, t), \qquad \forall t \geq 0.$$

So, x is a common best proximity point of the mappings S and T. If x' is another common best proximity point of the mappings S and T, in other words

$$M(x', Sx', t) = M(A, B, t) = M(x', Tx', t), \qquad \forall t \geq 0,$$

then by using the same argument as above we can show that $x = x'$.

Second, we claim that there exists an element $u \in A_0$ such that $Su = Tu$. To support the claim, let x_0 be a fixed point element in A_0. By the hypothesis, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. This process can be carried on. Having chosen $x_n \in A_0$, by the hypothesis, we can find an element $x_{n+1} \in A_0$ such that $Sx_n = Tx_{n+1}$. By the condition (iv), there exists an element $u_n \in A_0$ such that $M(u_n, Sx_n, t) = M(A, B, t)$ for all $t \geq 0$ and $n \in \mathbb{N}$.

Further, it follows from the choice x_n and u_n that

$$M(u_{n+1}, Sx_{n+1}, t) = M(A, B, t) = M(u_n, Tx_{n+1}, t), \qquad \forall\ t \geq 0.$$

So, by the condition (i), we have

$$M(u_n, u_{n+1}, \alpha t) \geq \min\{M(u_{n-1}, u_n, t), M(u_{n-1}, u_n, t), M(u_{n-1}, u_{n+1}, t), \\ M(u_n, u_n, t)\},$$

for all $t \geq 0$. Thus, we have

$$M(u_n, u_{n+1}, \alpha t) \geq \min\{M(u_{n-1}, u_n, t), M(u_{n-1}, u_{n+1}, t)\}, \tag{2.3}$$

for all $t \geq 0$. In the following we show by induction that for each $n \in \mathbb{N}$ and for each $t \geq 0$, there exists $1 \leq m \leq n+1$ such that

$$M(u_n, u_{n+1}, t) \geq M(u_0, u_m, \alpha^{-n} t). \tag{2.4}$$

If $n = 1$, then by (2.3), we have

$$M(u_1, u_2, \alpha t) \geq \min\{M(u_0, u_1, t), M(u_0, u_2, t)\} = M(u_0, u_m, t),$$

for some $1 \leq m \leq 2$ and for all $t \geq 0$. Thus (2.4) holds for $n = 1$. Assume towards a contradiction that (2.4) is not true and take $n_0 > 1$, be the least natural number such that (2.4) does not hold. So there exists $t_0 > 0$, such that for all $1 \leq m \leq n_0 + 1$, we have

$$M(u_{n_0}, u_{n_0+1}, t_0) < M(u_0, u_m, \alpha^{-n_0} t_0). \tag{2.5}$$

If $\min\{M(u_{n_0-1}, u_{n_0}, \alpha^{-1} t_0),\ M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1} t_0)\} = M(u_{n_0-1}, u_{n_0}, \alpha^{-1} t_0)$, then by the hypothesis we have

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0-1}, u_{n_0}, \alpha^{-1} t_0) \geq M(u_0, u_m, \alpha^{-n_0} t_0),$$

for some $1 \leq m \leq n_0$, a contradiction. Thus

$$\min\{M(u_{n_0-1}, u_{n_0}, \alpha^{-1} t_0), M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1} t_0)\} = M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1} t_0),$$

and form (2.3), we have

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1} t_0). \tag{2.6}$$

By the condition (i), we get

$$\begin{aligned} M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1} t) \geq \min\{&M(u_{n_0-2}, u_{n_0}, \alpha^{-2} t), M(u_{n_0-2}, u_{n_0-1}, \alpha^{-2} t), \\ &M(u_{n_0-2}, u_{n_0+1}, \alpha^{-2} t), M(u_{n_0}, u_{n_0-1}, \alpha^{-2} t)\}, \end{aligned}$$

for all $t \geq 0$. If

$$\begin{aligned} \min\{&M(u_{n_0-2}, u_{n_0}, \alpha^{-2} t_0), M(u_{n_0-2}, u_{n_0-1}, \alpha^{-2} t_0), M(u_{n_0-2}, u_{n_0+1}, \alpha^{-2} t_0), \\ &M(u_{n_0}, u_{n_0-1}, \alpha^{-2} t_0)\} = M(u_{n_0}, u_{n_0-1}, \alpha^{-2} t_0), \end{aligned}$$

then from (2.6) and the above, we have

$$\begin{aligned} M(u_{n_0}, u_{n_0+1}, t_0) \geq& M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1} t_0) \geq M(u_{n_0}, u_{n_0-1}, \alpha^{-2} t_0) \\ =& M(u_{n_0-1}, u_{n_0}, \alpha^{-2} t_0) \geq M(u_0, u_m, \alpha^{-(n_0+1)} t_0) \\ \geq& M(u_0, u_m, \alpha^{-n_0} t_0), \end{aligned}$$

for some $1 \leq m \leq n_0 \leq n_0+1$, a contradiction. Therefore

$$M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1}t_0) \geq \min\{M(u_{n_0-2}, u_{n_0}, \alpha^{-2}t_0), M(u_{n_0-2}, u_{n_0-1}, \alpha^{-2}t_0), M(u_{n_0-2}, u_{n_0+1}, \alpha^{-2}t_0)\},$$

from (2.6) and the above, we get

$$\begin{aligned} M(u_{n_0}, u_{n_0+1}, t_0) \geq & M(u_{n_0-1}, u_{n_0+1}, \alpha^{-1}t_0) \\ \geq & \min\{M(u_{n_0-2}, u_{n_0}, \alpha^{-2}t_0), M(u_{n_0-2}, u_{n_0-1}, \alpha^{-2}t_0), \\ & M(u_{n_0-2}, u_{n_0+1}, \alpha^{-2}t_0)\} \\ = & M(u_{n_0-2}, u_m, \alpha^{-2}t_0), \end{aligned} \tag{2.7}$$

for some $1 \leq m \in \{n_0-1, n_0, n_0+1\} \leq n_0+1$. Again by the condition (i), we have

$$M(u_{n_0-2}, u_m, \alpha^{-2}t) \geq \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t), M(u_{n_0-3}, u_m, \alpha^{-3}t), M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t)\},$$

for all $t \geq 0$. If $m = n_0 - 1$, then

$$\begin{aligned} & \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0), M(u_{n_0-3}, u_m, \alpha^{-3}t_0), \\ & \quad M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t_0)\} \\ & = \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0), M(u_{n_0-3}, u_m, \alpha^{-3}t_0)\}. \end{aligned}$$

If $m = n_0$, $m \neq n_0 - 1$ and

$$\begin{aligned} & \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0), M(u_{n_0-3}, u_m, \alpha^{-3}t_0), \\ & \quad M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t_0)\} \\ & = M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t_0) = M(u_{n_0-2}, u_{n_0-1}, \alpha^{-3}t_0), \end{aligned}$$

then from (2.7) and the above, we have

$$\begin{aligned} M(u_{n_0}, u_{n_0+1}, t_0) \geq & M(u_{n_0-2}, u_m, \alpha^{-2}t_0) \geq M(u_{n_0-2}, u_{n_0-1}, \alpha^{-3}t_0) \\ \geq & M(u_0, u_{m'}, \alpha^{-(n_0-2)}(\alpha^{-3}t_0)) \\ \geq & M(u_0, u_{m'}, \alpha^{-n_0}t_0), \end{aligned}$$

for some $1 \leq m' \leq n_0+1$, a contradiction. Therefore

$$\begin{aligned} & \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0), M(u_{n_0-3}, u_m, \alpha^{-3}t_0), \\ & \quad M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t_0)\} \\ & = \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0), M(u_{n_0-3}, u_m, \alpha^{-3}t_0)\}. \end{aligned}$$

If $m = n_0+1$, $m \neq n_0-1$ and $m \neq n_0$, then

$$M(u_{n_0-2}, u_{n_0}, \alpha^{-2}t_0) \geq M(u_{n_0-2}, u_m, \alpha^{-2}t_0).$$

Now if

$$\begin{aligned}\min\{&M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0)M(u_{n_0-3}, u_m, \alpha^{-3}t_0),\\ &M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t_0)\}\\ =&M(u_{n_0-2}, u_{m-1}, \alpha^{-3}t_0) = M(u_{n_0-2}, u_{n_0}, \alpha^{-3}t_0),\end{aligned}$$

then from the above, we have

$$\begin{aligned}M(u_{n_0-2}, u_{n_0}, \alpha^{-2}t_0) \geq& M(u_{n_0-2}, u_m, \alpha^{-2}t_0) \geq M(u_{n_0-2}, u_{n_0}, \alpha^{-3}t_0)\\ \geq& M(u_{n_0-2}, u_{n_0}, \alpha^{-2}t_0),\end{aligned}$$

a contradiction, since if the above inequality becomes equality, then we can assume that $m = n_0$. Therefore from the above, we get

$$\begin{aligned}M(u_{n_0}, u_{n_0+1}, t_0) \geq& M(u_{n_0-2}, u_m, \alpha^{-2}t_0)\\ \geq& \min\{M(u_{n_0-3}, u_{m-1}, \alpha^{-3}t_0), M(u_{n_0-3}, u_{n_0-2}, \alpha^{-3}t_0),\\ &M(u_{n_0-3}, u_m, \alpha^{-3}t_0)\}\\ =& M(u_{n_0-3}, u_{m'}, \alpha^{-3}t_0),\end{aligned}$$

for some $1 \leq m' \leq n_0 + 1$. Therefore by continuing this process, we see that for each $1 \leq k \leq n_0$, there exists $1 \leq m \leq n_0 + 1$ such that

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0-k}, u_m, \alpha^{-k}t_0). \tag{2.8}$$

If $k = n_0$ in (2.8), then this is a contradiction by (2.5). So (2.4) holds for all $n \in \mathbb{N}$. Suppose that $(d_\lambda)_{\lambda\in(0,1)}$ is the family of pseudometrics in Definition 1.8, by Theorem 1.12 the family of pseudometrics $(d_\lambda)_{\lambda\in(0,1)}$ generates the topology induced by M on X. We obtain by (2.4) that for u_n and every $\lambda \in (0, 1)$,

$$d_\lambda(u_n, u_{n+1}) \leq \alpha^n \max_{1\leq m\leq n+1}\{d_\lambda(u_0, u_m)\}. \tag{2.9}$$

Indeed, if $\max_{1\leq m\leq n+1}\{d_\lambda(u_0, u_m)\} < r$, then $M(u_0, u_m, r) > 1 - \lambda$, for all $m \in \{1, \cdots, n + 1\}$ and (2.4) implies $M(u_n, u_{n+1}, \alpha^n r) > 1 - \lambda$, which means that $d_\lambda(u_n, u_{n+1}) < \alpha^n r$. From (2.9) we get

$$d_\lambda(u_n, u_{n+1}) \leq \alpha^n \left(d_\lambda(u_0, u_1) + d_\lambda(u_1, u_2) + \cdots + d_\lambda(u_n, u_{n+1})\right).$$

Let $a_n = d_\lambda(u_{n-1}, u_n)$ and let $s_n = \sum_{i=1}^n a_n$. So we have

$$a_n \leq \alpha^{n-1}s_n. \tag{2.10}$$

We now show that $\sum_{n=1}^\infty a_n = \lim_{n\to\infty} s_n < \infty$. Assume towards a contradiction that $\lim_{n\to\infty} s_n = \infty$. By the hypothesis we can assume without loss of generality that $s_n \neq 0$ for all $n \in \mathbb{N}$. So by the hypothesis the series

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n} < \infty, \tag{2.11}$$

is convergent. From (2.11), we get there exists $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$,

$$1 - \frac{s_n}{s_{n+m}} = \frac{s_{n+m} - s_n}{s_{n+m}} = \frac{a_{n+1} + \cdots + a_{n+m}}{s_{n+m}} \leq \sum_{j=1}^{m} \frac{a_{n+j}}{s_{n+j}} < \frac{1}{2},$$

taking the limit as $m \to \infty$, we get $1 \leq \frac{1}{2}$, a contradiction. Therefore for every $\lambda \in (0,1)$ and $p \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d_\lambda(u_n, u_{n+p}) = 0.$$

Then (u_n) is a Cauchy sequence and by the hypothesis there exists some element $u \in A_0$ such that $\lim_{n\to\infty} u_n = u$. By the hypothesis it is easy to see that $Su_n = Tu_{n+1}$, for all $n \in \mathbb{N}$, now by the continuity of the mappings S and T we get $Su = Tu$, so the desired result is obtained. □

The following corollary, is immediate.

Corollary 2.2. *Let A and B be nonempty subsets of a complete FM-space $(X, M, *_M)$ such that A_0 and B_0 are nonempty and A_0 is closed. If the mappings $T, S : A \to B$ satisfy the following conditions:*

(i) T dominates S proximally,

(ii) S and T commute proximally,

(iii) S and T are continuous,

(iv) $S(A_0) \subset B_0$ and $S(A_0) \subset T(A_0)$.

Then, there exists a unique element $x \in A$ such that

$$M(x, Sx, t) = M(A, B, t) = M(x, Tx, t),$$

for all $t \geq 0$.

Corollary 2.3. *Let $(X, M, *_M)$ be a complete FM-space, S be a self mapping on X and T be a continuous self mapping on X such that commutes with S. If $S(X) \subseteq T(X)$ and there exists a constant $\alpha \in (0,1)$ such that*

$$M(Sx, Sy, \alpha t) \geq \min\{M(Tx, Ty, t), M(Tx, Sx, t), M(Tx, Sy, t), M(Ty, Sx, t)\}, \tag{2.12}$$

for every $x, y \in X$ and $t \geq 0$. Then S and T have a unique common fixed point.

Proof: We used the assumption of continuity of S in Theorem 2.1 to show that

$$\lim_{n\to\infty} u_n = u, \qquad Tu_n = Su_{n-1}, \qquad \& \qquad \lim_{n\to\infty} Tu_n = Tu, \qquad \forall n \in \mathbb{N},$$

$$\Rightarrow \lim_{n\to\infty} Su_{n-1} = Su.$$

By (2.12), for all $t \geq 0$ we have

$$M(Su_n, Su, \alpha t) \geq \min\{M(Tu_n, Tu, t), M(Tu_n, Su_n, t), M(Tu_n, Su, t), M(Tu, Su_n, t)\}.$$

Since $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_{n+1} = Tu$, then $M(Tu, Su, \alpha t) \geq M(Tu, Su, t)$ for all $t \geq 0$. By Lemma 1.5, we get $Tu = Su$ or

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_{n+1} = Tu = Su.$$

Also, it is easy to see that $M(X, X, t) = 1$ for all $t \geq 0$, $X_0 = X$, S and T satisfy the condition (i) and (ii) of Theorem 2.1. So there exists $x \in X$ such that

$$M(x, Sx, t) = M(x, Tx, t) = M(X, X, t) = 1,$$

for all $t \geq 0$ or $Sx = x = Tx$, as required. □

If we take T to be the identity mapping in the above corollary, we get the following:

Corollary 2.4. *Let $(X, M, *_M)$ be a complete FM-space and S be a self mapping on X. If there exists a constant $\alpha \in (0, 1)$ such that*

$$M(Sx, Sy, \alpha t) \geq \min\{M(x, y, t), M(x, Sx, t), M(x, Sy, t), M(y, Sx, t)\}, \quad (2.13)$$

for every $x, y \in X$ and $t \geq 0$. Then S has a unique fixed point.

In what follows, we present some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results proved in this paper.

Example 2.5. *Let $X = [0, 1] \times [0, 1]$ and $d : X \times X \to [0, \infty)$ be given by $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and*

$$M((x_1, x_2), (y_1, y_2), t) = \frac{t}{t + d((x_1, x_2), (y_1, y_2))}.$$

Clearly, $(X, M, *_M)$ is a complete fuzzy metric space. Let $A = \{(0, x) : x \in [0, 1]\}$, $B = \{(1, x) : x \in [0, 1]\}$ and $S, T : A \to B$ be defined as $T(0, x) = (1, x)$ for all $x \in [0, 1]$ and

$$S(0, x) = \begin{cases} (1, \frac{1}{3}) \ ; & x < 1, \\ (1, \frac{1}{2}) \ ; & x = 1, \end{cases} \quad (\forall\ x \in [0, 1]).$$

It is easy to see that $A_0 = A$, $B_0 = B$, S, T commute proximally and by Example 1.25, T weakly dominates S proximally for $\alpha = 1/4$. Therefore, all the hypothesis of Theorem 2.1 are satisfied, then there exist unique element $x \in X$ such that

$$M(x, Sx, t) = M(A, B, t) = \frac{t}{t + 1} = M(x, Tx, t),$$

for all $t \geq 0$.

Example 2.6. *Let* $X = [0,1] \times [0,1]$ *and* $M(x,y,t) = \frac{t}{t+d(x,y)}$ *for all* $x, y \in X$, *where* $d(x,y)$ *is the Euclidean metric. Then* $(X, M, *_M)$ *is a complete FM-space. Suppose that*

$$A = \{(0,x) : 0 \leq x \leq 1\}, \qquad B = \{(1,y) : 0 \leq y \leq 1\}.$$

Then $M(A,B,t) = \frac{t}{t+1}$, $A_0 = A$ *and* $B_0 = B$. *Let* S *and* T *be define as* $T(0,y) = (1, \frac{y}{4})$ *and* $S(0,y) = (1, \frac{y}{32})$. *Thus all of the assumptions of Corollary 2.2 are satisfied, so there exist unique element* $x \in X$ *such that*

$$M(x, Sx, t) = M(A,B,t) = \frac{t}{t+1} = M(x, Tx, t),$$

for all $t \geq 0$.

Example 2.7. *Let* $X = \mathbb{R}^2$ *and* $M(x,y,t) = \frac{t}{t+d(x,y)}$ *for all* $x, y \in X$, *where* $d(x,y)$ *is the Euclidean metric. Then* $(X, M, *_M)$ *is a complete FM-space. Suppose that*

$$A = \{(x,y) : x \leq -1\}, \qquad B = \{(x,y) : x \geq 1\}.$$

Then $M(A,B,t) = \frac{t}{t+2}$, $A_0 = \{(-1,y) : y \in \mathbb{R}\}$ *and* $B_0 = \{(1,y) : y \in \mathbb{R}\}$. *Let* S *and* T *be define as* $T(x,y) = (-x, \frac{y}{4})$ *and* $S(x,y) = (-x, \frac{y}{5})$. *Further, the non-self mappings* S *and* T *commute proximally, and* T *dominates* S *proximally.*
It is easy to see that the other hypotheses of the Corollary 2.2 are satisfied, so there exist unique element $x \in X$ *such that*

$$M(x, Sx, t) = M(A,B,t) = \frac{t}{t+2} = M(x, Tx, t),$$

for all $t \geq 0$.

Example 2.8. *Consider* $X = [0,3]$ *and define* $M(x,y,t) = \frac{t}{t+|x-y|}$ *for all* $x, y \in X$. *Then* $(X, M, *_M)$ *is a complete FM-space. Define continuous self mappings* S *and* T *on* X *as*

$$Sx = \frac{1}{6}x + 1, \qquad Tx = \frac{1}{3}(x + \frac{12}{5}), \qquad (x \in X).$$

It is easy to see that the condition (2.12) in Corollary 2.3 for $\alpha = 1/2$ *hold,* $ST = TS$ *and* $S(X) \subseteq T(X)$. *Therefore, all the hypothesis of Corollary 2.3 are satisfied, then* S *and* T *have a unique common fixed point* $x = \frac{6}{5}$.

Example 2.9. *Consider* $X = [-1,1]$ *and define* $M(x,y,t) = \frac{t}{t+|x-y|}$ *for all* $x, y \in X$. *Then* $(X, M, *_M)$ *is a complete FM-space. Define self mapping* S *on* X *as follows:*

$$Sx = \begin{cases} 0\ ; & -1 \leq x < 0, \\ \frac{x}{16(1+x)}\ ; & 0 \leq x < \frac{4}{5} \text{ or } \frac{7}{8} < x \leq 1, \\ \frac{x}{16}\ ; & \frac{4}{5} \leq x \leq \frac{7}{8}. \end{cases}$$

To verify condition (2.13) in Corollary 2.4 we need to consider several possible cases.

Case 1. Let $x, y \in [-1, 0)$. Then

$$d(Sx, Sy) = |Sx - Sy| = 0 \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Case 2. Let $x \in [-1, 0)$ and $y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Sx, Sy) = |Sx - Sy| = \frac{y}{16(1 + y)} \leq \frac{1}{8}|y - 0| = \frac{1}{8}d(y, Sx).$$

Case 3. Let $x \in [-1, 0)$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Sx, Sy) = |Sx - Sy| = \frac{y}{16} \leq \frac{1}{8}|y - 0| = \frac{1}{8}d(y, Sx).$$

Case 4. Let $x, y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Sx, Sy) = |Sx - Sy| = |\frac{x}{16(1 + x)} - \frac{y}{16(1 + y)}| \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Case 5. Let $x \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Sx, Sy) = |Sx - Sy| = |\frac{x}{16(1 + x)} - \frac{y}{16}| \leq \frac{1}{16}(\frac{x}{1 + x} + y) \leq \frac{1}{16}(\frac{1}{2} + \frac{7}{8}) \leq \frac{11}{128},$$

and

$$\frac{123}{160} = \frac{4}{5} - \frac{1}{16}\frac{1}{2} \leq y - \frac{x}{16(1 + x)} \leq |y - \frac{x}{16(1 + x)}| = d(y, Sx).$$

Thus

$$d(Sx, Sy) \leq \frac{11}{128} \leq \frac{123}{1280} = \frac{1}{8} \times \frac{123}{160} \leq \frac{1}{8}d(y, Sx).$$

Case 6. Let $x, y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Sx, Sy) = |Sx - Sy| = |\frac{x}{16} - \frac{y}{16}| \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Hence, we obtain

$$d(Sx, Sy) \leq \frac{1}{8}\max\{d(x, y), d(x, Sx), d(x, Sy), d(y, Sx)\}, \qquad (x, y \in [-1, 1]),$$

or in other words

$$M(Sx, Sy, \frac{1}{8}t) \geq \min\{M(x, y, t), M(x, Sx, t), M(x, Sy, t), M(y, Sx, t)\},$$

for every $x, y \in X$ and $t \geq 0$. Then S has a unique fixed point 0 in X, by Corollary 2.4.

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