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On the Stability of a Class of Cosine Type Functional Equations

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abstract: The aim of this paper is to investigate the stability problem for the pexiderized trigonometric functional equation

$$f_1(xy) + f_2(x\sigma(y)) = 2g_1(x)g_2(y), \quad x, y \in G, \tag{E}$$

where G is an arbitrary group, f_1, f_2, g_1 and g_2 are complex valued functions on G and σ is an involution of G. Results of this paper also can be extended to the setting of monoids (that is, a semigroup with identity) that need not be abelian.

Key Words: Stability, Superstability, D'Alembert's equation, Trigonometric functional equation.

Contents

1. Introduction

Let V be a vector space. In [4], Baker et al. have been proved that the functional equation

$$f(x + y) = f(x)f(y), \quad x, y \in V, \tag{1.1}$$

is superstable in the class of functions $f : V \to \mathbb{R}$ i.e. every such function satisfying the inequality

$$|f(x + y) - f(x)f(y)| \leq \varepsilon, \quad x, y \in V,$$

where ε is a fixed positive real number, either is bounded or satisfies (1.1).

The superstability of the cosine functional equation (also called classical d'Alembert's equation)

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G, \tag{C}$$

is studied by J. Baker [3], Badora [2] and Gàvruta [6], and the superstability problem for the mixed trigonometric functional equations

$$\begin{aligned} f(x + y) - f(x - y) &= 2f(x)g(y), \quad x, y \in G, \\ f(x + y) - f(x - y) &= 2g(x)f(y), \quad x, y \in G, \end{aligned}$$

on the abelian group $(G, +)$, is investigated by Kim and Lee ([10], [11]).

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The cosine functional equation (C) is generalized to the following functional equations

$$\begin{aligned} f(x+y)+f(x-y) &= 2f(x)g(y), \quad x,y\in G,\\ f(x+y)+f(x-y) &= 2g(x)f(y), \quad x,y\in G,\\ f(x+y)+f(x-y) &= 2g(x)g(y), \quad x,y\in G, \end{aligned}$$

and their stabilities are explored by Kannappan and Kim ([9], [12], [13]) and Tyrala [23]. The superstability problem for the pexiderized cosine type functional equation

$$f_1(x+y)+f_2(x-y)=2g_1(x)g_2(y), \quad x,y\in G,$$

on the abelian group $(G,+)$ is investigated by Kim [13] and Kusollerschariya and Nakmahachalasint [14].

In this paper, let G be any group, e denotes its neutral element, $\mathbb{C}$ the field of complex numbers. We may assume that f_1, f_2, g_1 and g_2 are nonzero *complex valued functions on G, δ is a nonnegative real constant, and σ is an involution of G, i.e. $\sigma(xy)=\sigma(y)\sigma(x)$ and $\sigma(\sigma(x))=x$ for all $x,y\in G$.* For any complex-valued function F on G we use the notation $\check{F}(x):=F(\sigma(x)),\ x\in G$.

The functional equation

$$f_1(xy)+f_2(x\sigma(y))=2g_1(x)g_2(y), \quad x,y\in G, \tag{E}$$

contain four unknown functions. However, if we put $F_1:=\frac{f_1+f_2}{2}$ and $F_2:=\frac{f_1-f_2}{2}$ then (E) is equivalent to the two equations

$$\begin{aligned} F_1(xy)+F_1(x\sigma(y)) &= 2g_1(x)\frac{g_2(y)+g_2(\sigma(y))}{2}, \quad x,y\in G,\\ F_2(xy)-F_2(x\sigma(y)) &= 2g_1(x)\frac{g_2(y)-g_2(\sigma(y))}{2}, \quad x,y\in G. \end{aligned}$$

So the study of (E) reduces to a study of equations of the forms

$$f(xy)\mp f(x\sigma(y))=2g(x)h(y), \quad x,y\in G,$$

each containing only three unknown functions [19]. Studying these equations is based on the solution of d'Alembert's functional equation

$$f(xy)+f(x\sigma(y))=2f(x)f(y), \quad x,y\in G, \tag{A}$$

that has been solved, on an arbitrary group, by Davison in [5] and Stetkær in [20] each in his own way. Their formulas of solutions involve harmonic analysis on G and these make sense to solve other functional equations see e.g. [33].

In [18], Roukbi, Zeglami and Kabbaj proved the superstability of Wilson's functional equation

$$f(xy)+f(x\sigma(y))=2f(x)g(y), \quad x,y\in G, \tag{W}$$

where G is any group and σ is an involuon of G. Namely, the following Theorem holds true.

Theorem 1.1. *([18]) Let $\delta > 0$ be given. Assume that functions $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta \quad \textit{for all } x, y \in G.$$

Then one of the following statements holds:
(i) f, g are bounded,
(ii) f is unbounded and g satisfies d'Alembert's long functional equation

$$g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) = 4g(x)g(y), \ x, y \in G, \tag{L.A}$$

(iii) g is unbounded and the pair (f, g) satisfies Eq. (W).

The stability of the following equation

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G,$$

is investigated on an arbitrary group G by Zeglami, Kabbaj and Roukbi [25] in the following Theorem:

Theorem 1.2. *([25]) Let $\delta > 0$ be given. Assume that functions $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \delta \quad \textit{for all } x, y \in G.$$

Then one of the following statements holds:
(i) f, g are bounded,
(ii) g is unbounded and $\tilde{f} = \frac{1}{f(e)}f$ satisfies Eq. (L.A),
(iii) f is unbounded and (f, g) satisfies the equation

$$g(xy) + g(x\sigma(y)) = 2g(x)\tilde{f}(y) \textit{ for all } x, y \in G.$$

The stability problem of the trigonometric type functional equation

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y), \ x, y \in G, \tag{T}$$

is also studied by Zeglami and Kabbaj in [27].

Theorem 1.3. *([27]) Let $\delta > 0$ be given. Assume that functions $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) - f(x\sigma(y)) - 2f(x)g(y)| \leq \delta \qquad \textit{for all } x, y \in G.$$

Then one of the following statements holds:
(i) f, g are bounded,
(ii) f is unbounded and g satisfies the functional equation

$$g(xy) - g(x\sigma(y)) - g(yx) + g(\sigma(y)x) = 0, \ x, y \in G,$$

(iii) g is unbounded and the pair (f, g) satisfies Eq. (T).

In the present paper our approach is more general. We study, on any group, not necessarily abelian, the stability problem of the pexiderezed cosine functional equation

$$f_1(xy) + f_2(x\sigma(y)) = 2g_1(x)g_2(y), \quad x, y \in G, \tag{E}$$

where f_1, f_2, g_1 and g_2 are complex valued functions and σ is an involution of G.

As consequences, we obtain the stability of the following functional equations

$$\begin{aligned}
f(xy) \pm f(x\sigma(y)) &= 2f(x)f(y), && x, y \in G,\\
f(xy) \mp f(x\sigma(y)) &= 2g(x)f(y), && x, y \in G,\\
f(xy) \pm f(x\sigma(y)) &= 2g(x)g(y), && x, y \in G,\\
f(xy) \mp f(x\sigma(y)) &= 2g(x)h(y), && x, y \in G,\\
f(xy) + g(x\sigma(y)) &= 2g(x)f(y), && x, y \in G,\\
f(xy) + g(x\sigma(y)) &= 2f(x)g(y), && x, y \in G.
\end{aligned}$$

The interested reader should refer to [1-4, 6-18, 21-32] for a thorough account on the subject of stability of functional equations.

2. Stability of pexiderized trigonometric equations

Lemma 2.1. *Let $\delta > 0$ be given. Assume that the function $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(xy) - f(x\sigma(y)) - 2f(x)f(y)| \leq \delta,$$

for all $x, y \in G$. Then f is bounded.

Proof: [[27], Corollary 1]. □

Lemma 2.2. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$ with $g_2(e) \neq 0$ satisfy the inequality*

$$|f_1(xy) + f_2(x\sigma(y)) - 2g_1(x)g_2(y)| \leq \delta, \tag{I}$$

for all $x, y \in G$. Then $f_1 + f_2$ is unbounded if and only if g_1 is also unbounded.

Proof: Putting $y = e$ in (I) we obtain

$$|f_1(x) + f_2(x) - 2g_2(e)g_1(x)| \leq \delta \quad \text{for all } x \in G,$$

which shows, since $g_2(e) \neq 0$, that $f_1 + f_2$ is unbounded is equivalent to g_1 is also unbounded. □

Lemma 2.3. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$ with $g_2(e) = 0$ satisfy the inequality (I). Then $f_1 + f_2$ is bounded and for all $x, y \in G$ we have*

$$|f_1(xy) - f_1(x\sigma(y)) - 2g_1(x)g_2(y)| \leq 2\delta\ , \tag{2.1}$$

and

$$|f_2(xy) - f_2(x\sigma(y)) + 2g_1(x)g_2(y)| \leq 2\delta\ .$$

Proof: The proof of each inequality are very similar, so it suffices to show the proof of (2.1). Putting $y = e$ in (I) we obtain

$$|f_1(x) + f_2(x)| \le \delta \quad \text{for all } x \in G, \tag{2.2}$$

which shows that $f_1 + f_2$ is bounded and

$$\begin{aligned}&|f_1(xy) - f_1(x\sigma(y)) - 2g_1(x)g_2(y)|\\ &\le |f_1(xy) + f_2(x\sigma(y)) - 2g_1(x)g_2(y)| + |f_1(x\sigma(y)) + f_2(x\sigma(y))|.\end{aligned}$$

By (2.2) and (I) we get that

$$|f_1(xy) - f_1(x\sigma(y)) - 2g_1(x)g_2(y)| \le 2\delta,$$

for all $x, y \in G$. □

Lemma 2.4. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$ with $g_2(e) = 1$ satisfy the inequality (I). Then*

$$|g_1(xy) + g_1(x\sigma(y)) - 2g_1(x)h(y)| \le 2\delta \quad \textit{for all } x, y \in G,$$

where

$$h = \frac{g_2 + \check{g}_2}{2}.$$

Proof: Assume that $g_2(e) = 1$. Putting $y = e$ in the inequality (I). It is easy to show that

$$|f_1(x) + f_2(x) - 2g_1(x)| \le \delta \quad \text{for all } x \in G. \tag{2.3}$$

By virtue of inequalities (I) and (2.3) we have

$$\begin{aligned}&|g_1(xy) + g_1(x\sigma(y)) - 2g_1(x)h(y)|\\ &= \left|g_1(xy) + g_1(x\sigma(y)) - 2g_1(x)\frac{g_2(y) + g_2(\sigma(y))}{2}\right|\\ &= |g_1(xy) + g_1(x\sigma(y)) - g_1(x)g_2(y) - g_1(x)g_2(\sigma(y))|\\ &\le \left|-\frac{1}{2}f_1(xy) - \frac{1}{2}f_2(xy) + g_1(xy)\right|\\ &+ \left|-\frac{1}{2}f_1(x\sigma(y)) - \frac{1}{2}f_2(x\sigma(y)) + g_1(x\sigma(y))\right|\\ &+ \frac{1}{2}|f_1(xy) + f_2(x\sigma(y)) - 2g_1(x)g_2(y)|\\ &+ \frac{1}{2}|f_1(x\sigma(y)) + f_2(xy) - 2g_1(x)g_2(\sigma(y))|\\ &\le \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2} = 2\delta.\end{aligned}$$

□

In the following theorem the stability of the pexiderized trigonometric functional equation (E) will be investigated on an arbitrary group.

Theorem 2.1. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$ with $g_2(e) \neq 0$ satisfy the inequality (I). Then one of the following statements holds:*

(i) g_1 (or $f_1 + f_2$) and $g_2 + \breve{g}_2$ are bounded,

(ii) g_1 (or $f_1 + f_2$) is unbounded and $h = \frac{g_2+\breve{g}_2}{2g_2(e)}$ satisfies Eq. ($L.A$), i.e.

$$h(xy) + h(x\sigma(y)) + h(yx) + h(\sigma(y)x) = 4h(x)h(y), \ x, y \in G,$$

(iii) $g_2 + \breve{g}_2$ is unbounded and the pair (g_1, g_2) satisfies the equation

$$g_1(xy) + g_1(x\sigma(y)) = g_1(x)\frac{g_2(y) + \breve{g}_2(y)}{g_2(e)}, \ x, y \in G.$$

Proof: Assume that f_1, f_2, g_1 and g_2 satisfy the inequality (I) such that $g_2(e) \neq 0$. Dividing the two sides of the inequality (I) by $\alpha = g_2(e)$ we find that

$$\left|\tilde{f}_1(xy) + \tilde{f}_2(x\sigma(y)) - 2g_1(x)\tilde{g}_2(y)\right| \leq \frac{\delta}{|\alpha|} \text{ for all } x, y \in G,$$

where $\tilde{f}_1 = \frac{f_1}{\alpha}$, $\tilde{f}_2 = \frac{f_2}{\alpha}$ and $\tilde{g}_2 = \frac{g_2}{\alpha}$. We see that $\tilde{g}_2(e) = 1$. By using Lemma 2.4 we obtain that

$$|g_1(xy) + g_1(x\sigma(y)) - 2g_1(x)h(y)| \leq 2\frac{\delta}{|\alpha|} \text{ for all } x, y \in G,$$

where

$$h = \frac{g_2 + \breve{g}_2}{2g_2(e)}.$$

Using Theorem 1.1, we conclude that each functions f_1, f_2, g_1 and g_2 satisfying (I) with $g_2(e) \neq 0$ fall into one of the categories (i)-(iii) of Theorem 2.1. □

Corollary 2.2. *Let $\delta > 0$ be given. Assume that functions $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + g(x\sigma(y)) - 2f(x)g(y)| \leq \delta,$$

for all $x, y \in G$.

1) If $g(e) \neq 0$, then:

i) Either f is bounded or $h = \frac{g+\breve{g}}{2g(e)}$ satisfies Eq. ($L.A$).

ii) Either $g + \breve{g}$ is bounded or the pair (f, g) satisfies the equation

$$f(xy) + f(x\sigma(y)) = f(x)\frac{g(y) + \breve{g}(y)}{g(e)} \textit{ for all } x, y \in G.$$

2) If $g(e) = 0$, then either g (or f) is bounded or

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y) \textit{ for all } x, y \in G.$$

Proof: 1) (i) and (ii) are immediate consequences from Theorem 2.1 by taking $f_1 = g_1 = f$ and $f_2 = g_2 = g$.

2) If g is an unbounded function such that $g(e) = 0$ then f is also unbounded. By virtue of Lemma 2.3, we have

$$|f(xy) - f(x\sigma(y)) - 2f(x)g(y)| \leq 2\delta \ ,$$

for all $x, y \in G$. Applying (iii) of Theorem 1.3 we have

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y) \text{ for all } x, y \in G.$$

□

As another result of Theorem 2.1 we have the following corollary

Corollary 2.3. *Let $\delta > 0$ be given. Assume that functions f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq \delta,$$

for all $x, y \in G$.

1) If $f(e) \neq 0$, then:

i) Either g is bounded or $h = \frac{f+\check{f}}{2f(e)}$ satisfies Eq. (L.A).

ii) Either $f + \check{f}$ is bounded or the pair (f, g) satisfies the equation

$$g(xy) + g(x\sigma(y)) = g(x)\frac{f(y) + \check{f}(y)}{f(e)} \textit{ for all } x, y \in G.$$

2) If $f(e) = 0$, then either f (or g) is bounded or

$$-g(xy) + g(x\sigma(y)) = 2g(x)f(y) \textit{ for all } x, y \in G.$$

Proof: 1) The assertions (i) and (ii) are immediate consequences from Theorem 2.1 by taking $f_1 = g_2 = f$ and $f_2 = g_1 = g$.

2) If f is an unbounded function such that $f(e) = 0$ then g is also unbounded. By virtue of Lemma 2.3, we have

$$|g(xy) - g(x\sigma(y)) - 2g(x)(-f(y))| \leq 2\delta,$$

for all $x, y \in G$. So, by (iii) of Theorem 1.3, we have

$$g(xy) - g(x\sigma(y)) = -2g(x)f(y), \quad \forall x, y \in G.$$

□

In the next four corollaries, the stability problem of the considered functional equations will be obtained without any assumption on $g_2(e)$.

Corollary 2.4. *([30]) Let $\delta > 0$ be given. Suppose that functions f, g_1 and $g_2 : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2g_1(x)g_2(y)| \leq \delta \text{ for all } x, y \in G. \tag{2.4}$$

Then one of the following statements holds:
(i) f, g_1 and g_2 are bounded,
(ii) g_1 (or f) is unbounded and $\frac{g_2}{g_2(e)}$ satisfies Eq. (L.A),
(iii) $g_2 + \check{g}$ is unbounded and the pair (g_1, g_2) satisfies the equation

$$g_1(xy) + g_1(x\sigma(y)) = g_1(x)\frac{g_2(y) + \check{g}_2(y)}{g_2(e)} \text{ for all } x, y \in G.$$

Proof: If $g_2(e) = 0$. Putting $y = e$ in (2.4) we get $|f(x)| \leq \frac{\delta}{2}$ for all $x \in G$ i.e. f is bounded. Let $M = \sup |f|$ and choose $a, b \in G$ such that $g_1(a) \neq 0$ and $g_2(b) \neq 0$ then we get from the inequality (2.4) that

$$|g_2(x)| \leq \frac{1}{2\,|g_1(a)|}(2M + \delta) \text{ and } |g_1(x)| \leq \frac{1}{2\,|g_2(b)|}(2M + \delta),$$

for all $x \in G$, i.e. g_1 and g_2 are also bounded. So we are in the case (i).

Suppose that g_1 (or f) is unbounded. According to previous discussions $g_2(e) \neq 0$. Replacing y by $\sigma(y)$ in (2.4) results in

$$|f(x\sigma(y)) + f(xy) - 2g_1(x)g_2(\sigma(y))| \leq \delta \text{ for all } x, y \in G.$$

From this last inequality and (2.4) we get

$$|2g_1(x)|\,|g_2(y) - g_2(\sigma(y))| \leq 2\delta \text{ for all } x, y \in G,$$

and then it follows also that $g_2 \circ \sigma = g_2$ because g_1 is unbounded. Using notation of Theorem 2.1, the function h shall be equal to $\frac{g_2}{g_2(e)}$. Finally, if $g_2 + g_2 \circ \sigma$ is unbounded then $g_2(e) \neq 0$ and so the rest of the proof is an immediate consequence of Theorem 2.1. □

Corollary 2.5. *([30]) Let $\delta > 0$ be given. Suppose that functions f and $g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2g(x)g(y)| \leq \delta \text{ for all } x, y \in G. \tag{2.5}$$

Then either g (or f) is bounded or the function g satisfies the equation

$$g(xy) + g(x\sigma(y)) = 2g(x)\frac{g(y)}{g(e)} \text{ for all } x, y \in G.$$

Proof: It is obvious that g is bounded is equivalent to f is also bounded. Suppose that g is an unbounded function satisfying (2.5) then $g(e) \neq 0$. Replacing y by $\sigma(y)$ in (2.5) and adding the result to (2.5) we arrive at the inequality

$$|g(x)(g(y) - g(\sigma(y)))| \leq \delta \text{ for all } x, y \in G,$$

from which it follows that $g \circ \sigma = g$ because g is unbounded. The rest of the proof follows on putting $g_1 = g_2 = g$ on Corollary 2.4 (iii). □

Corollary 2.6. *([18]) Let $\delta > 0$ be given. Suppose that the function $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \delta \quad \text{for all } x, y \in G. \tag{2.6}$$

Then either f is bounded or f is a solution of Eq. (A).

Proof: If we replace y by e in (2.6) we see that if f is an unbounded function satisfying (2.6) then $f(e) = 1$ and the proof follows on putting $f = g$ on Corollary 2.5. □

Corollary 2.7. *Let $\delta > 0$ be given. Suppose that functions $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \delta \quad \text{for all } x, y \in G. \tag{2.7}$$

If f (or g) fails to be bounded, then g satisfies the equation

$$g(xy) + g(x\sigma(y)) = 2g(x)\frac{f(y)}{f(e)} \quad \text{for all } x, y \in G.$$

If, additionally the group G is abelian then $f(xy) + f(x\sigma(y)) = 2g(x)f(y)$ and g is a solution of Eq. (A).

Proof: Suppose that f (or g) is unbounded, it is easy to see that $f(e) \neq 0$ and $f = f \circ \sigma = \frac{f + f\circ\sigma}{2}$. On putting $g_1 = g$ and $g_2 = f$ in Corollary 2.4 (iii) we get that the pair (f, g) satisfies the equation

$$g(xy) + g(x\sigma(y)) = 2g(x)\frac{f(y)}{f(e)} \text{ for all } x, y \in G. \tag{2.8}$$

From (2.8) we deduce that $\tilde{f}(xy) + \tilde{f}(x\sigma(y)) + \tilde{f}(yx) + \tilde{f}(\sigma(y)x) = 4\tilde{f}(x)\tilde{f}(y)$ for all $x, y \in G$ where $\tilde{f} = \frac{f}{f(e)}$. If G is abelian or at least f is central (i.e. $f(xy) = f(yx)$ for all $x, y \in G$) then we have

$$\tilde{f}(xy) + \tilde{f}(x\sigma(y)) = 2\tilde{f}(x)\tilde{f}(y), \ \forall x, y \in G. \tag{2.9}$$

Dividing the two sides of the inequality (2.7) by $\alpha = f(e)$ we find that

$$\left|\tilde{f}(xy) + \tilde{f}(x\sigma(y)) - 2g(x)\tilde{f}(y)\right| \leq \frac{\delta}{|\alpha|}. \tag{2.10}$$

When we substitute (2.9) into (2.10) we get that

$$\left|2\tilde{f}(y)(\tilde{f}(x) - g(x))\right| \leq \frac{\delta}{|\alpha|}. \tag{2.11}$$

Since f is unbounded then so is $\tilde{f}$. Consequently (2.11) implies $g = \tilde{f}$ thus g satisfies (A). Substituting $\tilde{f}$ by g on the first factor of the right hand side of (2.9) the expression reduces to $f(xy) + f(x\sigma(y)) = 2g(x)f(y)$ for all $x, y \in G$. $\square$

Proposition 2.8. *Let $\delta > 0$ be given. Suppose that functions $f, g_1, g_2 : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) - f(x\sigma(y)) - 2g_1(x)g_2(y)| \leq \delta \text{ for all } x, y \in G. \tag{2.12}$$

i) If g_1 is unbounded then $g_2 + g_2 \circ \sigma = 0$.
ii) If $g_2 + g_2 \circ \sigma$ is unbounded then $g_1 = 0$.

Proof: i) Suppose that g_1 is unbounded. If we replace y by $\sigma(y)$ in (2.12) we get

$$|f(x\sigma(y)) - f(xy) - 2g_1(x)g_2(\sigma(y))| \leq \delta \text{ for all } x, y \in G.$$

From this inequality and (2.12) we obtain

$$|2g_1(x)|\,|g_2(y) + g_2(\sigma(y))| \leq 2\delta,$$

for all $x, y \in G$. Hence $g_2 + g_2 \circ \sigma = 0$ because g_1 is unbounded.
(ii) Follows from the previous discussion. $\square$

Corollary 2.9. *Let $\delta > 0$ be given. Suppose that functions $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) - f(x\sigma(y)) - 2g(x)f(y)| \leq \delta \text{ for all } x, y \in G.$$

i) If g is unbounded then $f + \check{f} = 0$.
ii) If $f + \check{f}$ is unbounded then $g = 0$.

Proof: The proof follows on putting $g_1 = g$ and $g_2 = f$ on Proposition 2.8. $\square$

Lemma 2.5. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$ with $f_2 = f_2 \circ \sigma$ satisfy the inequality (I).*

1) If $g_1(e) = 1$ then

$$|g_2(xy) + g_2(x\sigma(y)) - 2g_1(x)h(y)| \leq 2\delta \text{ for all } x, y \in G,$$

where

$$h = \frac{g_2 + \check{g}_2}{2}.$$

2) If $g_1(e) = 0$ then

$$|f_1(xy) - f_1(x\sigma(y)) - 2g_1(x)g_2(y)| \leq 2\delta \text{ for all } x, y \in G, \tag{2.13}$$

and

$$|-f_2(xy) + f_2(x\sigma(y)) - 2g_1(x)g_2(y)| \leq 2\delta \text{ for all } x, y \in G.$$

Proof: 1) Assume that $g_1(e) = 1$ and $f_2 = f_2 \circ \sigma$. Putting $x = e$ in the inequality (I). It is easy to show that

$$|f_1(y) + f_2(y) - 2g_2(y)| \le \delta \quad \text{for all } x \in G. \tag{2.14}$$

By the use of (2.14) and (I) we get for all $x, y \in G$ that

$$\begin{aligned}
&|g_2(xy) + g_2(x\sigma(y)) - 2g_1(x)h(y)| \\
&= |g_2(xy) + g_2(x\sigma(y)) - g_1(x)g_2(y) - g_1(x)g_2(\sigma(y))| \\
&\le \left|-\frac{1}{2}f_1(xy) - \frac{1}{2}f_2(xy) + g_2(xy)\right| \\
&+ \left|-\frac{1}{2}f_1(x\sigma(y)) - \frac{1}{2}f_2(x\sigma(y)) + g_2(x\sigma(y))\right| \\
&+ \frac{1}{2}\left|f_1(xy) + f_2(x\sigma(y)) - 2g_1(x)g_2(y)\right| \\
&+ \frac{1}{2}\left|f_1(x\sigma(y)) + f_2(xy) - 2g_1(x)g_2(\sigma(y))\right| \\
&\le \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2} = 2\delta.
\end{aligned}$$

2) The proof of each inequality are very similar so it suffices to show the proof of (2.13). Putting $x = e$ in (I) we obtain

$$|f_1(x) + f_2(x)| \le \delta \ \text{ for all } x \in G,$$

which shows, using (I), that

$$\begin{aligned}
&|f_1(xy) - f_1(x\sigma(y)) - 2g_1(x)g_2(y)| \\
&\le |f_1(xy) + f_2(x\sigma(y)) - 2g_1(x)g_2(y)| + |f_1(x\sigma(y)) + f_2(x\sigma(y))| \\
&\le 2\delta,
\end{aligned}$$

for all $x, y \in G$. □

Theorem 2.10. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$, with $f_2 = \check{f}_2$ and $g_1(e) \ne 0$, satisfy the inequality (I). Then*

i) Either g_1 (or g_2) (or $f_1 + f_2$) is bounded or $\frac{g_2 + \check{g}_2}{2g_2(e)}$ satisfies Eq. (L.A).

ii) Either $g_2 + \check{g}_2$ is bounded or the pair (g_1, g_2) satisfies the equation

$$g_1(xy) + g_1(x\sigma(y)) = g_1(x)\frac{g_2(y) + \check{g}_2(y)}{g_2(e)} \ \textit{for all } x, y \in G.$$

Assume that f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$, with $f_2 = \check{f}_2$ and $g_1(e) \ne 0$, satisfy the inequality (I). It is easy to see that g_2 is bounded is equivalent to $f_1 + f_2$ is also bounded. Dividing the two sides of the inequality (I) by $\beta = g_1(e)$ we find that

$$\left|\tilde{f}_1(xy) + \tilde{f}_2(x\sigma(y)) - 2\tilde{g}_1(x)g_2(y)\right| \le \frac{\delta}{|\beta|} \text{ for all } x, y \in G,$$

where $\tilde{f}_1 = \frac{f_1}{\beta}$, $\tilde{f}_2 = \frac{f_2}{\beta}$ and $\tilde{g}_1 = \frac{g_1}{\beta}$. We see that $\tilde{f}_2(\sigma(y)) = \tilde{f}_2(y)$ for all $y \in G$ and $\tilde{g}_1(e) = 1$. According to Lemma 2.5. we have

$$|g_2(xy) + g_2(x\sigma(y)) - 2\tilde{g}_1(x)h(y)| \leq 2\frac{\delta}{|\beta|} \text{ for all } x, y \in G,$$

where $h = \frac{g_2 + g_2 \circ \sigma}{2}$ and the rest of the proof follows from Corollary 2.4.

Corollary 2.11. *Let $\delta > 0$ be given. Assume that functions f_1, f_2, g_1 and $g_2 : G \to \mathbb{C}$, with $f_2 = \check{f}_2$, $g_1(e) \neq 0$ and $g_2 = \check{g}_2$, satisfy the inequality (I). Then either g_2 (or g_1) is bounded or the pair (g_1, g_2) satisfies the equation*

$$g_1(xy) + g_1(x\sigma(y)) = 2g_1(x)\tilde{g}_2(y) \text{ for all } x, y \in G.$$

Furthermore in the latter case the function $\tilde{g}_2 = \frac{1}{g_2(e)}g_2$ satisfies Eq. (L.A).

Proof: The proof is an immediate consequence of Theorem 2.10. □

Corollary 2.12. *Let $\delta > 0$ be given. Assume that functions $f, g : G \to \mathbb{C}$ with $g = \check{g}$ satisfy the inequality*

$$|f(xy) + g(x\sigma(y)) - 2f(x)g(y)| \leq \delta,$$

for all $x, y \in G$.

1) If $f(e) \neq 0$, then:

i) Either g (or f) is bounded or $\frac{g}{g(e)}$ satisfies Eq. (L.A).

ii) Either g is bounded or the pair (f, g) satisfies the equation

$$f(xy) + f(x\sigma(y)) = 2f(x)\frac{g(y)}{g(e)} \text{ for all } x, y \in G.$$

2) If $f(e) = 0$, then either g (or f) is bounded or

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y) \text{ for all } x, y \in G.$$

Proof: 1) The assertions (i) and (ii) are immediate consequences from Theorems 2.10 by taking $f_1 = g_1 = f$ and $f_2 = g_2 = g$.

2) If g is an unbounded function such that $f(e) = 0$ then f is also unbounded. By virtue of Lemma 2.5, we have

$$|f(xy) - f(x\sigma(y)) - 2f(x)g(y)| \leq 2\delta\ ,$$

for all $x, y \in G$. Applying (iii) of Theorem 1.3 we have

$$f(xy) - f(x\sigma(y)) = 2f(x)g(y) \text{ for all } x, y \in G.$$

□

As another result of Theorem 2.10 we have the following corollary

Corollary 2.13. *Let $\delta > 0$ be given. Assume that functions f and $g : G \to \mathbb{C}$ with $g = \check{g}$ satisfy the inequality*

$$|f(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq \delta \text{ for all } x, y \in G.$$

1) If $g(e) \neq 0$, then:
i) Either g (or f) is bounded or $\frac{f+\check{f}}{2f(e)}$ satisfies Eq. (L.A).
ii) Either $f + \check{f}$ is bounded or the pair (f, g) satisfies the equation

$$g(xy) + g(x\sigma(y)) = g(x)\frac{f(y) + \check{f}(y)}{f(e)} \text{ for all } x, y \in G.$$

2) If $g(e) = 0$, then either f (or g) is bounded or

$$-g(xy) + g(x\sigma(y)) = 2g(x)f(y) \text{ for all } x, y \in G.$$

Proof: 1) The assertions (i) and (ii) are immediate consequences from Theorem 2.10 by taking $f_1 = g_2 = f$ and $f_2 = g_1 = g$.

2) If f is an unbounded function such that $g(e) = 0$ then g is also unbounded. By virtue of Lemma 2.5, we have

$$|g(xy) - g(x\sigma(y)) - 2g(x)(-f(y))| \leq 2\delta,$$

for all $x, y \in G$. So, by (iii) of Theorem 1.3, we have

$$g(xy) - g(x\sigma(y)) = -2g(x)f(y),$$

for all $x, y \in G$. □

Remark 2.14. *i) All the results of this paper are also true if we suppose that G is a semigroup with identity because in demonstrations we never needed the inversion in G.*

ii) The results of this paper can also be extended to the stability of the considered equations controlled even by variable bounds.

References

1. M. Akkouchi, *Generalized Ulam-Hyers-Rassias stability of a Cauchy type functional equation*, Proyecciones J. of Math. **32** (2013), no. 1, 15-29.

2. R. Badora, *On Hyers–Ulam stability of Wilson's functional equation*, Aequationes Math. **60** (2000), 211-218.

3. J. A. Baker, *The stability of the cosine equation,* Proc. Amer. Math. Soc. **80** (1980), 411–416.

4. J. A. Baker, J. Lawrence, and F. Zorzitto, *The stability of the equation $f(x+y) = f(x)f(y)$*, Proc. Amer. Math. Soc. **74** (1979), 242-246.

5. T. M. K. Davison, *D'Alembert's functional equation on topological monoids,* Publ. Math. Debrecen 75/1-2 (2009), 41-66.

6. P. Gàvruta, *On the stability of some functional equations*, in: Th. M. Rassias and J. Tabor, Stability of mappings of Hyers-Ulam type, Hadronic Press, 1994, pp. 93–98.

7. D. H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222-224.

8. D. H. Hyers, G. I. Isac, and Th. M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications, 34. Birkhauser Boston, Inc., Boston, MA, 1998.

9. Pl. Kannappan and G. H. Kim, *On the stability of the generalized cosine functional equations*, Ann. Acad. Pedagog. Crac. Stud. Math., **1** (2001), 49-58.

10. G. H., Kim, *On the Stability of the mixed trigonometric functional equations*, Banach J. Math. Anal. **1** (2007), no. 2, 227-236.

11. G. H., Kim and Whan Y. Lee, *Boundedness of approximate trigonometric functions*, Applied Math. Letters **22** (2009), 439-443.

12. G. H. Kim, *The stability of the d'Alembert and Jensen type functional equations*, J. Math. Anal. Appl. **325** (2007), 237-248

13. G. H. Kim, *Superstability of some Pexider-type functional equation*, J. of Ineq. and Appl. Vol. 2010, Article ID 985348, 16 pages.

14. C. Kusollerschariya and P. Nakmahachalasint, *The stability of the pexiderized cosine functional equation*, Thai J. of math. Special Issue (Annual Meeting in Mathematics, 2008) : 39-44.

15. J.M. Rassias, *Solution of a problem of Ulam*, J. Approx. Theory, **57** (1989) 268-273.

16. Th.M. Rassias, *On the stability of linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297-300.

17. Th.M. Rassias: *Handbook of functional equations: Stability theory*, Springer Optimization and Its Applications vol. 96 (2014). doi:10.1007/ 978- 1- 4939- 1286- 5_17

18. A. Roukbi, D. Zeglami and S. Kabbaj, *Hyers–Ulam stability of Wilson's functional equation,* J. Math. Sci. Adv. Appl. **22** (2013), 19-26.

19. H. Stetkær, *Wilson's functional equations on groups*, Aequationes Math. **49** (1995), 252-275.

20. H. Stetkær, *Functionals equations on groups*, World Scientific Publishing Company (2013).

21. L. Székelyhidi, *On a stability theorem*, C. R. Math. Rep. Acad. Sci. Canada **3** (1981), 253-255.

22. M. Tial, D. Zeglami, S. Kabbaj*, On a Hilbert Gołab-Schinzel type functional equation*, Int. J. Nonlinear Anal. Appl. **6** (2015) No. 2, 149-159.

23. I. Tyrala, *The stability of the second generalization of d'Alembert's functional equation*, Scientic Issues, Mathematics XIII, Czestochowa 2008.

24. S.M. Ulam, *A collection of mathematical problems*, Interscience Publ. New York, 1961. Problems in Modern Mathematics, Wiley, New York 1964.

25. D. Zeglami, S. Kabbaj and A. Roukbi, *On the superstability of a generalization of the cosine equation*, British J. of Math. & Computer Science, 4 (5) (2014), 719-734.

26. D. Zeglami, A. Roukbi and S. Kabbaj, *Hyers-Ulam Stability of Generalized Wilson's and d'Alembert's Functional equations,* Afr. Mat. **26** (2015), 215-223.

27. D. Zeglami, S. Kabbaj, *On the superstability of trigonometric type functional equations* , British J. of Math. & Computer Science, **4** (8) (2014): 1146-1155.

28. D. Zeglami, *The superstability of a variant of Wilson's functional equation on an arbitrary group,* Afr. Mat. **26** (2015), 609-617.

29. D. Zeglami, A. Charifi and S. Kabbaj, *On the superstability of the pexider type generalized trigonometric functional equations*, Acta Math. Sci. Ser. B Engl. Ed. **34** (2014), no. 6, 1749-1760.

30. D. Zeglami, S. Kabbaj and A. Charifi, *On the stability of the generalized mixed trigonometric functional equations*, Adv. Pure. Appl. Math. **5** (2014), no. 4, 209-222.

31. D. Zeglami, B. Fadli, Ch. Park, Th.M. Rassias, *On a $\mu-$ mixed trigonometric functional equation,* J. Computational Analysis and Applications, **19** (2015), No. 6, 1003-1011.

32. D. Zeglami, A. Charifi and S. Kabbaj, *Superstability problem for a large class of functional equations*, Afr. Mat., **27** (2016), 469-484.

33. D. Zeglami and B. Fadli, *Integral functional equations on locally compact groups with involution,* Aequationes Math., 90 (2016), 967-982.

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