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Sliding window convergence and lacunary statistical convergence for measurable functions via modulus function

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abstract: In this paper we study the concepts of sliding window convergence for real valued measurable functions defined on $[0, \infty)$ via modulus function. We also establish some inclusions and consistency theorems for sequential methods along with examples. Finally, we give a Cauchy convergence criterion.

Key Words: modulus function; sliding window method; statistical convergence; lacunary statistical convergence; measurable functions; Cauchy criteria.

Contents

1. Introduction and Preliminaries

A function $M : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfy the following conditions:

1. $M(x) = 0$ if and only if $x = 0$,
2. $M(x + y) \leq M(x) + M(y)$, for all $x, y \geq 0$,
3. M is increasing,
4. M is continuous from the right at 0.

It follows that M must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $M(x) = \frac{x}{x+1}$, then $M(x)$ is bounded. If $M(x) = x^p, 0 < p < 1$ then the modulus function $M(x)$ is unbounded. For more details about modulus function and sequence spaces one may refer to ([4], [6], [10], [22], [25], [26]) and references therein.
The concept of statistical convergence was introduced by Steinhaus [28] and Fast [10] and later reintroduced by Schoenberg [27] independently. In recent years, statistical convergence was discussed in the theory of Fourier analysis, ergodic theory,

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number theory, measure theory, trigonometric series and Banach spaces, e.g. ([1]–[3], [16]–[18]) .The corresponding notion of convergence for function of a real variable was established in ([7], [8]) and recently investigated by Mòrlicz [23] . Fridy and Orhan ([14], [15]) introduced lacunary statistical convergence, with some of their result constructing on the work of Freedman et al. [12]. For latest work on the related topic can be found in [5], [19], [20], [21]. In this paper we encompassed Fridy and Orhan's work into more general settings of functions of a real variable by using an modulus function.
In this article we denote (γ, η) as a sliding window pair provided:
(1) γ and η are both nondecreasing, nonnegative real-valued measurable functions defined on $[0, \infty)$,
(2) $\gamma(r) < \eta(r)$ for every positive real number r, and $\eta(r)$ tends to infinity as r tends to infinity,
(3) $\liminf_r(\eta(r) - \gamma(r)) > 0$ and
(4) $(0, \eta(r)] = \cup\{(\gamma(s) - \eta(s)] : s \leq r\}$ for all $r > 0$.
Suppose $I_r = (\gamma(r), \eta(r)]$ and $\eta(r)-\gamma(r) = \mu(I_r)$, where $\mu(A)$ denotes the Lebesgue measure of the set A.

Let M be a modulus function, p be positive real number then we define the following definitions:

Definition 1.1. *Let (γ, η) as a sliding window pair and $g : [0, \infty) \to \mathbb{R}$ a measurable function. Then:*
(1) The function g is $N(\gamma, \eta, M, p)$ summable to L and write $N(\gamma, \eta, p, M,)-\lim g = L($or $g \to LN(\gamma, \eta, M, p))$ if and only if

$$\lim_{r\to\infty} \frac{1}{\mu(I_r)} \int_{I_r} M(|g(t) - L|)^p dt = 0.$$

(2) The function g is statistically (γ, η, M, p) convergent to L and we write $S(\gamma, \eta, M, p) - \lim g = L($or $g \to LS(\gamma, \eta, M, p))$ if and only if

$$\lim_{r\to\infty} \frac{1}{\mu(I_r)} \mu\big(t \in I_r : M(|g(t) - L|)^p \geq \epsilon\big) = 0$$

for all $\epsilon > 0$. In this case we write that g is $S(\gamma, \eta, M, p)$ convergent. We call either of the methods defined above a Sliding window method.

Definition 1.2. *For a sequence (x_p), lacunary sequence $\theta = \{k_n\}$ is $S_\theta-$summable to L provided*

$$\lim_{n\to\infty} \frac{1}{k_n - k_{n-1}} |\{k_{n-1} < p \leq k_n : M(|x_p - L|)^p \geq \epsilon\}| = 0$$

for all $\epsilon > 0$ (see [14]).

Note that the averages are taken over the disjoint intervals $(k_{n-1}, k_n]$, the preceding definition for statistical (γ, η, M, p) convergence does not require the intervals

$(\gamma(r), \eta(r)]$ to be disjoint. For instant, if $\gamma(r) = 0$ and $\eta(r) = r$, we have that $N(\gamma, \eta, M, p)$ and $S(\gamma, \eta, M, p)$ are strong Cesàro summability and statistical convergence for measurable functions as consider in [23] and [8].
Let $N(\gamma, \eta, M, p)$ and $S(\gamma, \eta, M, p)$ are strong Cesàro summability and statistical convergence for measurable functions by means of an modulus function. A function g is statistical convergent to L provided

$$\lim_{r\to\infty} \frac{1}{r}\mu\big(t \leq r : M(|g(t) - L|)^p \geq \epsilon\big) = 0$$

for all $\epsilon > 0$. Throughout this paper by $S - \lim g$ denote the statistical limit of g. If (γ, η) is a sliding window pair such that there is a function $\theta : \mathbb{N} \to (0, \infty)$ such that $\theta(n+1) - \theta(n)$ tends to infinity and a sequence (r_n) of real numbers for which, given $s \in (r_n, r_{n+1}]$, $\gamma(s) = \theta(n)$ and $\eta(s) = \theta(n + 1)$, then $N(\gamma, \eta, M, p)$ and $S(\gamma, \eta, M, p)$ will be denoted by $S_\theta(M, p)$ and $N_\theta(M, p)$. Let $I_n = (\theta(n), \theta(n + 1)]$ and observe that g is $S_\theta(M, p)-$ statistically convergent to L if and only if

$$\lim_{n\to\infty} \frac{1}{\mu(I_n)}|\{t \in I_n : M(|g(t) - L|)^p \geq \epsilon\}| = 0$$

for all $\epsilon > 0$. Note that the intervals I_n are pairwise disjoint in this special case. In keeping with the sequential method, the method $S_\theta(M, p)$ will be called lacunary statistical convergence. A similar construction of a pair (γ, η) can be used to show that $\lambda-$statistical convergence and $\lambda-$strong summability, as defined in [24], can be also be viewed as sliding window methods.
The objective of this paper is to introduce a class of summability methods that can be applied to measurable functions defined on $[0, \infty)$ by means of modulus function. These methods are known as sliding window methods are demonstrated on the methods of statistical convergence and lacunary statistical convergence by means of modulus function. We also establish some results for sequential summability to the setting of real valued functions defined on $[0, \infty)$.

2. Correlation between Strong summability and Statistical convergence

In this section we establish relationship between strong summability and statistical convergence.

Theorem 2.1. *Let M be a modulus function and p be a positive real numbers. Let (γ, η) be a sliding window pair, g be a measurable function and L be a real number.*
(i) If g is a $N(\gamma, \eta, M, p)$ summable to L, then g is $S(\gamma, \eta, M, p)$ convergent to L.
(ii) If g is bounded, then g is $N(\gamma, \eta, M, p)$ summable to L if and only if it is $S(\gamma, \eta, M, p)$ convergent to L.

Proof: Firstly we show that $N(\gamma, \eta, M, p) - \lim g = L$ implies $S(\gamma, \eta, M, p) - \lim g = L$. For $\epsilon > 0$, we have

$$\frac{1}{\mu(I_r)}\int_{I_r} M(|g(t) - L|)^p dt \geq \epsilon\frac{1}{\mu(I_r)}\mu(t \in I_r : M(|g(t) - L|)^p \geq \epsilon).$$

If g is bounded by B, then we have

$$\frac{1}{\mu(I_r)}\int_{I_r} M(|g(t)-L|)^p dt \leq \epsilon\frac{B}{\mu(I_r)}\mu(t\in I_r : M(|g(t)-L|)^p \geq \epsilon) + \frac{1}{\mu(I_r)}\int_{I_r}\epsilon dt.$$

As the first term of the right-hand side tends to 0 as r tends to infinity, it follows that $S(\gamma,\eta,M,p)-\lim g = L$ implies $N(\gamma,\eta,M,p)-\lim g = L$ when g is bounded. □

Theorem 2.2. *Let M be a modulus function and p be a positive real numbers. If g is $S(\gamma,\eta,M,p)$ convergent to L, then there is a measurable set A such that $S(\gamma,\eta,M,p)-\lim\chi_A = 1$ and $\lim M(g(t)-L)^p\chi_A(t) = 0$.*

Theorem 2.3. *Let M be a modulus function and p be a positive real numbers. Let (γ,η) be a sliding window pair. Then there is a measurable function g that is not $S(\gamma,\eta,M,p)$ convergent.*

Proof: We construct a set A such that its characteristic function, i.e., a function that only takes the value 0 or 1, is not $S(\gamma,\eta,M,p)$ convergent. Let r_1 be given and set $A_1 = (\gamma(r_1),\eta(r_1)]$. As A_1 is bounded, there is an $r_2 > r_1$ such that for all $s \geq r_2$ we have that

$$\mu(I_s\cap A_1) < \frac{1}{3}\mu(I_s).$$

Set $A_2 = (\gamma(r_2),\eta(r_2)]\backslash A_1$. Proceed inductively: given r_{n-1} and A_{n-1} select r_n such that $s \geq r_n$ implies that $\mu(I_s\cap\bigcup_{j=1}^{n-1}A_j) < (\frac{1}{3^{n-1}})\mu(I_s)$ and set $A_n = (\gamma(r_n),\eta(r_n)]\backslash\bigcup_{j=1}^{n-1}A_j$. Observe that

$$\mu(I_{r_n}\cap A_1) > (1-\frac{1}{3^{n-1}})\mu(I_{r_n}).$$

Now set $A = \bigcup_{j=1}^{\infty}A_{2j-1}$. Note that if n is odd, then

$$\frac{1}{\mu(I_{r_n})}\mu(I_{r_n}\cap A) \geq \frac{1}{\mu(I_{r_n})}\mu(I_{r_n}\cap A_n) > 1-\frac{1}{3^{n-1}}$$

and if n is even

$$\frac{1}{\mu(I_{r_n})}\mu(I_{r_n}\cap A) = \frac{1}{\mu(I_{r_n})}\mu\big(I_{r_n}\cap\bigcup_{j=1}^{\frac{n}{2}}A_{2j-1}\big) < \frac{1}{3^n}.$$

Define g by $g(t) = \chi_A(t)$ and select ϵ such that $0<\epsilon<1$. The above calculation shows that

$$\lim_n \frac{1}{\mu(I_{r_{2n-1}})}\mu(t\in I_{r_{2n-1}} : M(|g(t)-1|)^p \geq \epsilon) = 0$$

and that

$$\lim_n \frac{1}{\mu(I_{r_{2n}})}\mu(t\in I_{r_{2n}} : M(|g(t)|)^p \geq \epsilon) = 0$$

and thus, g is not $S(\gamma,\eta,M,p)$ convergent. □

Next we show that in general $S(\gamma,\eta,M,p)$ convergence is stronger than ordinary convergence and that $S(\gamma,\eta,M,p)$ convergence does not imply $N(\gamma,\eta,M,p)$ summability.

Theorem 2.4. *Let M be a modulus function and p be a positive real numbers. Let (γ,η) be a sliding window pair. Then there is a function g such that $\lim_t g(t)$ does not exist and $S(\gamma,\eta,M,p)-\lim g=0$ and an unbounded function h such that $S(\gamma,\eta,M,p)-\lim h=0$ and $N(\gamma,\eta,M,p)-\lim h$ does not exist.*

Proof: First we suppose that $0<\liminf_s(\eta(s)-\gamma(s))=b<\infty$. Select a sequence (s_j) that is increasing to infinity and
1. $s>s_j$ implies that $\eta(s)-\gamma(s))>b-\frac{1}{j}$,
2. $\eta(s_j)-\gamma(s_j)\le 2b$ and
3. $\eta(s_j)<\gamma(s_{j+1})$.
Note that the last choice is possible since $\eta(s_j)\to\infty$ as $j\to\infty$, $\eta(s_j)-\gamma(s_j)$ is bounded as function of j and hence $\gamma(s_j)\to\infty$ as $j\to\infty$. Also let δ_j be a sequence of positive numbers decreasing to 0 and such that $(\delta_j+\delta_{j+1})\backslash(b-\frac{1}{j})\to 0$ as $j\to\infty$. Set $H_j=(\gamma(j),\gamma(j)+\delta_j]$ and define the function g by $g(t)=1$ for $t\in\bigcup_j H_j$ and 0 otherwise. Note that $\lim_t g(t)$ does not exist.
We show that $S(\gamma,\eta,M,p)-\lim g=0$. Let $r\in[s_j,s_{j+1})$ and note that $\gamma(s_j)\le\gamma(r)\le\eta(r)\le\eta(s_{j+1})$ and hence for $\epsilon>0$,

$$\{t\in I_r : M(|g(t)|)^p\ge\epsilon\}\subset H_j\cup H_{j+1}.$$

As $\mu(H_j\cup H_{j+1})=\delta_j+\delta_{j+1}$ and $\mu(I_r)>b-\frac{1}{j}$, it follows that

$$\frac{1}{\mu(I_r)}\mu(\{t\in I_r : M(|g(t)|)^p\ge\epsilon\})\ge(\delta_j+\delta_{j+1})\setminus\big(b-\frac{1}{j}\big),$$

which tends to 0 as j tends to infinity. It follows that $S(\gamma,\eta,M,p)-\lim g=0$. Similarly we can show that $S(\gamma,\eta,M,p)-\lim h=0$ for any function h that has its support contained in $\bigcup_j H_j$. Now define a function h by

$$h(t)=\begin{cases}\delta_j^{-1}(\mu(I_{s_j})+2)^j, & t\in(\gamma(j),\gamma(j)+\delta_j];\\ 0, & \textit{otherwise}.\end{cases}$$

Observe that since for any L there is a J such that $j>J$ implies

$$\frac{1}{\mu(I_{s_j})}\int_{\gamma(s_j)}^{\eta(s_j)}M(|h(t)-L|)^p dt\ge\frac{1}{\mu(I_{s_j})}((\mu(I_{s_j})+2)^j-L)$$

which tends to infinity as j tends to infinity. Hence $N(\gamma,\eta,M,p)-\lim h=0$ does not exist and $S(\gamma,\eta,M,p)-\lim h=0$. Next we consider the case $\liminf_s(\eta(s)-\gamma(s))=\infty$. Select a sequence (s_j) increasing to infinity such that $\eta(s_1)>1$, $\eta(s_{j+1})>2\eta(s_j)$ and $r\ge s_j$ implies that $\eta(r)-\gamma(r)\ge 2^j$. Set $K_j=(\eta(s_j)-1,\eta(s_j)]$

and define the function g by $g(t) = 1$ for $t \in \bigcup_j K_j$ and 0 otherwise. Note that $\lim_t g(t)$ does not exist. We now demonstrate that $S(\gamma, \eta, M, p) - \lim g = 0$. Let $r \in [s_j, s_{j+1})$ and $\epsilon > 0$. Note that $\eta(r) \leq \eta(s_{j+1})$ implies that

$$\{t \in I_r : M(|g(t)|)^p dt \geq \epsilon\} \subset \bigcup_{l=1}^{j+1} K_l$$

and hence $\mu(t \in I_r : M(|g(t)|)^p \geq \epsilon) \leq j + 1$. Since $\mu(I_r) > 2^j$, it follows that

$$\frac{1}{\mu(I_r)} \mu(t \in I_r : M(|g(t)|)^p \geq \epsilon) < \frac{j+1}{2^j}$$

which tends to 0 as j tends to infinity. Hence $S(\gamma, \eta, M, p) - \lim g = 0$. As before, $S(\gamma, \eta, M, p) - \lim h = 0$ for any function that has its support contained in $\bigcup_{j=1} K_j$.

Define the function h by $h(t) = j(\eta(s_j) - \gamma(s_j))$ when $t \in K_j$ for some j and 0 otherwise. Observe that for any L there is a J such that $j > J$ implies

$$\frac{1}{\mu(I_{s_j})} \int_{\gamma(s_j)}^{\eta(s_j)} M(|h(t) - L|)^p dt \geq \frac{1}{\mu(I_{s_j})} \int_{\eta(s_j)-1}^{\eta(s_j)} M(|h(t) - L|)^p dt > j - 1.$$

Hence $N(\gamma, \eta, M, p) - \lim h$ does not exist and $S(\gamma, \eta, M, p) - \lim h = 0$. □

Next we compare sliding window methods to statistical convergence. The next examples establishes that in general statistical sliding window convergence is not equivalent to statistical convergence.

Example 2.5. *Let M be a modulus function and p be a positive real numbers. Let (γ, η) be a sliding window pair and a bounded function g with the property that g is $S(\gamma, \eta, M, p)$ convergent but neither strongly Cesàro summable nor statistical convergence. Let $M = I$ (identity) and $p = 1$. By using an inductive argument we constructed sliding window pair to generate a sequence (a_n). Let $a_1 = 1$ and select a_2 such that*

$$\frac{1}{a_2}(a_2 - a_1) > 1 - \frac{1}{2}$$

and a_3 such that

$$\frac{1}{a_3}(a_2 - a_1) < \frac{1}{2}.$$

Proceeding inductively: if $n - 1$ is odd, select a_n such that

$$\frac{1}{a_n} \sum_{j=1}^{\frac{n}{2}} (a_{2j} - a_{2j-1}) > 1 - \frac{1}{n}.$$

and if $n - 1$ is even select a_n such that

$$\frac{1}{a_n - a_{n-3}} \sum_{j=1}^{\frac{n-1}{2}} (a_{2j} - a_{2j-1}) < \frac{1}{n}.$$

Define the sliding window pair (γ, η) using the function $\gamma(t) = a_{2n}$ and $\eta(t) = a_{2n+3}$ for $t \in (n, n+1]$ and the function g by $g(t) = 1$ for $t \in (a_{n-1}, a_n]$ when n is even and 0 otherwise. We establish that the function g is $S(\gamma, \eta, M, p)$ convergent to 0 by observing, that for $s \in (p, p+1]$ and $\epsilon > 0$, we have that

$$\frac{1}{\eta(s) - \gamma(s)}\mu(t \in I_s : M(|g(t)|)^p \geq \epsilon) = \frac{a_{2p+2} - a_{2p+1}}{a_{2p+3} - a_{2p}} < \frac{1}{2p+3}.$$

which tends to 0 as s hence p tends to infinity. Recall that strong Cesàro summability and statistical convergence correspond respectively to $N(\gamma', \eta', M', p')$ and $S(\gamma', \eta', M', p')$ convergence when $M'(r) = I(identity)$, $p' = 1$, $\gamma'(r) = 0$, and $\eta'(r) = r$ for all positive real r. First we establish g is not strongly Cesàro summable. Observe that

$$\frac{1}{a_{2p}} \int_0^{2p} g(t)dt = \frac{1}{a_{2p}} \sum_{j=1}^{p} (a_{2j} - a_{2j-1}) > 1 - \frac{1}{2p}$$

which tends to 1 as p tends to infinity. Next note that

$$\frac{1}{a_{2p} + 3} \int_0^{2p+3} g(t)dt = \frac{1}{a_{2p+3}} \sum_{j=1}^{p+1} (a_{2j} - a_{2j-1}) < \frac{1}{2p+3}$$

which tends to 0 as p tends to infinity. Hence g is not strongly Cesàro summable. As g is bounded, it follows from Theorem 2.1 that g is not statistical convergent.

Lemma 2.6. *[9] Let $V = (0, v]$ be an interval and let $\mathfrak{U} = \{U_z : z \in Z\}$ be a collection of half-open, half-closed intervals such that $V = \bigcup\{U_z : z \in Z\}$ and there is a $b > 0$ such that $b < \mu(U_z)$ for all $z \in Z$. Then, for any $\epsilon > 0$, there is a finite, disjoint subcollection $\{U_1, ..., U_n\}$ of $\mathfrak{U}$ such that*

$$\sum_{i=1}^{n} \mu(U_i) > \frac{v}{3} - \epsilon.$$

Lemma 2.7. *[9] Let (γ, η) be a sliding window pair.*
1. If $Q < r$, then $(\eta(Q), \eta(r)] \subset \bigcup\{I_s : Q < s \leq r\}$.
2. If Q is sufficiently large and $\epsilon > 0$, then there is a finite disjoint subcollection $\{I_{s_i}\}$ of $\{I_s : Q < s \leq r\}$ such that

$$\sum_{i=1}^{n} \mu(I_{s_i}) > \frac{\eta(r)}{3} - \eta(Q) - \epsilon.$$

Note that example 2.5 shows that it is necessary to assume that the function is convergent with respect to both methods.

Theorem 2.8. *Let M be a modulus function and p be a positive real numbers. If g is statistically (γ, η) convergent and statistically convergent, then $S(\gamma, \eta, M, p) - \lim g = S - \lim g$.*

Proof: We assume without loss of generality, that g is a function such that $S(\gamma,\eta,M,p)-\lim g=1$ and $S-\lim g=0$ and derive a contradiction. First select Q such that $s>Q$ implies that $\mu(t\in I_s : M(|g(t)-1|)^p \geq \frac{3}{4}) < (\frac{1}{6})\mu(I_s)$ and hence $\mu(t\in I_s : M(|g(t)|)^p > \frac{1}{4}) \geq \mu(I_s)-(\frac{1}{6})\mu(I_s)=(\frac{5}{6})\mu(I_s)$. Since $\eta(r)\to\infty$, we may select $R>Q$ such that $r>R$ implies $\frac{\eta(Q)}{\eta(r)}<\frac{1}{12}$.
Let $0<\epsilon<\frac{1}{12}$, be given and using Lemma 2.7 select a finite disjoint collection of intervals $\{I_s : Q<s_i\leq r; i=1,......n\}$ such that

$$\sum_{i=1}^{n}\mu(I_{s_i})\geq\frac{\eta(r)}{3}-\eta(Q)-\epsilon.$$

Now

$$\begin{aligned}\mu\Big(t\leq\eta(r) : M(|g(t)|)^p>\frac{1}{4}\Big) &\geq \sum_{i=1}^{n}\mu\Big(t\in I_{s_i} : M(|g(t)|)^p>\frac{1}{4}\Big)\\ &\geq \Big(\frac{5}{6}\Big)\sum_{i=1}^{n}\mu(I_{s_i})\\ &> \Big(\frac{5}{6}\Big)\Big(\frac{\eta(r)}{3}-\eta(Q)-\epsilon\Big).\end{aligned}$$

Since $\epsilon<\frac{1}{12}$, we have

$$\frac{1}{\eta(r)}\mu\Big(t\leq\eta(r) : M(|g(t)|)^p>\frac{1}{4}\Big)>\frac{1}{9}.$$

Consequently, the statistical limit of g is not equal to 0 which contradicts the hypothesis on g. □

Theorem 2.9. *Let M be a modulus function and p be a positive real numbers. Let (γ,η) be a sliding window pair. The following are equivalent:*
(1) *If a function g is statistically convergent, then g is statistically (γ,η,M,p) convergent.*
(2) $\liminf\frac{\eta(r)}{\gamma(r)}>1$.

Proof: Firstly, we need to show (2) implies (1). By hypothesis we have $\liminf\frac{\eta(r)}{\gamma(r)}>1$ which yields that there are positive numbers Q and δ such that $r>Q$ implies that $\frac{\eta(r)}{\gamma(r)}>1+\delta$. Note that

$$\frac{\eta(r)-\gamma(r)}{\eta(r)}=1-\frac{\gamma(r)}{\eta(r)}>\frac{\delta}{1+\delta}$$

and hence

$$\frac{1}{\eta(r)}>\frac{\delta}{1+\delta}\frac{1}{\eta(r)-\gamma(r)}.$$

Now let $\epsilon > 0$ and $r > Q$ be given. Observe that

$$\frac{1}{\eta(r)}\mu\Big(t \leq \eta(r) : M(|g(t) - L|)^p \geq \epsilon\Big) \geq \frac{1}{\eta(r)}\mu\Big(t \in I_r : M(|g(t) - L|)^p \geq \epsilon\Big)$$
$$> \frac{\delta}{1+\delta}\frac{1}{\eta(r) - \gamma(r)}\mu\Big(t \in I_r : M(|g(t) - L|)^p \geq \epsilon\Big).$$

It follows that if $S - \lim g = L$, then $S(\gamma, \eta, M, p) - \lim g = L$. Next we show (1) imply (2). Let us assume that $\liminf \frac{\eta(r)}{\gamma(r)} = 1$ and construct a function g such that $S - \lim g = 0$ and $S(a, b) - \lim g$ does not exist. We start a number of increasing sequence (r_j) such that $r_j \to \infty$ and $\frac{\eta(r_j)}{\gamma(r_j)} \to 1$ as $j \to \infty$. We define ϵ_j by $\frac{\eta(r_j)}{\gamma(r_j)} = 1 + \epsilon_j$. Note that $\lim \epsilon_j = 0$ and $\gamma(r_j) = \frac{\eta(r_j)}{(1+\epsilon_j)}$ also observe that $\eta(r_j) \to \infty$ as $\gamma(r_j) \to \infty$. We construct a sequence of disjoint intervals $((\gamma_n, \eta_n])$ as follows. Set $\gamma_1 = \gamma(r_1)$ and $\eta_1 = \eta(r_1)$. Next select $s = r_j$ for some $j > 1$ such that

$$\frac{\eta_1}{\eta(s)} < \frac{\epsilon_1}{1+\epsilon_1} \text{ and } \gamma(s) > \eta_1.$$

Set $\gamma_2 = \gamma(s)$ and $\eta_2 = \eta(s)$. If γ_{n-1} and η_{n-1} have been selected, we choose $s = r_j$ for some j such that $\gamma(s) > \eta_{n-1}$ and $\frac{\eta_{n-1}}{\eta(s)} < \frac{\epsilon_{n-1}}{1+\epsilon_{n-1}}$. Set $\gamma_n = \gamma(s)$ and $\eta_n = \eta(s)$ Define g by $g(t) = 1$ if $t \in (\gamma_n, \eta_n]$ for some n and 0 otherwise. First we establish that $S - \lim g = 0$. Let $0 < \epsilon < 1$ and $r \in (0, \infty)$. Observe that if $r \in (\eta_{n-1}, \gamma_n]$ for some n then

$$\begin{aligned}
\frac{1}{r}\mu\Big(t \leq r : M(|g(t)|)^p \geq \epsilon\Big) &= \frac{1}{r}\sum_{j=1}^{n-1}(\eta_j - \gamma_j) \\
&\leq \frac{1}{\eta_{n-1}}\sum_{j=1}^{n-2}(\eta_j - \gamma_j) + \frac{1}{\eta_{n-1}}(\eta_{n-1} - \gamma_{n-1}) \\
&\leq \frac{\eta_{n-2}}{\eta_{n-1}} + \frac{\epsilon_{n-1}}{1+\epsilon_{n-1}} \leq \frac{\epsilon_{n-2}}{1+\epsilon_{n-2}} + \frac{\epsilon_{n-1}}{1+\epsilon_n}
\end{aligned}$$

which tends to 0 as n tends to infinity. Next suppose that $r \in (\gamma_n, \eta_n]$ for some n. In this case

$$\frac{1}{r}\mu\Big(t \leq r : M(|g(t)|)^p \geq \epsilon\Big) = \frac{1}{r}\sum_{j=1}^{n-1}(\eta_j - \gamma_j) + \frac{1}{r}(r - \gamma_n) \tag{2.1}$$

$$< \frac{\eta_{n-1}}{\gamma_n} + (1 - \frac{\gamma_n}{r}).$$

Now, since $\gamma_n = \frac{\eta_n}{1+\epsilon_n}$ and $r \leq \eta_n$, we have that

$$(2.1) \leq \frac{\eta_{n-1}}{\eta_n}(1+\epsilon_n) + \big(1 - \frac{\gamma_n}{\eta_n}\big) < \frac{\epsilon_{n-1}(1+\epsilon_n)}{1+\epsilon_{n-1}} + \frac{\epsilon_n}{1+\epsilon_{n-1}}$$

which also tends to 0 as n tends to infinity. As

$$\lim_{r} \frac{1}{r}\mu\Big(t \leq r : M(|g(t)|)^p \geq \epsilon\Big) = 0.$$

for all $\epsilon > 0$, we have that $S - \lim g = 0$. Now note that, by construction

$$\frac{1}{\eta_n - \gamma_n}\mu\Big(\gamma_n < t \leq \eta_n : M(|g(t)|)^p \geq \frac{1}{2}\Big) = 1$$

for all n, hence the $S(\gamma, \eta, M, p) - \lim g$ is either not equal to 0 or does not exist. By Theorem(2.8), since $S - \lim g = 0$, if $S(\gamma, \eta, M, p) - \lim g$ exists then $S(\gamma, \eta, M, p) - \lim g = 0$. Hence $S(\gamma, \eta, M, p) - \lim g$ does not exist. □

Theorem 2.10. *Let M be a modulus function and p be a positive real numbers. Let (γ, η) be a lacunary sliding window pair. The following are equivalent:*
(1) *If a function g is lacunary statistically convergent, then g is statistically convergent.*
(2) $\limsup \frac{\eta(r)}{\gamma(r)} < \infty$.

Proof: Since (γ, η) be a lacunary sliding window pair, there is a function $\theta : \mathbb{N} \to (0, \infty)$ such that $\theta(n+1) - \theta(n)$ tends to infinity and a sequence (r_n) of real numbers for which, given $s \in (r_n, r_{n+1}], \gamma(s) = \theta(n)$ and $\eta(s) = \theta(n+1)$. First we establish that (2) implies (1). Suppose that $S_\theta - \lim g = L$ and that $\limsup_r \frac{\eta(r)}{\gamma(r)} < \infty$, i.e., there is an $H > 0$ such that $\frac{\theta(l+1)}{\theta(l)} < H$ for all $l \in \mathbb{N}$. Throughout the proof we let $I_l = (\theta(l), \theta(l+1)]$. Let $\epsilon, \delta > 0$ be given. Select N such that $n > N$ implies that

$$\mu\Big(t \in I_n : M(|g(t) - L|)^p \geq \epsilon\Big) < \mu(I_n)\frac{\delta}{(2H)}$$

and Q such that $l > Q$ implies that $\theta(N+1) < \frac{\theta(l)\delta}{2}$. Now let $l > Q$ and $s \in (\theta(l), \theta(l+1)]$. Now that

$$0 \leq \frac{1}{s}\mu\Big(0 \leq t \leq s : M(|g(t) - L|)^p \geq \epsilon\Big) \tag{2.2}$$

$$\leq \frac{1}{\theta(l)}\mu\Big(0 \leq t \leq \theta(l+1) : M(|g(t) - L|)^p \geq \epsilon\Big)$$

and setting $J_i = \{t \in I_i : M(|g(t) - L|)^p \geq \epsilon\}$, we have that

$$0 \leq \frac{1}{\theta(l)}\mu\Big(0 \leq t \leq \theta(l+1) : M(|g(t) - L|)^p \geq \epsilon\Big) \tag{2.3}$$

$$= \tfrac{1}{\theta(l)}\mu\Big(\bigcup_{i=1}^{N} J_i + \bigcup_{i=N+1}^{l} J_i\Big) \leq \frac{\theta(N+1)}{\theta(l)} + \frac{1}{\theta(l)}\sum_{i=N+1}^{l} \mu(J_i).$$

Note that for $i > N$, $\mu(J_i) < (\frac{\delta}{2H})\mu(I_i)$ and $\mu(I_i) = \theta(i+1) - \theta(i)$. Thus,

$$\begin{aligned}
(2.3) \quad &\leq \frac{\theta(N+1)}{\theta(l)} + \frac{1}{\theta(l)}\frac{\delta}{2H}\sum_{i=N+1}^{l}(\theta(i+1) - \theta(i)) \\
&\leq \frac{\theta(N+1)}{\theta(l)} + \frac{\theta(l+1)}{\theta(l)}\frac{\delta}{2H} \\
&\leq \frac{\delta}{2} + H\frac{\delta}{2H} \\
&= \delta.
\end{aligned}$$

As $\delta > 0$ was arbitrary, it follows that $S - \lim g = L$. Next we need to show (2) implies (1). Let $S_\theta(M,p)$ be a lacunary method such that

$$\limsup_{l} \frac{\theta(l+1)}{\theta(l)} = \infty.$$

We will construct a function g such that $S_\theta - \lim g = 0$ and $S - \lim g = L$ does not exist. By assumption, there is a subsequence l_y such that $\frac{\theta(l_y+1)}{\theta(l_y)} > y$ for each y. Set $J_y = (\theta(l_y), 2\theta(l_y)]$ and define g by $g(t) = 1$ if $t \in J_y$ for some y and 0 otherwise. Note that $\mu(J_y) = \theta(l_y)$ and $\theta(l_y+1) > y\theta(l_y)$. Let $\epsilon > 0$ be given and suppose that $\theta(q) < s \leq \theta(q+1)$. If $q = l_y$ for some y, then

$$\frac{1}{\mu(I_s)}\mu\Big(t \leq I_s : M(|g(t)|)^p \geq \epsilon\Big) = \frac{\mu(J_q)}{\theta(q+1) - \theta(q)} = \frac{\theta(q)}{\theta(q+1) - \theta(q)}. \qquad (2.4)$$

Now note that $\theta(l_y+1) > y\theta(l_y)$ yield that $\theta(l_y+1) - \theta(l_y) > (y-1)\theta(l_y)$ hence if $q = l_y$, then (2.4) is less then $\frac{1}{y-1}$, which tends to 0 as q tends to ∞. Also note that if $q \neq l_y$, for all y, then $\mu\Big(t \leq I_s : M(|g(t)|)^p \geq \epsilon\Big) = 0$. Hence $S_\theta - \lim g = 0$. Next observe that for $0 < \epsilon < 1$,

$$\frac{1}{2\theta(l_y)}\mu\Big(0 < t \leq 2\theta(l_y) : M(|g(t)|)^p \geq \epsilon\Big) \geq \frac{\theta(l_y)}{2\theta(l_y)} = \frac{1}{2}$$

and hence $S_\theta - \lim g \neq 0$ or does not exist. As before, Theorem 2.8 yield that $S - \lim g$ does not exist. $\square$

3. Cauchy Criterion for $S(\gamma, \eta, M, p)-$convergence

In this section we make an effort to establish Cauchy criterion for $S(\gamma, \eta, M, p)$-convergence and a criterion for two sliding window methods to be equivalent for bounded functions.

Definition 3.1. *Let (γ, η) be a sliding window pair. The function g is said to be $S(\gamma, \eta, M, p)$-Cauchy if for every $r > 0$ there is an element $t_r \in I_r$ such that $\lim_r g(t_r)$ exists and*

$$\lim_{r\to\infty} \frac{1}{\mu(I_r)}\mu\Big(t \in I_r : M(|g(t) - g(t_r)|)^p \geq \epsilon\Big) = 0$$

for all $\epsilon > 0$.

Theorem 3.2. *Let* M *be a modulus function and* p *be a positive real numbers. Let* (γ, η) *be a sliding window pair and* g *be a measurable function. Then* $S(\gamma, \eta, M, p)-\lim g$ *exists if and only if* g *is* $S(\gamma, \eta, M, p)$*-Cauchy.*

Proof: First we establish that $S(\gamma, \eta, M, p)$-convergent functions are $S(\gamma, \eta, M, p)$-Cauchy. Let g be a function such that $S(\gamma, \eta, M, p) - \lim g = L$ and $j \in \mathbb{N}$. Set $K^j = \{0 \leq t < \infty : M(|g(t) - L|)^p < j^{-1}\}$ and observe that

$$(\frac{1}{\mu(I_r)})\mu(I_r \cap K^j) \to 1$$

as r tends to infinity for each $j \in \mathbb{N}$. Hence there is an increasing sequence of indices $R_1 < R_2 < ....$ such that $r > R_j$ implies $K^j \cap I_r \neq 0$, Let $t_r \in I_r \cap K^j$ for $R_j \leq r < R_{j+1}$. It is clear, by construction, that $\lim_r M(|g(t_r) - L|)^p = 0$. Now note that if $M(|g(t) - g(t_r)|)^p \geq \epsilon$ and $M(|g(t_r) - L|)^p < \frac{\epsilon}{2}$, then $M(|g(t) - L|)^p \geq \frac{\epsilon}{2}$, and hence $\{t : M(|g(t) - g(t_r)|)^p \geq \epsilon\} \subset \{t : M(|g(t) - L|)^p \geq \epsilon\}$. Select J such that $\frac{1}{J} < \frac{\epsilon}{2}$ and let $\delta > 0$ be given. Next select R' such that $r > R'$ implies that $M(|g(t_r) - L|)^p < \frac{\epsilon}{2}$ and $\mu\Big(t \leq I_r : M(|g(t) - L)^p \geq \frac{\epsilon}{2}\Big) < \delta\mu(I_r)$. It follows that

$$\frac{1}{\mu(I_r)}\mu\Big(t \in I_r : M(|g(t) - g(t_r)|)^p \geq \epsilon\Big)$$

$$\leq \frac{1}{\mu(I_r)}\mu\Big(t \in I_r : M(|g(t) - L|)^p \geq \frac{\epsilon}{2}\Big) < \delta$$

and hence g is $S(\gamma, \eta, M, p)$-Cauchy.
Conversely, suppose that if g is $S(\gamma, \eta, M, p)$-Cauchy, (t_r) as given in the definition and $\lim_r g(t_r) = L$. Let R' such that $r > R'$ implies $M(|g(t_r) - L|)^p < \frac{\epsilon}{2}$. Note that

$$0 \leq \frac{1}{\mu(I_r)}\mu\Big(t \in I_r : M(|g(t) - L|)^p \geq \epsilon\Big)$$

$$\leq \frac{1}{\mu(I_r)}\mu\Big(t \in I_r : M(|g(t) - g(t_r)|)^p \geq \frac{\epsilon}{2}\Big)$$

for $r > R'$. Hence $\lim_r \Big(\frac{1}{m_r}\Big)\mu\Big(t \leq I_r : M(|g(t) - L|)^p \geq \epsilon\Big) = 0$ and thus, $S(\gamma, \eta, M, p) - \lim g = L$. □

Theorem 3.3. *Let* M *be a modulus function and* p *be a positive real numbers. Let* (γ, η) *and* (γ', η') *be two sliding window pairs and* g *be a bounded measurable function. If there is a* $B > 0$ *such that*

$$|\gamma(t) - \gamma'(t)| + |\eta(t) - \eta'(t)| \leq B$$

for all t *and* $\lim_s(\eta(s) - \gamma(s)) = \infty$*, then* $g \to LN(\gamma, \eta, M, p)$ *if and only if* $g \to LN(\gamma', \eta', M, p)$ *and* $g \to LS(\gamma, \eta, M, p)$ *if and only if* $g \to LS(\gamma', \eta', M, p)$.

Proof: By Theorem 2.1, it suffices to demonstrate that $g \to LN(\gamma, \eta, M, p)$ if and only if $g \to LN(\gamma', \eta', M, p)$ and then it suffices to consider the case $L = 0$. First we establish that

$$\frac{1}{\mu(I'_r)} \int_{\gamma'(r)}^{\eta'(r)} M(|g(t)|)^p dt - \frac{1}{\mu(I_r)} \int_{\gamma(r)}^{\eta(r)} M(|g(t)|)^p dt \tag{3.1}$$

$$\longrightarrow 0 \text{ as } r \longrightarrow \infty.$$

Note that

$$\begin{aligned} (3.1) \quad &= \quad \frac{1}{\mu(I'_r)} \left(\int_{\gamma'(r)}^{\gamma(r)} M(|g(t)|)^p dt - \int_{\eta'(r)}^{\eta(r)} M(|g(t)|)^p dt \right) \\ &+ \quad \left(\frac{\mu(I_r) - \mu(I'_r)}{\mu(I'_r)} \right) \frac{1}{\mu(I_r)} \int_{\gamma(r)}^{\eta(r)} M(|g(t)|)^p dt. \end{aligned}$$

Now suppose that $M(|g(t)|)^p \leq V$ for all $t \geq 0$. Then

$$\begin{aligned} \frac{1}{\mu(I'_r)} \left(\int_{\gamma'(r)}^{\gamma(r)} M(|g(t)|)^p dt - \int_{\eta'(r)}^{\eta(r)} M(|g(t)|)^p dt \right) &\leq \frac{V}{\mu(I'_r)} \Big(|\gamma(t) - \gamma'(t)| + |\eta(t) - \eta'(t)| \Big) \\ &\leq \frac{BV}{\mu(I'_r)} \end{aligned}$$

and

$$\left(\frac{\mu(I_r) - \mu(I'_r)}{\mu(I'_r)} \right) \frac{1}{\mu(I_r)} \int_{\gamma(r)}^{\eta(r)} M(|g(t)|)^p dt \leq \frac{BV}{\mu(I'_r)}.$$

Since $\lim_r \mu(I'_r) = \infty$, both terms tends to 0 as r tends to ∞. It follows that $g \to 0N(\gamma, \eta, M, p)$ if and only if $g \to 0N(\gamma', \eta', M, p)$. $\square$

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