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Existence of entropy solutions for degenerate elliptic unilateral problems with variable exponents

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ABSTRACT: In this article, we study the following degenerate unilateral problems:

$$-\operatorname{div}(a(x,\nabla u)) + H(x,u,\nabla u) = f,$$

which is subject to the weighted Sobolev spaces with variable exponent $W_0^{1,p(x)}(\Omega,\omega)$, where ω is a weight function on Ω, (ω is a measurable, a.e. strictly positive function on Ω and satisfying some integrability conditions). The function $H(x,s,\xi)$ is a nonlinear term satisfying some growth condition but no sign condition and the right hand side $f \in L^1(\Omega)$.

Key Words: Entropy solutions, weighted variable exponent Sobolev spaces, unilateral problem.

Contents

1. Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ and $p \in \mathcal{C}^+(\overline{\Omega})$.
This paper will be concerned with the existence of entropy solutions of the following nonlinear unilateral elliptic problems

$$\begin{cases} u \text{ is a measurable function such that } u \geq \psi \quad \text{a.e. in } \Omega,\ T_k(u) \in W_0^{1,p(x)}(\Omega,\omega), \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(\varphi - u)dx + \int_\Omega H(x,u,\nabla u)T_k(\varphi - u)dx \geq \int_\Omega f T_k(\varphi - u)dx \\ \forall \varphi \in K_\psi \cap L^\infty(\Omega), \end{cases} \tag{1.1}$$

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where

$$K_\psi = \left\{u \in W_0^{1,p(x)}(\Omega,\omega),\, u \geq \psi \text{ a.e. in } \Omega\right\}$$

with a measurable function ψ such that

$$\psi^+ \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega). \tag{1.2}$$

We make the following assumptions on a, H and f:

The function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

$$|a(x,\xi)| \leq \beta\omega(x)^{\frac{1}{p(x)}}(k(x) + \omega(x)^{\frac{1}{p'(x)}}|\xi|^{p(x)-1}), \tag{1.3}$$

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) > 0 \quad \forall\xi \neq \eta, \tag{1.4}$$

$$a(x,\xi)\xi \geq \alpha\omega(x)|\xi|^{p(x)}, \tag{1.5}$$

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$, where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying:

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)\omega(x)|\xi|^{p(x)}, \tag{1.6}$$

where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and $\gamma(x)$ belongs to $L^1(\Omega)$.

Furthermore, we suppose that

$$f \in L^1(\Omega). \tag{1.7}$$

In various applications (such as in elasticity, non-Newtonian fluids the flow through porous media and image processing), we can meet boundary value obstacle problems like problem (1.1) for elliptic equations whose ellipticity is "disturbed" in the sense that some degeneration or singularity appears. This "bad" behavior can be caused by the coefficients of the corresponding differential operator. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces. Many of these models have already been analyzed for constant exponents of nonlinearity but it seems to be more realistic to assume the exponent to be variable.

Under our assumptions, problem (1.1) does not admit, in general, a weak solution since the term $a(x,\nabla u)$ may not belong to $(L^1_{loc}(\Omega))^N$. In order to overcome this difficulty, we work with the framework of entropy solutions. This notion was first introduced by Sanchón and Urbano [20] who studied a Dirichlet problem of $p(x)$-Laplace equation and obtained the existence and the uniqueness of entropy solutions for L^1 data. The paper of Sanchón and Urbano showed the way to study the notion of entropy solutions to problems in variable exponent spaces with Dirichlet homogeneous boundary-value conditions (see e.g. [3,4,5,6,17,23]). At the same time, the theory regarding the weighted Sobolev spaces with variable exponent $p(x)$, i.e. $W_0^{1,p(x)}(\Omega,\omega)$ have been introduced in [15] and [1].

The first goal of this paper is to show the existence of entropy solutions for (1.1) in the weighted variable exponent Sobolev spaces, using the approximation ways under the conditions on a, H, f introduced above and certain assumptions on ω that will be specified later. We shall make use of the properties for the weighted variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega,\omega)$ proven in [14]. However, This manuscript generalized the results in [12,22] to the obstacle case and generalized the results in [18] to the weighted case.

The main difficulty in proving the existence of a solution stems from the fact that $H(x,u,\nabla u)$ does not assume the sign condition (i.e. $H(x,s,\xi)s \geq 0$). Otherwise, the term $H(x,u,\nabla u)$ is said to be an absorption term, in this case a detailed picture of what happens is available (see e.g. [4,6,8,9,10,11]).

The plan of our paper is as follows: In Section 2, we give some preliminaries and notations. In Section 3, the existence of entropy solutions of (1.1) is obtained.

2. Abstract framework

In this section, we will introduce an adequate functional space where problems of type (1.1) can be studied. Such a space will be called weighted Sobolev spaces with variable exponent $W^{1,p(x)}(\Omega,\omega)$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$.

Set $\mathcal{C}_+(\overline{\Omega}) = \{p | p \in \mathcal{C}(\overline{\Omega}),\ p(x) > 1 \text{ for any } x \in \bar{\Omega}\}$.

Let ω be a measurable positive and a.e. finite function defined in R^N. Further, in all this section, we suppose that the following integrability conditions are satisfied

$$(H1) : \omega \in L^1_{loc}(\Omega),$$

$$(H2) : \omega^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega),$$

$$(H3) : \omega^{-s(x)} \in L^1_{loc}(\Omega), \text{where} \quad s(x) \in \Big(\frac{N}{p(x)}, \infty\Big) \cap \Big[\frac{1}{p(x)-1}, \infty\Big).$$

The reasons that we assume $(H1)$, $(H2)$ and $(H3)$ can be found in [14]. By $L^{p(x)}(\Omega,\omega)$ we denote the weighted space of measurable functions $u(x)$ on Ω such that

$$\int_\Omega |u(x)|^{p(x)}\omega(x)dx < \infty,$$

where $p \in \mathcal{C}_+(\overline{\Omega})$, $1 \leq p_- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p_+ := \sup_{x\in\Omega} p(x) \leq \infty$ and ω is the weight function. This is a Banach function space with respect to the norm

$$\|u\|_{p(x),\omega} = \inf\Big\{\lambda > 0,\ \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\omega(x)dx \leq 1\Big\}.$$

We denote by $L^{p'(x)}(\Omega,\omega^*)$ the conjugate space of $L^{p(x)}(\Omega,\omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and where $\omega^*(x) = \omega(x)^{1-p'(x)}$.

Proposition 2.1. *Denote*

$$I_\omega(u) = \int_\Omega |u|^{p(x)}\omega(x)dx, \quad \forall u \in L^{p(x)}(\Omega,\omega).$$

Then the following assertions hold:

(i) $\|u\|_{p(x),\omega} < 1$ *(resp. = 1 or* > 1*) if and only if* $I_\omega(u) < 1$ *(resp. = 1 or* > 1*)*

(ii) $\|u\|_{p(x),\omega} > 1$ *implies* $\|u\|^{p-}_{p(x),\omega} \leq I_\omega(u) \leq \|u\|^{p+}_{p(x),\omega}$*, and* $\|u\|_{p(x),\omega} < 1$ *implies* $\|u\|^{p+}_{p(x),\omega} \leq I_\omega(u) \leq \|u\|^{p-}_{p(x),\omega}$

(iii) $\|u\|_{p(x),\omega} \to 0$ *if and only if* $I_\omega(u) \to 0$*, and* $\|u\|_{p(x),\omega} \to \infty$ *if and only if* $I_\omega(u) \to \infty$.

Proof. By taking $I_\omega(u) = I(\omega^{\frac{1}{p(x)}}u)$, where $I(u) = \int_\Omega |u|^{p(x)}dx$ and $\|\omega^{\frac{1}{p(x)}}u\|_{p(x)} = \|u\|_{p(x),\omega}$, we can prove Proposition 2.1 as a consequence of the corresponding one in [13]. □

We define the weighted Sobolev space with variable exponent by

$$W^{1,p(x)}(\Omega,\omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega,\omega)\}.$$

with the norm

$$\|u\|_{1,p(x),\omega} = \|u\|_{p(x)} + \|\nabla u\|_{p(x),\omega} \quad \forall u \in W^{1,p(x)}(\Omega,\omega).$$

We denote by $W_0^{1,p(x)}(\Omega,\omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega,\omega)$ and $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Remark 2.2. *Under the assumptions (H1)–(H3), we can prove the following results which will be used later. It is worth pointing out that the conditions (H1) and (H2) are essential. Without it the space* $W^{1,p(x)}(\Omega,\omega)$ *is not necessarily a Banach space even though* $p(x)$ *is a constant.*

Proposition 2.3. *[14] Let* $\Omega \in \mathbb{R}^N$ *be an open set,* $p \in \mathcal{C}_+(\overline{\Omega})$*. If (H1) and (H2) holds, then*

$$L^{p(x)}(\Omega,\omega) \hookrightarrow L^1_{loc}(\Omega)$$

Proposition 2.4. *[14]. If* $(H1)$ *and* $(H2)$ *holds, then* $W^{1,p(x)}(\Omega,\omega)$ *is a separable and reflexive Banach space.*

For $p, s \in \mathcal{C}^+(\overline{\Omega})$, set

$$p_s(x) = \frac{p(x)s(x)}{1+s(x)} < p(x),$$

where $s(x)$ is given in $(H3)$. Put

$$p_s^*(x) = \begin{cases} \dfrac{p(x)s(x)N}{(1+s(x))N - p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \leq p_s(x). \end{cases}$$

for almost all $x \in \Omega$.

Proposition 2.5 ([14]). *Let $p, s \in \mathcal{C}^+(\overline{\Omega})$, $1 < p_- \leq p_+ < \infty$ and let $(H1), (H2)$ and $(H3)$ be satisfied, then we have the continuous embedding*

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow W^{1,p_s(x)}(\Omega).$$

Moreover, we have the compact embedding

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$$

provided that $r \in \mathcal{C}^+$ and $1 \leq r(x) < p_s^*(x)$ for all $x \in \Omega$.

3. Some technical Lemmas

Lemma 3.1 ([7]). *Let $g \in L^{p(x)}(\Omega, \omega)$ and $g_n \in L^{p(x)}(\Omega, \omega)$ with $\|g_n\|_{p(x),\omega} \leq C$ for $1 < p(x) < \infty$. If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ in $L^{p(x)}(\Omega, \omega)$.*

Lemma 3.2. *Assume that (1.3)–(1.5), and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega, \omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega, \omega)$ and*

$$\int_\Omega [a(x, \nabla u_n) - a(x, \nabla u)]\nabla(u_n - u)dx \to 0. \tag{3.1}$$

Then $u_n \to u$ strongly in $W_0^{1,p(x)}(\Omega, \omega)$.

Proof. Let $D_n = [a(x, \nabla u_n) - a(x, \nabla u)]\nabla(u_n - u)$. We have D_n is a positive function, and by (3.1) $D_n \to 0$ in $L^1(\Omega)$. Extracting a subsequence, still denoted by u_n, we can write $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega, \omega)$ which implies $u_n \to u$ a.e. in Ω, and since $D_n \to 0$ a.e. in Ω, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \setminus B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $k(x) < \infty$, $u_n(x) \to u(x)$, $D_n(x) \to 0$.

Defining $\xi_n = \nabla u_n(x)$, $\xi = \nabla u(x)$, we have

$$\begin{aligned} D_n(x) &= [a(x, \xi_n) - a(x, \xi)](\xi_n - \xi) \\ &= a(x, \xi_n)\xi_n + a(x, \xi)\xi - a(x, \xi_n)\xi - a(x, \xi)\xi_n \\ &\geq \alpha\omega(x)|\xi_n|^{p(x)} + \alpha\omega(x)|\xi|^{p(x)} - \beta\omega(x)^{\frac{1}{p(x)}}(k(x) + |\xi_n|^{p(x)-1})|\xi| \\ &\quad - \beta\omega(x)^{\frac{1}{p(x)}}(k(x)|\xi|^{p(x)-1})|\xi_n| \\ &\geq \alpha|\xi_n|^{p(x)} - C_x\big[1 + |\xi_n|^{p(x)-1} + |\xi_n|\big], \end{aligned} \tag{3.2}$$

where C_x is a constant depending on x, without dependence on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by the standard argument $|\xi_n|$ is bounded uniformly with respect to n, then we deduce that

$$D_n(x) \geq |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big). \tag{3.3}$$

If $|\xi_n| \to \infty$ (for a subsequence), which is absurd since $D_n(x) \to \infty$. Let now ξ^* be an accumulation point of ξ_n, we have $|\xi^*| < \infty$ and by the continuity of a we obtain

$$[a(x, \xi^*) - a(x, \xi)](\xi^* - \xi) = 0. \tag{3.4}$$

In view of (1.4), we have $\xi^* = \xi$, which implies that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega. \tag{3.5}$$

Since the sequence $a(x, \nabla u_n)$ is bounded in $(L^{p'(x)}(\Omega, \omega^*))^N$ and $a(x, \nabla u_n) \to a(x, \nabla u)$ a.e. in Ω, by Lemma 3.1, we can establish that

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \quad \text{in } (L^{p'(x)}(\Omega, \omega^*))^N \ \text{ a.e. in } \Omega. \tag{3.6}$$

We set $\bar{y}_n = a(x, \nabla u_n)\nabla u_n$ and $\bar{y} = a(x, \nabla u)\nabla u$. We can write

$$\bar{y}_n \to \bar{y} \text{ in } L^1(\Omega).$$

We have

$$a(x, \nabla u_n)\nabla u_n \geq \alpha\omega(x)|\nabla u_n|^{p(x)}.$$

Let $z_n = \omega(x)|\nabla u_n|^{p(x)}$, $z = \omega(x)|\nabla u|^{p(x)}$, $y_n = \frac{\bar{y}_n}{\alpha}$ and $y = \frac{\bar{y}}{\alpha}$. Then by Fatou's Lemma,

$$\int_\Omega 2y\, dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\, dx, \tag{3.7}$$

i.e., $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z| dx$. Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z| dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z| dx \leq 0, \tag{3.8}$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(\Omega, \omega))^N. \tag{3.9}$$

Hence $u_n \to u$ in $W_0^{1,p(x)}(\Omega, \omega)$, which completes the proof. □

Lemma 3.3 ([4]). *Let $F : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function with $F(0) = 0$ and $p \in \mathcal{C}_+(\bar{\Omega})$. If $u \in W_0^{1,p(x)}(\Omega, \omega)$, then $F(u) \in W_0^{1,p(x)}(\Omega, \omega)$, moreover, if D is the set of discontinuous points of F' is finite, then*

$$\frac{\partial (F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

The following Lemma is a direct deduction from Lemma 3.3.

Lemma 3.4. *Let $u \in W_0^{1,p(x)}(\Omega, \omega)$ then $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ lie in $W_0^{1,p(x)}(\Omega)$. Moreover*

$$\frac{\partial u^+}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} & \text{if } u > 0 \\ 0 & \text{if } u \leq 0, \end{cases} \qquad \frac{\partial u^-}{\partial x_i} = \begin{cases} 0 & \text{if } u \geq 0 \\ -\frac{\partial u}{\partial x_i} & \text{if } u < 0. \end{cases}$$

Remark 3.5. *We feel that the techniques needed to obtain the proofs of Lemma 3.1 and Lemma 3.3 can be done by a slight modifications of the corresponding ones in [7] and [4].*

4. Existence result of entropy solutions

In this section, we study the existence of entropy solutions to problem (1.1) when the right-hand side $f \in L^1(\Omega)$.
We first recall some notations. In the following let T_k denotes the truncation function at height $k \geq 0$: $T_k(r) = \min(k,\ \max(r,\ -k))$, and define

$$T_0^{1,p(x)}(\Omega,\omega) = \big\{u \text{ measurable in } \Omega : T_k(u) \in W_0^{1,p(x)}(\Omega,\omega),\ \forall\ k > 0\big\}.$$

Let us first define the entropy solution of our problem.

Definition 4.1. A measurable function $u \in T_0^{1,p(x)}(\Omega,\omega)$ is called an entropy solution of the obstacle problem (1.1) for $\{f,\psi\}$ if $u \geq \psi$ a.e. in Ω and for every $k \geq 0$,

$$\int_\Omega a(x,\nabla u)\nabla T_k(\varphi-u)dx + \int_\Omega H(x,u,\nabla u)T_k(\varphi-u)dx \geq \int_\Omega fT_k(\varphi-u)dx$$

for every $\varphi \in K_\psi \cap L^\infty(\Omega)$.

Now we shall prove the following existence theorem.

Theorem 4.2. *Assume that (1.2)–(1.7) and (H1)–(H3) hold. Then there exists at least one entropy solution of the problem (1.1).*

4.1. Approximate problem

To prove existence of a solution to (1.1) we introduce approximating problems for which existence is easy to prove. To this end, let Ω_n be a sequence of compact subsets of Ω such that Ω_n is increasing to Ω as $n \to \infty$, and let (f_n) be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. Then we consider the following approximate problems

$$\begin{gathered} u_n \in K_\psi \\ \int_\Omega a(x,\nabla u_n)\nabla(u_n-v)dx + \int_\Omega H_n(x,u_n,\nabla u_n)(u_n-v)dx \leq \int_\Omega f_n(u_n-v)dx \end{gathered} \tag{4.1}$$

for all $v \in K_\psi$, where

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1+\frac{1}{n}|H(x,s,\xi)|}\chi_{\Omega_n}$$

with χ_{Ω_n} is the characteristic function of Ω_n. Note that $|H_n(x,s,\xi)| \leq |H(x,s,\xi)|$ and $|H_n(x,s,\xi)| \leq n$.

Theorem 4.3. *For fixed n, the approximate problem (4.1) has at least one solution.*

Proof. Let $X = K_\psi$, we define the operator $G_n : X \to X^*$ by

$$\langle G_n u, v\rangle = \int_\Omega H_n(x,u,\nabla u)\, v dx.$$

We have for all $u, v \in X$,

$$\Big|\int_\Omega H_n(x,u,\nabla u)v\,dx\Big| \le \Big|\int_\Omega n|v|dx\Big| \le n\|v\|_{L^1(\Omega)} \le C\|v\|_{L^{p(x)}(\Omega)}.$$

Hence the operator G_n is bounded.

We may adopt the same procedure as in [21] to deduce that the operator $B_n = A + G_n$ is pseudo-monotone.

Next, for the coerciveness of B_n, we want to show that

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,p(x),\omega}} \to +\infty \quad \text{if } \|v\|_{1,p(x),\omega} \to \infty \text{ for } v, v_0 \in K_\psi.$$

For this, let $v_0 \in K_\psi$, we use Hölder inequality and the growth condition to have

$$\begin{aligned}\langle Av, v_0\rangle &= \int_\Omega a(x,\nabla v)\nabla v_0\,dx\\ &= \int_\Omega a(x,\nabla v)\omega(x)^{\frac{-1}{p(x)}}\nabla v_0\omega(x)^{\frac{1}{p(x)}}dx\\ &\le C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\Big(\int_\Omega |a(x,\nabla v)|^{p'(x)}\omega(x)^{\frac{-p'(x)}{p(x)}}\Big)^{\theta'}\|v_0\|_{W_0^{1,p(x)}(\Omega,\omega)}\\ &\le C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|v_0\|_{W_0^{1,p(x)}(\Omega,\omega)}\Big(\int_\Omega \beta(K(x)^{p'(x)}+|\nabla v|^{p(x)}\omega(x))\Big)^{\theta'}\\ &\le C_0(C_1+I_\omega(\nabla v))^{\theta'},\end{aligned}$$

where

$$\theta' = \begin{cases} \frac{1}{p'^-} & \text{if } \|a(x,\nabla v)\|_{L^{p'(x)}(\Omega,\omega^*)} \ge 1,\\ \frac{1}{p'^+} & \text{if } \|a(x,\nabla v)\|_{L^{p'(x)}(\Omega,\omega^*)} \le 1.\end{cases} \tag{4.2}$$

Relation (1.5), gives

$$\frac{\langle Av, v\rangle}{\|v\|_{1,p(x),\omega}} - \frac{\langle Av, v_0\rangle}{\|v\|_{1,p(x),\omega}} \ge \frac{1}{\|v\|_{1,p(x),\omega}}(\alpha I_\omega(\nabla v) - C_0(C_1+I_\omega(\nabla v))^{\theta'}). \tag{4.3}$$

Therefore $\dfrac{I_\omega(\nabla v)}{\|v\|_{1,p(x),\omega}} \to \infty$ as $\|v\|_{1,p(x),\omega} \to \infty$.

Since $\dfrac{< G_n v, v >}{\|v\|_{1,p(x),\omega}}$ and $\dfrac{< G_n v, v_0 >}{\|v\|_{1,p(x),\omega}}$ are bounded, then we can write

$$\frac{\langle B_n v, v-v_0\rangle}{\|v\|_{1,p(x),\omega}} = \frac{\langle Av, v-v_0\rangle}{\|v\|_{1,p(x),\omega}} + \frac{\langle G_n v, v\rangle}{\|v\|_{1,p(x),\omega}} - \frac{\langle G_n v, v_0\rangle}{\|v\|_{1,p(x),\omega}} \to \infty \qquad \text{as } \|v\|_{1,p(x),\omega} \to \infty.$$

Finally, we conclude that B_n is pseudo-monotone and coercive. As a consequence of [16, Theorem 8.2], there exists at least one solution of the approximate problem (4.1). □

4.1.1. A priori estimates.

Proposition 4.4. *Assume that (1.3)–(1.7) and (H1)–(H3) hold, and let u_n be a solution of the approximate problem (4.1). Then, there exists a constant C (which does not depend on n and k) such that*

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\omega(x)dx \leq Ck \quad \forall\ k > 0.$$

Proof. Let $v = u_n - \eta \exp(G(u_n))T_k(u_n^+ - \psi^+)$ where $G(s) = \int_0^s \frac{g(t)}{\alpha}dt$ and $\eta \geq 0$. As a consequence v belongs to $\in W_0^{1,p(x)}(\Omega, \omega)$, and for η small enough we obtain $v \geq \psi$ and then it is an admissible test function in (4.1). It follows that

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla\Big(\exp(G(u_n))T_k(u_n^+ - \psi^+)\Big)dx \\
&+ \int_\Omega H_n(x, u_n, \nabla u_n)\exp(G(u_n))T_k(u_n^+ - \psi^+)dx \\
&\leq \int_\Omega f_n \exp(G(u_n))T_k(u_n^+ - \psi^+)dx
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))T_k(u_n^+ - \psi^+)dx \\
&+ \int_\Omega a(x, \nabla u_n)\nabla T_k(u_n^+ - \psi^+)\exp(G(u_n))dx \\
&\leq -\int_\Omega H_n(x, u_n, \nabla u_n)\exp(G(u_n))T_k(u_n^+ - \psi^+)dx \\
&\quad + \int_\Omega f_n \exp(G(u_n))T_k(u_n^+ - \psi^+)dx \\
&\leq \int_\Omega (f_n + \gamma(x))\exp(G(u_n))T_k(u_n^+ - \psi^+)dx \\
&\quad + \int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)dx.
\end{aligned}$$

From (1.5) and the fact that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ and $\gamma \in L^1(\Omega)$, we deduce that

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n^+ - \psi^+)\exp(G(u_n))dx \\
&\leq \int_\Omega f_n \exp(G(u_n))T_k(u_n^+ - \psi^+)\,dx + \int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)\,dx \\
&\leq (\|f\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)})\exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})k \leq C_1 k,
\end{aligned} \tag{4.4}$$

where C_1 is a positive constant. Consequently,

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla u_n^+ \exp(G(u_n))dx$$
$$\le \int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla \psi^+ \exp(G(u_n))dx + C_1 k.$$

Using Young's inequality together with assumption (1.5) yield

$$\int_{\{|u_n^+-\psi^+|\le k\}} |\nabla u_n^+|^{p(x)}\omega(x)dx \le C_2 k. \tag{4.5}$$

Since $\{x\in\Omega, |u_n^+|\le k\}\subset\{x\in\Omega, |u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$, it follows that

$$\int_\Omega |\nabla T_k(u_n^+)|^{p(x)}\omega(x)dx = \int_{\{|u_n^+|\le k\}} |\nabla u_n^+|^{p(x)}\omega(x)dx \le \int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} |\nabla u_n^+|^{p(x)}\omega(x)dx$$

Moreover, (4.5) implies

$$\int_{\{u_n\ge 0\}} |\nabla T_k(u_n)|^{p(x)}\omega(x)dx \le C_3(k+\|\psi^+\|_\infty), \quad \forall k>0, \tag{4.6}$$

where C_3 is a positive constant.

On the other hand, taking $v = u_n + \exp(-G(u_n))T_k(u_n^-)$ as a test function in (4.1), we obtain

$$-\int_\Omega a(x,\nabla u_n)\nabla(\exp(-G(u_n))T_k(u_n^-))dx$$
$$-\int_\Omega H_n(x,u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n^-)dx$$
$$\le -\int_\Omega f_n \exp(-G(u_n))T_k(u_n^-)dx.$$

Using (1.6), we have

$$\int_\Omega a(x,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(-G(u_n))T_k(u_n^-)dx$$
$$-\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))dx$$
$$\le \int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n^-)dx + \int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(-G(u_n))T_k(u_n^-)dx$$
$$-\int_\Omega f_n\exp(-G(u_n))T_k(u_n^-)dx.$$

By the same way as in (4.4), we get

$$-\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))dx$$
$$= \int_{\{u_n\le 0\}} a(x,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))dx \le C_3 k.$$

Using again (1.5), we deduce that

$$\int_{\{u_n\le 0\}} |\nabla T_k(u_n)|^{p(x)}\omega(x)dx \le C_4 k, \tag{4.7}$$

where C_4 is a constant positive. Combining (4.6) and (4.7), we conclude that

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\omega(x)dx \le Ck \quad \text{with} \quad C>0. \tag{4.8}$$

The above inequality implies that

$$\|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \le (Ck)^{\theta''}, \tag{4.9}$$

with

$$\theta'' = \begin{cases} 1/p^- & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \ge 1 \\ 1/p^+ & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \le 1. \end{cases} \tag{4.10}$$

□

4.1.2. Strong convergence of truncations.

Proposition 4.5. *There exist a measurable function u such that*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in } W_0^{1,p(x)}(\Omega,\omega).$$

The proof of the above proposition is done in two steps.

Step 1. First we will show that $(u_n)_n$ is a Cauchy sequence in measure. Let $k>0$ be large enough and B_R a ball. Combining the generalized Hölder inequality and Poincaré inequality, one has

$$\begin{aligned} k \text{ meas}(\{|u_n|>k\}\cap B_R) &= \int_{(\{|u_n|>k\}\cap B_R)} |T_k(u_n)|dx \\ &\le C\|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \\ & C\Big(\int_\Omega |T_k(u_n)|^{p(x)}\omega(x)dx\Big)^{\frac{1}{\gamma}} \\ &\le Ck^{1/\gamma}, \end{aligned} \tag{4.11}$$

where

$$\gamma = \begin{cases} 1/p^- & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \ge 1 \\ 1/p^+ & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \le 1. \end{cases} \tag{4.12}$$

Which yields,

$$\text{meas}(\{|u_n|>k\}\cap B_R) \le C\frac{1}{k^{1-\frac{1}{\gamma}}} \to 0 \quad \text{as } k\to\infty. \tag{4.13}$$

Moreover, we have, for every $\delta > 0$,

$$\begin{aligned} \text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) \leq \text{meas}(\{|u_n| > k\} \cap B_R) + \text{meas}(\{|u_m| > k\} \cap B_R) \\ + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \delta\} \cap B_R). \end{aligned}$$

By (4.13), we deduce that for all $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}(\{|u_n| > k\} \cap B_R) \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}(\{|u_m| > k\} \cap B_R) \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0. \quad (4.14)$$

Since (4.8), $T_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega, \omega)$, then there exists a subsequence still denoted $T_k(u_n)$ such that $T_k(u_n)$ converges to η_k weakly in $W_0^{1,p(x)}(\Omega, \omega)$, as n goes to ∞, strongly in $L^{p(x)}(\Omega)$ (because $p(x) \leq p_s^*(x)$), and a.e. in Ω. Thus, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure, then there exists n_0 which depend on δ and ε such that

$$\text{meas}(\{|T_k(u_n) - T_k(u_m)| > \delta\} \cap B_R) \leq \frac{\varepsilon}{3} \quad \forall m, n \geq n_0 \text{ and } k \geq k_0. \quad (4.15)$$

Let $\varepsilon > 0$. Then, by combining(4.14) and(4.15), we obtain

$$\text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) \leq \varepsilon \quad \forall n, \ m \geq n_0(k_0, \delta, R).$$

Then u_n is a Cauchy sequence in measure, thus, there exists a subsequence still denoted u_n which converges almost everywhere to some measurable function u, and by Lemma 3.1, we obtain

$$T_k(u_n) \to T_k(u) \text{ weakly in } W_0^{1,p(x)}(\Omega, \omega) \text{ and strongly in } L^{p(x)}(\Omega). \quad (4.16)$$

Step 2. In order to prove the strong convergence of truncation $T_k(u_n)$, let show the following intermediate result which is proved in the appendix.

Lemma 4.6. *There exist a subsequence of u_n solution of problem (4.1) satisfies, for any $k \geq 0$,*
Assertion (i):

$$\lim_{j \to \infty} \lim_{n \to \infty} \int_{\{j \leq |u_n| \leq j+1\}} a(x, \nabla u_n) \nabla u_n dx = 0. \quad (4.17)$$

Assertion(ii):

$$\lim_{j \to \infty} \lim_{n \to \infty} \int_\Omega a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u)) h_j(u_n) dx = 0. \quad (4.18)$$

Assertion(iii):

$$\lim_{j \to \infty} \lim_{n \to \infty} \int_\Omega a(x, \nabla T_k(u_n)) \nabla T_k(u_n)(1 - h_j(u_n)) dx = 0. \quad (4.19)$$

Note that h_j (j is a nonnegative real parameter) is a real variable function defined by

$$h_j(s) = \begin{cases} 1 & \text{if } |s| \leq j, \\ 0 & \text{if } |s| \geq j+1, \\ j+1-|s| & \text{if } j \leq |s| \leq j+1. \end{cases} \tag{4.20}$$

We can write

$$\begin{aligned} &\int_\Omega (a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))dx \\ &= \int_\Omega (a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))h_j(u_n)dx \\ &\quad + \int_\Omega (a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))(1-h_j(u_n))dx. \end{aligned}$$

Thanks to (4.18), the first integral of the right-hand side converges to zero as n and j tend to infinity. Concerning the second term, we have

$$\begin{aligned} &\int_\Omega a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))(1-h_j(u_n))dx, \\ &= \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx, \\ &\quad - \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)(1-h_j(u_n))\,dx, \\ &\quad - \int_\Omega a(x,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))(1-h_j(u_n))\,dx. \end{aligned}$$

According to (4.19), the first integral of the right-hand side approaches zero as n and j tend to infinity, and since $a(x,\nabla T_k(u_n))$ in $(L^{p'(x)}(\Omega,\omega^*))^N$ and $\nabla T_k(u)(1-h_j(u_n))$ converges to zero, then the second integral converges to zero. For the third integral, it converges to zero because $\nabla T_k(u_n) \to \nabla T_k(u)$ weakly in $(L^{p(x)}(\Omega,\omega))^N$. Finally we conclude that,

$$\lim_{n\to\infty} \int_\Omega \Big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))\Big)(\nabla T_k(u_n) - \nabla T_k(u))dx = 0.$$

Using (4.19) and Lemma 3.2, we deduce

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,p(x)}(\Omega,\omega) \text{ as n tends to } +\infty, \tag{4.21}$$

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{4.22}$$

4.1.3. Passing to the limit. In this step we claim that

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{4.23}$$

Let $v = u_n + \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}} ds$. Since $v \in W_0^{1,p(x)}(\Omega, \omega)$ and $v \geq \psi$, then v is an admissible test function in (4.1). Therefore,

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \nabla\Big(- \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\Big)\, dx \\
&+ \int_\Omega H(x, u_n, \nabla u_n)(- \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}} ds) dx \\
&\leq \int_\Omega f_n(- \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(-G(u_n))(\int_{u_n}^{0} g(s)\chi_{\{s<-h\}} ds) dx \\
&+ \int_\Omega a(x, \nabla u_n) \nabla u_n \exp(-G(u_n)) g(u_n) \chi_{\{u_n<-h\}} dx \\
&\leq \int_\Omega \gamma(x) \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx \\
&\quad + \int_\Omega g(u_n) |\nabla u_n|^{p(x)} \omega(x) \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx \\
&\quad - \int_\Omega f_n \exp(-G(u_n)) \int_{u_n}^{0} g(s)\chi_{\{s<-h\}}\, ds\, dx.
\end{aligned}$$

Using the initial condition (1.5) and the fact that $\int_{u_n}^{0} g(s)\chi_{\{s<-h\}} ds \leq \int_{-\infty}^{-h} g(s) ds$, we obtain

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n) \nabla u_n \exp(-G(u_n)) g(u_n) \chi_{\{u_n<-h\}} dx \\
&\leq \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}) \int_{-\infty}^{-h} g(s) ds (\|\gamma\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}) \\
&\leq \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}) \int_{-\infty}^{-h} g(s) ds (\|\gamma\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}),
\end{aligned}$$

we also have, by (1.5)

$$\int_{\{u_n<-h\}} g(u_n) |\nabla u_n|^{p(x)} \omega(x) dx \leq c \int_{-\infty}^{-h} g(s) ds \tag{4.24}$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to+\infty} \sup_n \int_{\{u_n<-h\}} g(u_n) |\nabla u_n|^{p(x)} \omega(x) dx = 0. \tag{4.25}$$

On the other hand, let

$$M = \exp(\frac{\|g\|_{L^1(R)}}{\alpha}) \int_0^{+\infty} g(s)ds$$

and $h \geq M + \|\psi^+\|_{L^\infty(\Omega)}$. Consider

$$v = u_n - \exp(G(u_n)) \int_0^{u_n} g(s)\chi_{\{s>h\}} ds.$$

Since $v \in W_0^{1,p(x)}(\Omega, \omega)$ and $v \geq \psi$, v is an admissible test function in (4.1). Then, similarly to (4.25), we obtain

$$\lim_{h\to+\infty} \sup_{n\in N} \int_{\{u_n>h\}} g(u_n)|\nabla u_n|^{p(x)}\omega(x)dx = 0. \tag{4.26}$$

Combining (4.21), (4.25), (4.26) and Vitali's theorem, we conclude (4.23). Now, let $\varphi \in K_\psi \cap L^\infty(\Omega)$ and take $v = u_n - T_k(u_n - \varphi)$ as a test function in (4.1). We obtain

$$\begin{aligned} & u_n \in K_\psi \\ & \int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - \varphi)dx + \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - \varphi)dx \\ & \leq \int_\Omega f_n T_k(u_n - \varphi)dx \quad \forall \varphi \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{aligned} \tag{4.27}$$

Finally, from (4.21) and (4.23), we can pass to the limit in (4.27). This completes the proof of Theorem 4.2.

5. Appendix

Proof of Lemma 4.6.

Proof of Assertion (i):

Consider the function

$$v = u_n - \eta \exp(G(u_n))T_1(u_n - T_j(u_n))^+.$$

For j large enough and η small enough, we can deduce that $v \geq \psi$ and since $v \in W_0^{1,p(x)}(\Omega, \omega)$, v is a admissible test function in (4.1). Then, we obtain

$$\begin{aligned} & \int_\Omega a(x, \nabla u_n)\nabla\Big(\exp(G(u_n))T_1(u_n - T_j(u_n))^+\Big)dx \\ & + \int_\Omega H_n(x, u_n, \nabla u_n)\exp(G(u_n))T_1(u_n - T_j(u_n))^+ dx \\ & \leq \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_j(u_n))^+ dx. \end{aligned}$$

From the growth conditions (1.5) and (1.6), we have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla(T_1(u_n-T_j(u_n))^+)\exp(G(u_n))dx\\&\le\int_\Omega \gamma(x)\exp(G(u_n))T_1(u_n-T_j(u_n))^+dx\\&\quad+\int_\Omega f_n\exp(G(u_n))T_1(u_n-T_j(u_n))^+dx.\end{aligned}\tag{5.1}$$

Since f_n converges to f strongly in $L^1(\Omega)$ and $\gamma\in L^1(\Omega)$, by Lebesgue's theorem, the right-hand side approaches zero as $n,j\to\infty$. Therefore, passing to the limit first in n, then in j, we obtain from (5.1)

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\le u_n\le j+1\}} a(x,\nabla u_n)\nabla u_n dx=0.\tag{5.2}$$

On the other hand, consider the test function $v=u_n+\exp(-G(u_n))T_1(u_n-T_j(u_n))^-$ in (4.1). Similarly to (5.2), it is easy to see that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{-j-1\le u_n\le -j\}} a(x,\nabla u_n)\nabla u_n dx=0\tag{5.3}$$

Finally, by (5.2) and (5.3) we obtain assertion (i).

Proof of Assertion (ii):

On one hand, let $v=u_n-\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)$ with h_j is defined in (4.20) and η small enough such that $v\in K_\psi$, then we take v as test function in (4.1), we obtain

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla\Big(\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)\Big)dx\\&+\int_\Omega H_n(x,u_n,\nabla u_n)\Big(\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)\Big)dx\\&\le\int_\Omega f_n\eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)dx.\end{aligned}$$

Similarly, using (1.5) and (1.6), we deduce

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+\exp(G(u_n))h_j(u_n)dx\\&\le\int_\Omega\gamma(x)\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)dx\\&+\int_{\{j\le u_n\le j+1\}} a(x,\nabla u_n)\nabla u_n\exp(G(u_n))(T_k(u_n)-T_k(u))^+dx\\&+\int_\Omega f_n\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)dx.\end{aligned}$$

In view of (5.2), the convergence f_n to f in $L^1(\Omega)$ and $\gamma \in L^1(\Omega)$, it is easy to see that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))^+ \times \exp(G(u_n))h_j(u_n)dx \leq 0. \tag{5.4}$$

Moreover, (5.4) becomes

$$\begin{aligned}&\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|\leq k\}} a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\\ &\quad\times \exp(G(u_n))h_j(u_n)dx\\ &-\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|> k\}} a(x,\nabla u_n)\nabla T_k(u)\\ &\quad\times \exp(G(u_n))h_j(u_n)dx \leq 0.\end{aligned}$$

Since $h_j(u_n)=0$ if $|u_n|>j+1$, we obtain

$$\begin{aligned}&\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|> k\}} a(x,\nabla u_n)\nabla T_k(u)\exp(G(u_n))h_j(u_n)dx\\ &=\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|> k\}} a(x,\nabla T_{j+1}(u_n))\nabla T_k(u)\\ &\quad\times \exp(G(u_n))h_j(u_n)dx\\ &=\lim_{j\to+\infty}\int_{\{|u|>k\}} X_j\nabla T_k(u)\exp(G(u))h_j(u)dx=0,\end{aligned}$$

where X_j is the limit of $a(x,\nabla T_{j+1}(u_n))$ in $(L^{p'(x)}(\Omega,\omega))^N$ as n goes to infinity and $\nabla T_k(u)\chi_{\{|u|>k\}}=0$ a.e. in Ω. Consequently,

$$\lim_{j,n\to\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}\Big(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u))\Big) \times(\nabla T_k(u_n)-\nabla T_k(u))h_j(u_n)=0. \tag{5.5}$$

On the other hand, taking $v=u_n+\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n)$ as test function in (4.1) and reasoning as in (5.5) we have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla(-\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n))dx\\ &+\int_\Omega H_n(x,u_n,\nabla u_n)(-\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n))dx\\ &\leq -\int_\Omega f_n(\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_j(u_n))dx.\end{aligned}$$

Similarly to (5.5), it is easy to see that

$$\lim_{j,n\to\infty}\int_{\{T_k(u_n)-T_k(u)\leq 0\}} a(x,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\exp(-G(u_n))h_j(u_n)dx=0. \tag{5.6}$$

Combining (5.5) and (5.6) we obtain the desired assertion (ii).

Proof of Assertion (iii):
Let $v = u_n + \exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))$ as test function in(4.1). Then we have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla\Big(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\Big)dx\\&+\int_\Omega H_n(x,u_n,\nabla u_n)\Big(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\Big)dx\\&\le-\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))dx\end{aligned}$$

Using(1.6) and (1.5), we deduce that

$$\begin{aligned}&\int_{\{u_n\le 0\}} a(x,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))(1-h_j(u_n))dx\\&\le-\int_{\{-1-j\le u_n\le -j\}} a(x,\nabla u_n)\nabla u_n\exp(-G(u_n))T_k(u_n)^-dx\\&\quad+\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))dx\\&\quad-\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))dx\end{aligned}$$

In view of (4.17), the second integral tends to zero as n and j approach infinity. By Lebesgue's theorem, it is possible to conclude that the third and the fourth integrals converge to zero as n and j approach infinity. Then

$$\lim_{j,n\to\infty}\int_{\{u_n\le 0\}} a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))dx=0. \tag{5.7}$$

On the other hand, we take $v = u_n - \eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))$ which is an admissible test function in (4.1), we have

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla\Big(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\Big)dx\\&+\int_\Omega H_n(x,u_n,\nabla u_n)\Big(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\Big)dx\\&\le\int_\Omega f_n\Big(\eta\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))\Big)dx\end{aligned}$$

Which takes, by using (1.6) and (1.5), the form

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n^+-\psi^+)\exp(G(u_n))(1-h_j(u_n))dx\\
&\le -\int_{\{j\le u_n\le j+1\}} a(x,\nabla u_n)\nabla u_n\exp(G(u_n))T_k(u_n^+-\psi^+)dx\\
&\quad+\int_{\{-j-1\le u_n\le -j\}} a(x,\nabla u_n)\nabla u_n\exp(G(u_n))T_k(u_n^+-\psi^+)dx\\
&\quad+\int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))dx\\
&\quad+\int_\Omega f_n\exp(G(u_n))T_k(u_n^+-\psi^+)(1-h_j(u_n))dx=\varepsilon_1(j,n)
\end{aligned}\tag{5.8}$$

By (4.17) and Lebesgue's theorem, we conclude that $\varepsilon_1(j,n)$ converges to zero as n and j approach infinity. From (5.8), we have

$$\begin{aligned}
&\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla u_n^+\exp(G(u_n))(1-h_j(u_n))dx\\
&\le\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla\psi^+\exp(G(u_n)(1-h_j(u_n)))dx+\varepsilon_1(j,n)
\end{aligned}$$

Thanks to (1.3) and Young's inequality, it is possible to conclude that

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla\psi^+\exp(G(u_n)(1-h_j(u_n)))dx\le\varepsilon_2(j,n),$$

where $\varepsilon_2(j,n)$ converges to zero as n and j go to infinity. Since $\exp(G(u_n))$ is bounded,

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,\nabla u_n)\nabla u_n^+(1-h_j(u_n)))dx\le\varepsilon_3(j,n).$$

Since $\{x\in\Omega,\quad |u_n^+|\le k\}\subset\{x\in\Omega,\quad |u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$, hence

$$\begin{aligned}
&\int_{\{|u_n^+|\le k\}} a(x,\nabla u_n)\nabla u_n(1-h_j(u_n)))dx\\
&\le\int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} a(x,\nabla u_n)\nabla u_n(1-h_j(u_n)))dx\le\varepsilon_3(j,n)
\end{aligned}$$

Which, for all $k\ge 0$, yields

$$\lim_{j,n\to\infty}\int_{\{u_n\ge 0\}} a(x,\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))dx=0,\tag{5.9}$$

Finally, using (5.7) and (5.9), we conclude assertion (iii). Which finish the proof of Lemma 4.6.

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