Bol. Soc. Paran. Mat. (3s.) **v. 35** 3 (2017): 9–24.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v35i3.29791

Some fixed point results in probabilistic Menger space*

Maryam Shams, Shahnaz Jafari

abstract: In this paper, we define the concept of $\alpha-\beta$-contractive mapping in probabilistic Menger space and prove some fixed point theorems for such mapping. Some examples are given to support the obtained results.

Key Words: Continuous t-norm, Probabilistic Menger space, Voltra integral equation, $\alpha-\beta$-contractive mapping

Contents

1. Introduction

Probabilistic metric spaces were introduced in 1942 by Menger [10]. In such spaces, the notion of distance between two points x and y is replaced by a distribution function $F_{x,y}(t)$. Sehgal, in his Ph.D.thesis [13], extended the notion of a contraction mapping to the setting of the Menger probabilistic metric spaces.
Then Sehgal and Bharuch-Reid in 1972 followed a generalization of Banach contraction principle on a complete Menger space [14]. In 1984 Khan et al, introduced the concept of altering distance functions [9]. This concept is extended to Menger spaces by Choudhury and Das in [5]. This extension of altering distance function, has been further used by many authors [6], [7] and [11]. In 2009 a probabilistic contraction mapping principle has been proved in G-complete Menger spaces [7]. Babacev defined nonlinear generalized contractive type condition involving altering distance in Menger spaces [3]. The existence of fixed points for mappings satisfying generalized contractive type conditions, defined on various spaces, studied by many authors [1], [2].
In this paper, we give a generalization of the concept contractive mapping and introduce the notion of $\alpha-\beta$-contractive mapping. Also we compare it with previous results in Menger space and prove some fixed point theorems for such contractive mapping. Our results generalize and improve the previous results in fixed point.
We first bring notion, definitions and known results, which are related to our work. For more details, we refer the reader to [8].

* The project is partially supported by the Center of Excellence for Mathematics, University of Shahrekord.
2000 *Mathematics Subject Classification:* Primary 47H10; 47H09; secondary 46S50.

Definition 1.1. *A distribution function is a function* $F : (-\infty, \infty) \to [0, 1]$, *that is non-decreasing and left continous on* $\mathbb{R}$, *moreover,* $\inf_{t\in\mathbb{R}} F(t) = 0$ *and* $\sup_{t\in\mathbb{R}} F(t) = 1$.

The set of all the distribution functions is denoted by D, and the set of those distribution functions such that $F(0) = 0$ is denoted by D^+. We will denote the specific distribution function by

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Definition 1.2. *A probabilistic metric space (briefly, PM-space) is an ordered pair* (X, F), *where* X *is a nonempty set and* F *is a mapping from* $X \times X$ *into* D^+ *such that, if* $F_{x,y}$ *denotes the value of* F *at the pair* (x, y), *the following conditions hold:*

(PM1) $F_{x,y}(t) = H(t)$ *if and only if* $x = y$,

(PM2) $F_{x,y}(t) = F_{y,x}(t)$,

(PM3) If $F_{x,y}(t) = 1$ *and* $F_{y,z}(s) = 1$, *then* $F_{x,z}(t + s) = 1$, *for all* $x, y, z \in X$ *and* $s, t \geq 0$.

Definition 1.3. *A binary operation* $T : [0, 1]\times[0, 1] \to [0, 1]$ *is a continuous t-norm if the following conditions hold:*

(a) T *is commutative and associative,*

(b) T *is continuous,*

(c) $T(a, 1) = a$ *for all* $a \in [0, 1]$,

(d) $T(a, b) \leq T(c, d)$ *whenever* $a \leq c$ *and* $b \leq d$, *for* $a, b, c, d \in [0, 1]$.

The following are three basic continuous t-norms.
(i) The minimum t-norm, say T_M, defined by $T_M(a, b) = \min\{a, b\}$.
(ii) The product t-norm, say T_p, defined by $T_p(a, b) = a.b$.
(iii)The Lukasiewicz t-norm, say T_L, defined by $T_L(a, b) = \max\{a + b - 1, 0\}$.
These t- norms are related in the following way: $T_L \leq T_P \leq T_M$.

Definition 1.4. *A Menger space is a triple* (X, F, T), *where* (X, F) *is a PM-space and* T *is a continuous t-norm such that for all* $x, y, z \in X$ *and* $s, t \geq 0$

$$F_{x,y}(t + s) \geq T(F_{x,z}(t), F_{z,y}(s)).$$

Definition 1.5. *Let* (X, F, T) *be a Menger space. Then*

(i) A sequence x_n *in* X *is said to be converge to* x *if, for every* $\epsilon > 0$ *and* $0 < \lambda < 1$, *there exists a positive integer* N *such that* $F_{x_n x}(\epsilon) > 1 - \lambda$, *whenever* $n \geq N$.

(ii) *A sequence x_n in X is called Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N scuh that $F_{x_n x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$.*

(iii) *A Menger space is said to be complete if and only if every Cauchy sequence in X is converge to a point in X.*

(iv) *A sequence x_n is called G-Cauchy if $\lim_{n\to\infty} F_{x_n x_{n+m}}(t) = 1$, for each $m \in \mathbb{N}$ and $t > 0$.*

(v) *The space (X, F, T) is called G-complete if every G-Cauchy sequence in X is convergent.*

It follows immediately that a Cauchy sequence is a G-Cauchy sequence. The convers is not always true. This has been established by an example in [16].

According to [12], the (ϵ, λ)-topology in Menger PM-space (X, F, T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = \{N_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1)\},$$

where

$$N_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The (ϵ, λ) -topology is a Hausdorff topology. In this topology, a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$, for every sequence $x_n \to x_0$.

Lemma 1.6. *[17] If (X, F, T) is a Menger space and T is continuous, then probabilistic distance function F is a low semi continuous function of points, i.e. for every fixed point $t > 0$, if $p_n \to p$ and $q_n \to q$ then lim inf $F_{p_n, q_n}(t) \to F_{p,q}(t)$.*

Φ-functions in Menger space introduced by Choudhury and Das in [5].

Definition 1.7. *A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:*

(i) *$\phi(t) = 0$ if and only if $t = 0$,*

(ii) *$\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,*

(iii) *ϕ is left continuous in $(0, \infty)$,*

(iv) *ϕ is continuous at 0.*

In the sequel, the class of all Φ-functions will be denoted by Φ.

2. Fixed point theorems for generalized $\alpha-\beta$-contractive mappings

In this section we introduce the notions of generalizd $\alpha-\beta$-contractive mapping in probabilistic Menger spaces.

Lemma 2.1. *[3] Let (X, F, T) be a complete Menger space and $\varphi : [0, \infty) \to [0, \infty)$ be a Φ-function. Then the following statement holds:*
If for $x, y \in X, c \in (0, 1)$, we have $F_{x,y}(\varphi(t)) \geq F_{x,y}(\varphi(\frac{t}{c}))$ for all $t > 0$, then $x = y$.

Theorem 2.2. *[3] Let (X, F, T) be a complete Menger space with continuous t-norm T which satisfies $T(a, a) \geq a$ for each $a \in [0, 1]$. Let $c \in (0, 1)$ be fixed. If for a Φ-function φ and a self-mapping f on X, we have*

$$F_{fx,fy}(\varphi(t)) \geq \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), F_{y,fy}(\varphi(\frac{t}{c})), F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\}, \tag{2.1}$$

for all $x, y \in X$ and for all $t > 0$, then f has a unique fixed point in X.

Now, we introduce the following definitions:

Definition 2.3. *Let (X, F, T) be a Menger PM-space and $f : X \to X$ be a given mapping and $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$, be two functions, we say that f is $\alpha - \beta$-admissible if*

(i) $x, y \in X$, for all $t > 0$, $\alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1$,

(ii) $x, y \in X$, for all $t > 0$, $\beta(x, y, t) \leq 1 \Rightarrow \beta(fx, fy, t) \leq 1$.

Definition 2.4. *Let (X, F, T) be a Menger space and $f : X \to X$ be a given mapping. We say that f is a generalized $\alpha - \beta$- contractive mapping if there exist two functions $\alpha, \beta : X \times X \times (0, \infty) \to (0, \infty)$ such that*

$$\beta(x, y, t)F_{fx,fy}(\varphi(t)) \geq \alpha(x, y, t) \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), F_{y,fy}(\varphi(\frac{t}{c})), F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\}, \tag{2.2}$$

for all $x, y \in X$ and for all $t > 0$, where $\varphi \in \Phi$ and $c \in (0, 1)$.

Remark 2.5. *If $\alpha(x, y, t) = \beta(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$, then the condition (2.2) reduce to condition (2.1), but the converse is not true always, (see Example 2.7).*

Theorem 2.6. *Let (X, F, T) be a complete Menger space with continuous t-norm T which satisfies $T(a, a) \geq a$ with $a \in [0, 1]$, let $f : X \to X$ be a generalized $\alpha - \beta$-contractive mapping satisfyings the following conditions:*

(i) f is $\alpha - \beta$-admissible,

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0, t) \geq 1$ *and* $\beta(x_0, fx_0, t) \leq 1$ *for all* $t > 0$,

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_n, x_{n+1}, t) \leq 1$, $\alpha(x_n, x_{n+1}, t) \geq 1$ *for all* $n \in \mathbb{N}$ *and for all* $t > 0$, *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x_n, x, t) \leq 1$ *and* $\alpha(x_n, x, t) \geq 1$ *for all* $n \in \mathbb{N}$ *and for all* $t > 0$,
then f *has a fixed point.*

Proof: Since T is continuous and $T(a, a) \geq a$, for all $a \in [0, 1]$, then we have

$$T(a, a) \geq T(\min\{a, b\}, \min\{a, b\}) \geq \min\{a, b\}$$

and by (PM3), we write $F_{x,y}(2t) \geq \min\{F_{x,z}(t), F_{z,y}(t)\}$, for all $x, y, z \in X$. Now, Let $x_0 \in X$ be such that (ii) holds and define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. First, we suppose $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, otherwise f has trivially a fixed point. Now, since f is $\alpha - \beta$-admissible, we have

$$\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \leq 1 \Longrightarrow \beta(x_1, x_2, t) = \beta(fx_0, fx_1, t) \leq 1$$

and

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Longrightarrow \alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq 1.$$

Consequently, by induction, we get $\beta(x_n, x_{n+1}, t) \leq 1$, and $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $t > 0$. From the properties of function φ and by (i), (iv) we can find $r > 0$ such that $t > \varphi(r)$ and therefore we have

$$\begin{aligned}
F_{x_n,x_{n+1}}(t) &\geq F_{fx_{n-1},fx_n}(\varphi(r)) \geq \beta(x_{n-1}, x_n, r) F_{fx_{n-1},fx_n}(\varphi(r)) \\
&\geq \alpha(x_{n-1}, x_n, r) \min\big\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_{n-1},x_n}(\varphi(r/c)), \\
&\quad F_{x_n,x_{n+1}}(\varphi(r/c)), F_{x_{n-1},x_{n+1}}(2\varphi(r/c)), F_{x_n,x_n}(2\varphi(r/c))\big\} \\
&\geq \min\big\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_{n-1},x_n}(\varphi(r/c)), \\
&\quad F_{x_n,x_{n+1}}(\varphi(r/c)), F_{x_{n-1},x_{n+1}}(2\varphi(r/c)), F_{x_n,x_n}(2\varphi(r/c))\big\} \\
&= \min\big\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_n,x_{n+1}}(\varphi(r/c)), F_{x_{n-1},x_{n+1}}(2\varphi(r/c))\big\} \\
&\geq \min\big\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_n,x_{n+1}}(\varphi(r/c)), \min\{F_{x_{n-1},x_n}(\varphi(r/c)), \\
&\quad F_{x_n,x_{n+1}}(\varphi(r/c))\}\big\} \\
&= \min\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_n,x_{n+1}}(\varphi(r/c))\}.
\end{aligned}$$

We shall prove that

$$F_{x_n,x_{n+1}}(\varphi(r)) \geq F_{x_{n-1},x_n}(\varphi(\frac{r}{c})). \tag{2.3}$$

If we assume that $F_{x_n,x_{n+1}}(\varphi(\frac{r}{c}))$ is the minimum, that from Lemma 2.1, we get that $x_n = x_{n+1}$, which leads to contradiction with the assumption $x_{n+1} \neq x_n$ and so $F_{x_{n-1},x_n}(\varphi(\frac{r}{c}))$ is the minimum and therefore (2.3) holds true. Since φ is strictly increasing, we have

$$F_{x_n,x_{n+1}}(t) \geq F_{x_n,x_{n+1}}(\varphi(r)) \geq F_{x_{n-1},x_n}(\varphi(\frac{r}{c})) \geq ... \geq F_{x_0,x_1}(\varphi(\frac{r}{c^n})),$$

that is, $F_{x_n,x_{n+1}}(t) \geq F_{x_0,x_1}(\varphi(\frac{r}{c^n}))$ for arbitrary $n \in \mathbb{N}$. Next, Let $m, n \in \mathbb{N}$ with $m > n$, then by (PM3) we have

$$\begin{aligned} F_{x_n,x_m}((m-n)t) &\geq \min\{F_{x_n,x_{n+1}}(t), ..., F_{x_{m-1},x_m}(t)\} \\ &\geq \min\{F_{x_0,x_1}(\varphi(\frac{r}{c^n})), ..., F_{x_0,x_1}(\varphi(\frac{r}{c^{m-1}}))\}. \end{aligned}$$

Since φ is strictly increasing and $\varphi(t) \to \infty$ as $t \to \infty$, then fixed $\epsilon \in (0,1)$, so there exists $n_0 \in \mathbb{N}$ such that $F_{x_0,x_1}(\varphi(\frac{r}{c^n})) > 1-\epsilon$, whenever $n \geq n_0$. This implies that, for every $m > n \geq n_0$, we get $F_{x_n,x_m}((m-n)t) \geq 1-\epsilon$. Since $t > 0$ and $\epsilon \in (0,1)$ is arbitrary, we deduce that $\{x_n\}$ is a Cauchy sequence in the complete Menger space (X, F, T). Then, $x_n \to u$ as $n \to \infty$ for some $u \in X$. We will show that u is a fixed point of f. By (PM3), we have

$$\begin{aligned} F_{fu,u}(t) &\geq T(F_{fu,x_n}(\varphi(r)), F_{x_n,u}(t-\varphi(r))) \\ &\geq \min\{F_{fu,x_n}(\varphi(r)), F_{x_n,u}(t-\varphi(r))\}. \end{aligned}$$

Notice that, if $x_n = fu$ for infinitely many values of n, then $u = fu$ and hence the proof finishes. Therefore, we assume that $x_n \neq fu$ for all $n \in \mathbb{N}$. Thus, since $\lim_{n\to\infty} x_n = u$, for any arbitrary $\epsilon \in (0,1)$ and n large enough, we get $F_{x_n,u}(t-\varphi(r)) > 1-\epsilon$ and hence, we have $F_{u,fu}(t) \geq \min\{F_{fu,x_n}(\varphi(r)), 1-\epsilon\}$. Since $\epsilon > 0$ is arbitrary, we can write $F_{fu,u}(t) \geq F_{fu,x_n}(\varphi(r))$. Next, we get

$$\begin{aligned} F_{fu,u}(t) &\geq F_{fu,x_n}(\varphi(r)) = F_{fu,fx_{n-1}}(\varphi(r)) \geq \beta(u, x_{n-1}, r) F_{fu,fx_{n-1}}(\varphi(r)) \\ &\geq \alpha(u, x_{n-1}, r) \min\big\{F_{u,x_{n-1}}(\varphi(\frac{r}{c})), F_{x_{n-1},x_n}(\varphi(\frac{r}{c})), \\ &\quad F_{fu,u}(\varphi(\frac{r}{c})), F_{fu,x_{n-1}}(2\varphi(\frac{r}{c})), F_{u,x_n}(2\varphi(\frac{r}{c}))\big\} \\ &\geq \min\big\{F_{u,x_{n-1}}(\varphi(\frac{r}{c})), F_{x_{n-1},x_n}(\varphi(\frac{r}{c})), F_{fu,u}(\varphi(\frac{r}{c})), \\ &\quad F_{fu,x_{n-1}}(2\varphi(\frac{r}{c})), F_{u,x_n}(2\varphi(\frac{r}{c}))\big\} \\ &\geq \min\big\{F_{u,x_{n-1}}(\varphi(\frac{r}{c})), F_{fu,u}(\varphi(\frac{r}{c})), F_{x_{n-1},x_n}(\varphi(\frac{r}{c}))\big\}. \end{aligned}$$

It follows that

$$\begin{aligned} F_{fu,u}(t) &\geq \liminf_{n\to\infty} F_{fu,x_n}(\varphi(r)) \\ &\geq \liminf_{n\to\infty} \min\big\{F_{u,x_{n-1}}(\varphi(\frac{r}{c})), F_{fu,u}(\varphi(\frac{r}{c})), F_{x_{n-1},x_n}(\varphi(\frac{r}{c}))\big\} \\ &\geq \min\{1-\epsilon, F_{fu,u}(\varphi(\frac{r}{c})), 1-\epsilon\}. \end{aligned}$$

Finally, since $\epsilon \in (0,1)$ is arbitrary, we have $F_{fu,u}(\varphi(r)) \geq F_{fu,u}(\varphi(\frac{r}{c}))$ and so, by Lemma 2.1, we deduce that $u = fu$. This completes the proof. □

The following examples show the usefulness of Definition 2.4.

Example 2.7. *Let* $X = [\frac{1}{4}, \infty)$, $T(a,b) = \min\{a,b\}$ *for all* $a, b \in [0,1]$ *and* $F_{x,y}(t) = \frac{t}{t+|x-y|}$ *for all* $x, y \in X$ *and for all* $t > 0$. *Clearly* (X, F, T) *is a complete Menger space. Define the mapping* $f : X \to X$ *by*

$$fx = \begin{cases} 1 & x \in [\frac{1}{4}, 1] \\ 2 & otherwise \end{cases}$$

and two functions $\alpha, \beta : X \times X \times (0,\infty) \to (0,\infty)$ *by*

$$\beta(x,y,t) = \begin{cases} 1 & x, y \in [\frac{1}{4}, 1] \\ \frac{2(t+1)}{2t+|x-y|} & otherwise \end{cases}, \quad \alpha(x,y,t) = \begin{cases} 1 & x, y \in [\frac{1}{4}, 1] \\ 0 & otherwise, \end{cases}$$

for all $t > 0$. *Now, consider* $\varphi : [0,\infty) \to [0,\infty)$ *defined by* $\varphi(t) = t$. *Let* $c = \frac{1}{2}$. *We show that* f *satisfies the hypotheses of Theorem 2.6. At first we prove* f *is* $\alpha - \beta$*-admissible. If* $\beta(x,y,t) \leq 1$, *this implies* $x, y \in [\frac{1}{4}, 1]$, *so by the definitions of* f *and* β, *we have* $\beta(fx, fy, t) = 1$. *Similarly when* $\alpha(x,y,t) \geq 1$, *then* $\alpha(fx, fy, t) \geq 1$. *Hence* f *is* $\alpha - \beta$*-admissible. On the other hand for* $x_0 = \frac{1}{2}$ *we have* $\alpha(\frac{1}{2}, f(\frac{1}{2}), t) = 1$ *and* $\beta(\frac{1}{2}, f(\frac{1}{2}), t) = 1$. *Finally we show that* f *satisfies* (2.2).
If $x, y \in [\frac{1}{4}, 1]$, *then* $\beta(x,y,t)F_{fx,fy}(t) = 1$ *and hence the inequality is true. If* $x, y \notin [\frac{1}{4}, 1]$, *then* $\alpha(x,y,t) = 0$ *and hence the inequality is obviously true. If* $x \in [\frac{1}{4}, 1]$ *and* $y \notin [\frac{1}{4}, 1]$, *then we have* $\alpha(x,y,t) = 0$ *and hence the inequality is obviously true. Thus all the conditions of Theorem 2.6 hold and* f *has two fixed points,* $x = 1$ *and* $x = 2$.
On the other hand, f *does not satisfy* (2.1). *Indeed for* $x = 1$ *and* $y = 2$, *we get*

$$\frac{t}{t+1} \geq \min\left\{\frac{t}{t+c}, 1, 1, \frac{t}{t+\frac{c}{2}}, \frac{t}{t+\frac{c}{2}}\right\} = \frac{t}{t+c},$$

which gives $c \geq 1$, *a contradiction.*

Corollary 2.8. *Let* (X, F, T) *be a complete Menger space with continuous* t*-norm* T *which satisfies* $T(a,a) \geq a$ *with* $a \in [0,1]$, *let there exists a function* $\beta : X \times X \times (0,\infty) \to (0,\infty)$ *such that* $f : X \to X$ *satisfies the following conditions:*

(i) $\beta(x,y,t)F_{fx,fy}(\varphi(t)) \geq \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), F_{y,fy}(\varphi(\frac{t}{c})),$
$F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\}$,
for all $x, y \in X$ *and for all* $t > 0$, *where* $\varphi \in \Phi$ *and* $c \in (0,1)$,

(ii) $x, y \in X$, *for all* $t > 0$, $\beta(x,y,t) \leq 1 \Rightarrow \beta(fx, fy, t) \leq 1$,

(iii) *there exists* $x_0 \in X$ *such that* $\beta(x_0, fx_0, t) \leq 1$ *for all* $t > 0$,

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_n, x_{n+1}, t) \leq 1$ *for all* $n \in \mathbb{N}$ *and for all* $t > 0$, *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x_n, x, t) \leq 1$ *for all* $n \in \mathbb{N}$ *and for all* $t > 0$,
then f *has a fixed point.*

Example 2.9. *Let* $X = [\frac{1}{4}, \frac{5}{4}] \cup \{\frac{3}{2}\}$, $T(a,b) = \min\{a,b\}$ *for all* $a, b \in [0,1]$ *and* $F_{x,y}(t) = \frac{t}{t+|x-y|}$ *for all* $x, y \in X$ *and for all* $t > 0$. *Clearly* (X, F, T) *is a complete Menger space. Define the mapping* $f : X \to X$ *by*

$$fx = \begin{cases} \frac{1}{4} & x = \frac{1}{4} \\ 1 & x \in (\frac{1}{4}, 1] \\ \frac{3}{2} & x \in (1, \frac{5}{4}] \cup \{\frac{3}{2}\} \end{cases}$$

and two functions $\alpha, \beta : X \times X \times (0, \infty) \to (0, \infty)$ *by*

$$\beta(x,y,t) = \begin{cases} \frac{1}{3} & x = y = 1 \\ 1 & x = y = \frac{1}{4} \\ \frac{1}{2} & x, y \in (\frac{1}{4}, 1) \\ 4 & \textit{otherwise} \end{cases}, \qquad \alpha(x,y,t) = \begin{cases} 0 & x \textit{ or } y \in (\frac{1}{4}, 1) \\ \frac{1}{3} & x = y = 1 \\ 1 & \textit{otherwise,} \end{cases}$$

for all $t > 0$. *Now, consider* $\varphi : [0, \infty) \to [0, \infty)$ *defined by* $\varphi(t) = t$ *and let* $c = \frac{1}{2}$. *We show that* f *satisfies the hypotheses of Theorem 2.6. At first we prove* f *is* $\alpha - \beta$*-admissible. If* $\beta(x,y,t) \le 1$, *this implies* $x, y \in [\frac{1}{4}, 1]$, *so by the definitions of* f *and* β, *we have* $\beta(fx, fy, t) \le 1$. *Similarly when* $\alpha(x,y,t) \ge 1$, *then* $\alpha(fx, fy, t) \ge 1$. *Hence* f *is* $\alpha - \beta$*-admissible. On the other hand for* $x_0 = \frac{1}{4}$ *we have* $\alpha(\frac{1}{4}, f(\frac{1}{4}), t) = 1$ *and* $\beta(\frac{1}{4}, f(\frac{1}{4}), t) = 1$. *Finally we show that* f *satisfies inequality (2.2).*
If $x = 1$ *and* $y \in (1, \frac{5}{4}]$, *then we get*

$$\frac{4t}{t+\frac{1}{2}} \ge \min\Big\{\frac{t}{t+\frac{y-1}{2}}, 1, \frac{t}{t+\frac{\frac{3}{2}-y}{2}}, \frac{t}{t+\frac{1}{8}}, \frac{t}{t+\frac{y-1}{4}}\Big\} = \frac{t}{t+\frac{\frac{3}{2}-y}{2}},$$

that is true for all $t > 0$. *If* $x = \frac{3}{2}$ *and* $y \in (1, \frac{5}{4})$, *then we have*

$$4 \ge \min\Big\{\frac{t}{t+\frac{\frac{3}{2}-y}{2}}, 1, \frac{t}{t+\frac{\frac{3}{2}-y}{2}}, 1, \frac{t}{t+\frac{y-\frac{3}{2}}{4}}\Big\} = \frac{t}{t+\frac{\frac{3}{2}-y}{2}},$$

that is true for all $t > 0$. *If* $x, y \in (\frac{1}{4}, 1)$, *then* $\beta(x,y,t) = \frac{1}{2}$ *and* $\alpha(x,y,t) = 0$, *hence the inequality holds trivially. If* $x \in (\frac{1}{4}, 1)$ *and* $y \notin (\frac{1}{4}, 1)$, *then* $\alpha(x,y,t) = 0$, *hence the inequality holds trivially. For the other cases inequality is also true. Thus all the conditions of Theorem 2.6 hold and* f *has three fixed points,* $x = 1, x = \frac{3}{2}$ *and* $x = \frac{1}{4}$.
We show that f *does not satisfy condition (i) of corollary 2.8. Indeed if* $x = y = 1$, *then* $\beta(x,y,t)F_{fx,fy}(t) = \frac{1}{3}$, *hence we have* $\frac{1}{3} \ge 1$, *that is not true.*

We prove, with next theorem, uniqueness of the fixed point.

Theorem 2.10. *With the same hypotheses of Theorem 2.6, if for all* $u \in Fix(f)$ *(The set of fixed point of* f *) and for all* $t > 0$ *there exists* $z \in X$ *such that* $\beta(z, fz, t) \le 1$ *with* $\beta(u, z, t) \le 1$ *and* $\alpha(z, fz, t) \ge 1$ *with* $\alpha(u, z, t) \ge 1$, *then* f *has a unique fixed point.*

Proof: Let $u, v \in X$ such that $fu = u$ and $fv = v$. From hypotheses, there exists $z \in X$ such that

$$\beta(z, fz, t) \le 1, with\ \ \beta(u, z, t) \le 1\ and\ \beta(v, z, t) \le 1$$

and

$$\alpha(z, fz, t) \ge 1, with\ \ \alpha(u, z, t) \ge 1\ and\ \alpha(v, z, t) \ge 1.$$

Since f is $\alpha - \beta$-admissible, then we have $\beta(fz, f^2z, t) \le 1$, $\alpha(fz, f^2z, t) \ge 1$. Also

$$\beta(u, fz, t) \le 1\ and\ \beta(v, fz, t) \le 1,\ \alpha(u, fz, t) \ge 1\ and\ \alpha(v, fz, t) \ge 1.$$

By induction, for all $t > 0$ we get

$$\beta(z_n, z_{n+1}, t) \le 1,\ \beta(u, z_n, t) \le 1,\ \beta(v, z_n, t) \le 1$$

and

$$\alpha(z_n, z_{n+1}, t) \ge 1,\ \alpha(u, z_n, t) \ge 1,\ \alpha(v, z_n, t) \ge 1,$$

where $z_n = f^n z$.
From the properties of function φ, we can find $r > 0$ such that $t > \varphi(r)$ and therefore we have

$$\begin{aligned} F_{u,z_{n+1}}(t) &\ge F_{u,fz_n}(\varphi(r)) = F_{fu,fz_n}(\varphi(r)) \ge \beta(u, z_n, r) F_{fu,fz_n}(\varphi(r)) \\ &\ge \alpha(u, z_n, r) \min \{F_{u,z_n}(\varphi(\frac{r}{c})), F_{u,fu}(\varphi(\frac{r}{c})), F_{z_n,z_{n+1}}(\varphi(\frac{r}{c})), \\ &\quad F_{u,z_{n+1}}(2\varphi(\frac{r}{c})), F_{z_n,fu}(2\varphi(\frac{r}{c}))\} \\ &\ge \min \{F_{u,z_n}(\varphi(\frac{r}{c})), F_{u,fu}(\varphi(\frac{r}{c})), F_{z_n,z_{n+1}}(\varphi(\frac{r}{c})), F_{u,z_{n+1}}(2\varphi(\frac{r}{c})), \\ &\quad F_{z_n,fu}(2\varphi(\frac{r}{c}))\}, \end{aligned}$$

which implies $F_{u,z_{n+1}}(t) \ge \min \{F_{u,z_n}(\varphi(\frac{r}{c})), F_{z_n,z_{n+1}}(\varphi(\frac{r}{c}))\}$. Now, we have two cases:
(i) We assume that $F_{z_n,z_{n+1}}(\varphi(\frac{r}{c}))$ is the minimum. Then, by applying (2.2), we can write

$$F_{u,z_{n+1}}(\varphi(r)) \ge F_{z_n,z_{n+1}}(\varphi(\frac{r}{c})) \ge \min \{F_{z_{n-1},z_n}(\varphi(\frac{r}{c^2})), F_{z_n,z_{n+1}}(\varphi(\frac{r}{c^2}))\}.$$

Now, if $F_{z_n,z_{n+1}}(\varphi(\frac{r}{c^2}))$ is the minimum for some $n \in \mathbb{N}$, by Lemma 2.1, we deduce that $z_n = z_{n+1} = u$. Consequently, we deduce that $\beta(v, u, t) \le 1$ and $\alpha(v, u, t) \ge 1$ and so by (2.2) we have

$$\begin{aligned} F_{u,v}(\varphi(t)) &\ge min\{F_{u,v}(\varphi(\frac{t}{c})), F_{v,v}(\varphi(\frac{t}{c})), F_{u,u}(\varphi(\frac{t}{c})), F_{v,u}(2\varphi(\frac{t}{c})), \\ &\quad F_{u,v}(2\varphi(\frac{t}{c}))\} = F_{v,u}(\varphi(\frac{t}{c})). \end{aligned}$$

Again, by Lemma 2.1, we conclude that $u = v$. On the other hand, if $F_{z_{n-1},z_n}(\varphi(\frac{r}{c^2}))$ is the minimum, then

$$F_{z_n,z_{n+1}}(\varphi(\frac{r}{c})) \geq F_{z_{n-1},z_n}(\varphi(\frac{r}{c^2})) \geq ... \geq F_{z_0,z_1}(\varphi(\frac{r}{c^{n+1}}))$$

and, letting $n \to \infty$, we get

$$F_{z_n,z_{n+1}}(\varphi(\frac{r}{c})) \to 1.$$

Therefore $F_{u,z_{n+1}}(t) \to 1$ as $n \to \infty$, which implies $z_{n+1} \to u$ as $n \to \infty$.
(ii) Suppose that $F_{u,z_n}(\varphi(\frac{r}{c}))$ is the minimum, then we have

$$F_{u,z_{n+1}}(\varphi(r)) \geq F_{u,z_n}(\varphi(\frac{r}{c})) \geq F_{u,z_{n-1}}(\varphi(\frac{r}{c^2})) \geq ... \geq F_{u,z_0}(\varphi(\frac{r}{c^{n+1}})).$$

Letting $n \to \infty$, we obtain $F_{u,z_{n+1}}(\varphi(r)) \to 1$ as $n \to \infty$, i.e, $z_{n+1} \to u$ as $n \to \infty$. A similar argument shows that $z_{n+1} \to v$, for $n \to \infty$. Now, uniqueness of the limit, gives us $u = v$ and the proof is complete. □

Theorem 2.11. *Let (X,F,T) be a complete Menger space with continuous t-norm T and $f : X \to X$. Assume that there exist $\alpha, \beta : X \times X \times (0,\infty) \to (0,\infty)$ such that*

$$\begin{aligned} \beta(x,y,t)F_{fx,fy}(\varphi(t)) \quad \geq \quad & \alpha(x,y,t)\min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), \\ & F_{y,fy}(\varphi(\frac{t}{c})), F_{y,fx}(\varphi(\frac{t}{c}))\}, \end{aligned} \tag{2.4}$$

for all $x, y \in X$ and for all $t > 0$, where $\varphi \in \Phi$ and $c \in (0,1)$. Also suppose that the following conditions hold:

(i) f is $\alpha - \beta$-admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$,

(iii) for each sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}, t) \leq 1$, $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\beta(x_{m-1}, x_{n-1}, t) \leq 1$ and $\alpha(x_{m-1}, x_{n-1}, t) \geq 1$ for all $m, n \in \mathbb{N}$ with $m > n \geq k_0$ and for all $t > 0$,

(iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$, $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $t > 0$, and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$ and $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$. Then f has a fixed point.

Proof:

Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Define a sequence $\{x_n\}$ in X so that $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $u = x_n$ is a fixed point of f. Assume that $x_n \neq x_{n+1}$ for all

$n \in \mathbb{N}$. Then, by using the fact that f is $\alpha - \beta$- admissible, we write

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Rightarrow \alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq 1.$$

Similarly, we write

$$\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \leq 1 \Rightarrow \beta(x_1, x_2, t) = \beta(fx_0, fx_1, t) \leq 1.$$

By induction, it follows that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$, for all $n \in \mathbb{N}$.
From the properties of function φ, we can find $r > 0$ such that $t > \varphi(r)$ and therefore we have

$$\begin{aligned} F_{x_n,x_{n+1}}(t) &\geq F_{x_n,x_{n+1}}(\varphi(r)) \geq \beta(x_{n-1}, x_n, r) F_{fx_{n-1},fx_n}(\varphi(r)) \\ &\geq \alpha(x_{n-1}, x_n, r) \min\big\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_{n-1},x_n}(\varphi(r/c)), \\ &\quad F_{x_n,x_{n+1}}(\varphi(r/c)), F_{x_n,x_n}(\varphi(r/c))\big\} \\ &\geq \min\big\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_n,x_{n+1}}(\varphi(r/c)), \\ &\quad F_{x_n,x_n}(\varphi(r/c))\big\} \\ &= \ min\{F_{x_{n-1},x_n}(\varphi(r/c)), F_{x_n,x_{n+1}}(\varphi(r/c))\}. \end{aligned}$$

Next, if $F_{x_n,x_{n+1}}(\varphi(r/c))$ is the minimum, then

$$F_{x_n,x_{n+1}}(\varphi(r)) \geq F_{x_n,x_{n+1}}(\varphi(r/c))$$

and, from Lemma 2.1, we have $x_n = x_{n+1}$, which leads to contradiction with the assumption that $x_n \neq x_{n+1}$ for all $n \in N$.
On the other hand, if $F_{x_{n-1},x_n}(\varphi(r/c))$ is the minimum, then

$$F_{x_n,x_{n+1}}(t) \geq F_{x_{n-1},x_n}(\varphi(r/c)) \geq F_{x_{n-2},x_{n-1}}(\varphi(r/c^2)) \geq ... \geq F_{x_0,x_1}\varphi(r/c^n),$$

which, letting $n \to \infty$, gives us

$$F_{x_n,x_{n+1}}(t) \to 1. \tag{2.5}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not a Cauchy sequence, then for any $\varepsilon > 0$ and $\lambda \in (0, 1)$ there are subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $m(k) < n(k)$ with

$$F_{x_{m(k)},x_{n(k)}}(\varepsilon) \leq 1 - \lambda$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\varepsilon) > 1 - \lambda.$$

Since φ is continuous at 0 and strictly monotone increasing with $\varphi(0) = 0$, then there exists $\varepsilon_1 > 0$ such that $\varphi(\varepsilon_1) \leq \varepsilon$.

Thus, by the above arguments it is possible to obtain increasing sequences of integers $m(k)$ and $n(k)$, with $m(k) < n(k)$, such that

$$F_{x_{m(k)},x_{n(k)}}(\varphi(\varepsilon_1)) \leq 1-\lambda \tag{2.6}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\varphi(\varepsilon_1)) > 1-\lambda. \tag{2.7}$$

Since $\varepsilon \in (0,1)$ and $\varphi \in \Phi$, we can choose $\eta > 0$ such that

$$0 < \eta < \varphi(\varepsilon_1/c) - \varphi(\varepsilon_1),$$

that is,

$$\varphi(\varepsilon_1/c) - \eta > \varphi(\varepsilon_1)$$

and so, from (2.7), we get

$$F_{x_{m(k)},x_{n(k)-1}}(\varphi(\varepsilon_1/c)-\eta)) > F_{x_{m(k)},x_{n(k)-1}}(\varphi(\varepsilon_1)) > 1-\lambda. \tag{2.8}$$

Then, for any $0 < \lambda_1 < \lambda < 1$, by (2.5) it is possible to find a positive integer N_1 such that, for all $k > N_1$, we have

$$F_{x_{m(k)-1},x_{m(k)}}(\varphi(\eta)) > 1-\lambda_1 \tag{2.9}$$

and

$$F_{x_{n(k)-1},x_{n(k)}}(\varphi(\eta)) > 1-\lambda_1. \tag{2.10}$$

Now, by (PM3), we get

$$\begin{aligned} F_{x_{m(k)-1},x_{n(k)-1}}(\varphi(\varepsilon_1/c)) \quad \geq \quad & T(F_{x_{m(k)-1},x_{m(k)}}(\eta), \\ & F_{x_{m(k)},x_{n(k)-1}}(\phi(\varepsilon_1/c)-\eta)). \end{aligned} \tag{2.11}$$

Let λ_2 such that $0 < \lambda_2 < \lambda_1 < \lambda < 1$ be arbitrary. Then by (2.5) there exists a positive integer N_2 such that for all $k > N_2$, we have

$$F_{x_{m(k)-1},x_{m(k)}}(\eta) > 1-\lambda_2. \tag{2.12}$$

Now, using (2.8), (2.11) and (2.12), for all $k > max\{N_1, N_2\}$ we have

$$F_{x_{m(k)-1},x_{n(k)-1}}(\varphi(\varepsilon_1/c)) \geq T(1-\lambda_2, 1-\lambda).$$

Since λ_2 is arbitrary and T is continuous, we get

$$F_{x_{m(k)-1},x_{n(k)-1}}(\varphi(\varepsilon_1/c)) \geq 1-\lambda. \tag{2.13}$$

By using the above inequalities, we have

$$\begin{aligned}1-\lambda &\geq F_{x_{m(k)},x_{n(k)}}(\varphi(\varepsilon_1)) \geq F_{fx_{m(k)-1},fx_{n(k)-1}}(\varphi(\varepsilon_1))\\ &\geq \beta(x_{m(k)-1},x_{n(k)-1},\varepsilon_1)F_{fx_{m(k)-1},fx_{n(k)-1}}(\phi(\varepsilon_1))\\ &\geq \min\big\{F_{x_{m(k)-1},x_{n(k)-1}}(\varphi(\varepsilon_1/c)),F_{x_{m(k)-1},x_{m(k)}}(\varphi(\varepsilon_1/c)),\\ &\quad F_{x_{n(k)-1},x_{n(k)}}(\varphi(\varepsilon_1/c)),F_{x_{n(k)-1},x_{m(k)}}(\varphi(\varepsilon_1/c))\big\}\\ &> min\{1-\lambda,1-\lambda,1-\lambda,1-\lambda\}=1-\lambda,\end{aligned}$$

which is a contradiction, therefore $\{x_n\}$ is a Cauchy sequence in the complete Menger space. Thus $x_n \to u$ as $n \to \infty$ for some $u \in X$.
Now, we show that u is a fixed point of f. Since $\varphi \in \Phi$, we have that for all $x, y \in X$ and $t > 0$ there exists $r > 0$ such that $t > \varphi(r)$ and therefore we have

$$F_{fu,u}(t) \geq T(F_{fu,x_n}(\phi(r)),F_{x_n,u}(t-\phi(r))). \tag{2.14}$$

Since $t > \varphi(r)$, thus $t - \varphi(r) > 0$. Also, since $u = lim_{n\to\infty}x_n$, for arbitrary $\delta \in (0, 1)$ there exists $n_0 \in N$ such that for all $n \geq n_0$ we get

$$F_{x_n,u}(t-\varphi(r)) \geq 1-\delta. \tag{2.15}$$

Hence, from (2.14) and (2.15), we have

$$F_{fu,u}(t) \geq T(F_{fu,x_n}(\varphi(r)),1-\delta).$$

Notice that, if $x_n = fu$ for infinitely many values of n, then $u = fu$ and hence the proof finishes. Therefore, we assume that $x_n \neq fu$ for all $n \in N$. Consequently, since $\delta > 0$ is arbitrary and the $t-$norm T is continuous, we get

$$\begin{aligned}F_{fu,u}(t) &\geq F_{x_n,fu}(\varphi(r)) \geq F_{fx_{n-1},fu}(\varphi(r))\\ &\geq min\big\{F_{x_{n-1},u}(\varphi(r/c)),F_{x_{n-1},fx_{n-1}}(\varphi(r/c)),\\ &\quad F_{u,fu}(\varphi(r/c)),F_{u,fx_{n-1}}(\varphi(r/c))\big\}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the fact that T is continuous, we have

$$F_{fu,u}(\varphi(r)) \geq F_{fu,u}(\varphi(r/c))$$

and hence, by using Lemma 2.1, we get $fu = u$. This completes the proof. □

3. Application to integral equation

We give a typical application of fixed point methods to integral equations, by using our results in Section 2. Precisely, we consider the following Volterra type integral equation:

$$x(t) = g(t) + \int_0^t \Omega(t,s,x(s))ds, \quad \forall t \in [0,k], \quad k > 0. \tag{3.1}$$

Let $C([0,K],\mathbb{R})$ be the Banach space of all continuous functions defined on $[0,k]$ endowed with the sup norm

$$\|x\|_\infty = max_{t\in[0,k]} \mid x(t) \mid, \quad x \in C([0,k],\mathbb{R}).$$

Alternatively the Banach space $C([0,K],\mathbb{R})$ can be endowed with the Bielecki norm

$$\|x\|_B = max_{t\in[0,k]}(\mid x(t) \mid e^{-Lt}), \quad x \in C([0,k],\mathbb{R}), L > 0$$

and the induced metric $d_B(x,y) = \|x-y\|_B$, for all $x,y \in C([0,k],\mathbb{R})$, see [4]. We know that the two norms above are equivalent [15].
Next, we define the mapping $F : C([0,k],\mathbb{R}) \times C([0,k],\mathbb{R}) \to D$ by

$$F_{x,y}(t) = H(t - d_B(x,y)), \;\; t > 0, \;\; x,y \in C([0,k],\mathbb{R}).$$

V. M, Sehgal showed that the space $(C([0,k],\mathbb{R},F,T_M)$ is the (ϵ,λ)-complete Menger space induced by the Banach space $C([0,K],R)$, See [14].
Now, we discuss the existence of solution for Volterra type integral (3.1).

Theorem 3.1. *Let $(C([0,k],\mathbb{R},F,T_M)$ be the Menger space induced by the Banach space $C([0,K],\mathbb{R})$ and let $\Omega \in C([0,K]\times[0,K]\times\mathbb{R},\mathbb{R})$ be an operator satisfying the following conditions:*

(i) $\|\Omega\|_\infty = \sup_{t,s\in[0,k],x\in c([0,k],\mathbb{R})} |\Omega(t,s,x(s))| < \infty$,

(ii) there exists $L > 0$ such that for all $x,y \in C([0,K],\mathbb{R})$ and all $t,s \in [0,K]$ we get

$$\begin{aligned}|\Omega(t,s,fx(s)) - \Omega(t,s,fy(s))| \quad &\le L \quad \max\big\{|x(s)-y(s)|, |x(s)-fx(s)|,\\ &\qquad\qquad |y(s)-fy(s)|, |y(s)-fx(s)|\big\},\end{aligned}$$

where $f : C([0,K],\mathbb{R}) \to C([0,K],\mathbb{R})$ is defined by

$$fx(t) = g(t) + \int_0^t \Omega(t,s,fx(s))ds, \quad g \in C([0,K],\mathbb{R}).$$

Then, the Volterra type integral equation (3.1) has a unique solution $u \in C([0,K],\mathbb{R})$.

Proof: As mentioned above, $(C([0,k],R,F,T_M)$ is a complete Menger space. Let us consider the norm $\|x\|_B = max(\mid x(t) \mid e^{-Lt})$, where L satisfies condition (ii). Then, for all $x,y \in C([0,K],\mathbb{R})$, we get

$$\begin{aligned} d_B(fx,fy) \quad &\le \quad \max_{t\in[0,k]} \int_0^t |\Omega(t,s,fx(s)) - \Omega(t,s,fy(s))| e^{L(s-t)} e^{-Ls} ds \\ &\le \quad L \max\{d_B(x,y), d_B(x,fx), d_B(y,fy), \\ &\qquad d_B(y,fx)\} \max_{t\in[0,k]} \int_0^t e^{L(s-t)} ds \\ &\le \quad (1-e^{-Lk}) \max\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\}. \end{aligned}$$

Putting $c = 1 - e^{-Lk}$, by using the definition of $F_{x,y}$, for any $r \geq 0$ we have

$$\begin{aligned} F_{fx,fy} &\geq H(r - c\max\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\}) \\ &= H(\frac{r}{c} - \max\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\}) \\ &= \min\{F_{x,y}(\frac{r}{c}), F_{x,fx}(\frac{r}{c}), F_{y,fy}(\frac{r}{c}), F_{y,fx}(\frac{r}{c})\}, \end{aligned}$$

for all $x, y \in C([0,K],\mathbb{R})$. Therefore, by theorem 2.11 with $\varphi(r) = r$ for all $r \geq 0$ and $\beta(x,y,r) = \alpha(x,y,r) = 1$ for all $x, y \in C([0,K],\mathbb{R})$ and $r > 0$, we deduce that the operator f has a unique fixed point $u \in C([0,K],\mathbb{R})$, which is the unique solution of the integral equation (3.1).

□

References

[1] M. S. Asgari, Z. Badehian, *fixed point theorems for $\alpha-\beta, \psi$-contractive mappings in partially ordered sets,* J. Nonlinear Sci. Appl. 8 (2015) 518-528.

[2] W. A. Kutbi, W. Sintunavarat, *On new fixed point results for (α, ψ, ζ)-contractive multi-valued mappings on α-complete metric spaces and their consequences,* Fixed point Theory and Applications, (2015) 2015:2.

[3] N. A. Babacev, *Nonlinear generalized contraction on Menger PM-spaces,* Appl. Anal. Discrete Math. 6 (2012) 257-264.

[4] A. Bielecki, *Une remarque sur la methode de Banach–Cacciopoli–Tikhonov dans la theorie des equations differentielles ordinaires,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom Phys. 4 (1956) 261-264.

[5] B. S. Choudhury, K. P. Das, *A new contraction principle in Menger spaces,* Acta Math. Sin. 24 (2008) 1379-1386, .

[6] B. S. Choudhury, K. P. Das, S. K. Bhandari,*Fixed point theorem for mappings with cyclic contraction in Menger spaces,*Int. J. Pure Appl. Sci. Technol. 4 (2011) 1-9.

[7] P. N. Dutta, B. S. Choudhury, K. P. Das, *Some fixed point results in Menger spaces using a control function,* Surv. Math. Appl. 4 (2009) 41-52.

[8] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht 2001.

[9] M. S. Khan, M. Swaleh, S. Sessa, *Fixed points theorems by altering distances between the points,* Bull. Aust. Math. Soc. 30 (1984) 1-9.

[10] K. Menger, *Statistical metric,* Proc. Natl. sci. 28 (1942) 535-538.

[11] D. Mihet, *Altering distances in probabilistic Menger spaces,* Nonlinear Anal. 71 (2009) 2734-2738.

[12] B. Schweizer, A. Sklar, *Probabilistic Metric Space*, Elsevier, North-Holland, New York 1983.

[13] V. M. Sehgal, *Some fixed point theorems in functional analysis and probability*, Ph. D. Dissertation, Wayne State Univ, Detorit, Michigan 1966.

[14] V. M. Sehgal, T. A. Bharucha-Reid, *Fixed point of contraction mappigs in PM-spaces,* Math. Syst. Theory. 6 (1972) 97-102.

[15] Z. Shisheng, *On the theory of probabilistic metric spaces with application,* Acta Math. Sin. 1 (1985) 366-377.

[16] R. Vasuki, P. Veeramani, *Fixed point theorems and Cauchy sequences in fuzzy metric spaces,* Fuzzy Sets and Systems. 135 (2003) 415-417.

[17] M. Grabiec, *Fixed points in fuzzy metric spaces,* Fuzzy Sets and Systems. 27 (1983) 385-389.

M. Shams,
Faculty of Mathematical Sciences, Department of Pure Mathematics,
University of Shahrekord, P. O. Box 88186-34141, Shahrekord, Iran.
E-mail address: Maryam.shams@sci.sku.ac.ir
E-mail address: jafari.shahnaz@yahoo.com