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Non-linear Elliptic Unilateral Problems in Musielak-Orlicz spaces with L^1 data

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ABSTRACT: We prove an existence result of solutions for nonlinear elliptic unilateral problems having natural growth terms and L^1 data in Musielak-Orlicz-Sobolev space W^1L_φ, under the assumption that the conjugate function of φ satisfies the Δ_2-condition.

Key Words: Musielak-Orlicz spaces, non-linear problems, unilateral problems, truncations.

Contents

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$. Consider the following non-linear Dirichlet problem

$$A(u) + g(x, u, \nabla u) = f, \tag{1.1}$$

where $A(u) = -\text{div } a(x, u, \nabla u)$ is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1}L_\psi(\Omega)$ with φ and ψ are two complementary Musielak-Orlicz functions, and g is a non-linearity with sign condition and satisfying, for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$ and almost all $x \in \Omega$, the following natural growth condition:

$$|g(x, s, \xi)| \leq b(|s|)(a_0(x) + \varphi(x, |\xi|)),$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function and $a_0(.)$ is a given non-negative function in $L^1(\Omega)$.
The right-hand side f is assumed to belongs to $L^1(\Omega)$.

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On Orlicz spaces and in the variational case, it is well known that Gossez and Mustonen solved in [20] the following obstacle problem

$$\begin{cases} u \in K_\phi \\ \langle A(u), u-v\rangle + \int\limits_\Omega g(x,u)(u-v)\,dx \le \langle f, u-v\rangle \\ \text{for all } v \in K_\phi \cap L^\infty(\Omega). \end{cases} \tag{1.2}$$

where K_ϕ is a convex subset in $W_0^1 L_M(\Omega)$ given by $K_\phi = \{v \in W_0^1 L_M(\Omega) : v \ge \phi \text{ a.e in } \Omega\}$, with ϕ is a measurable function satisfying some regularity condition. An existence result has been proved in [2] by Aharouch, Benkirane and Rhoudaf where the nonlinearity g depend on x, u and ∇u and without assuming the Δ_2-condition on the N-function.

In the case where $f \in L^1(\Omega)$, the unilateral problem corresponding to (1.1) has been studied in [3] by Aharouch and Rhoudaf and in [16] by Elmahi and Meskine without assuming the Δ_2-condition on the N-function.

In the framework of variable exponent Sobolev spaces, Azroul, Redwane and Yazough have shown in [6] the existence of solutions for the unilateral problem associated to (1.1) where the second member f is in $L^1(\Omega)$.

In the setting of Musielak-Orlicz spaces and in variational case, Benkirane and Sidi El vally [12] proved the existence of solutions for the obstacle problem (1.2), they generalized the work of Gossez and Mustonen in [20].

The purpose of this paper is to prove, in the setting of Musielak spaces, an existence result for unilateral problem corresponding to (1.1) in the case where $f \in L^1(\Omega)$ under the assumption that the conjugate function of the Musielak-Orlicz function φ satisfies the Δ_2-condition and by assuming

$$\int_1^\infty \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt = \infty \quad \text{for a.e. } x \in \Omega. \tag{1.3}$$

This assumption (1.3) allows us to use a Poincaré type inequality in the proof of the main result of this work (Theorem 3.3). Remark that this condition corresponds, in the classical Sobolev spaces $W^{1,p}$ to the case $p < N$, which is the interesting case in these spaces.
Further works for the unilateral problem corresponding to (1.1) in the L^p case can be found in [13,14,15].

2. Preliminaries

Musielak-Orlicz function. Let Ω be an open subset of $\mathbb{R}^N$ $(N \ge 2)$, and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

i) $\varphi(x,.)$ is an N-function for a.a. $x \in \Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0 \ \ \forall\, t > 0$, $\lim\limits_{t\to 0} \sup\limits_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0$ and $\lim\limits_{t\to\infty} \inf\limits_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$);

ii) $\varphi(.,t)$ is a measurable function for all $t \geq 0$.

A function φ which satisfies the conditions *i)* and *ii)* is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is $\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$.

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $C > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq \; C\varphi(x,t) + h(x) \quad \text{for all} \quad x \in \Omega \text{ and all } t \geq 0. \tag{2.1}$$

when (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exists two positive constants c and t_0 such that for almost all $x \in \Omega : \gamma(x,t) \leq \varphi(x,c\,t)$ for all $t \geq t_0$ (resp. for all $t \geq 0$ i.e. $t_0 = 0$).

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, if for every positive constant c, we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,c\,t)}{\varphi(x,t)}\right) = 0 \quad (\text{resp.} \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,c\,t)}{\varphi(x,t)}\right) = 0).$$

Remark 2.1. *[12] If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have $\gamma(x,t) \leq k(\varepsilon)\,\varphi(x,\varepsilon t)$ for all $t \geq 0$.*

Musielak-Orlicz space. For a Musielak-Orlicz function φ and a measurable function
$u : \Omega \to \mathbb{R}$ we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)\,dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \varrho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (or generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} : \varrho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0\right\}.$$

For a Musielak-Orlicz function φ we put $\psi(x,s) = \sup_{t\geq 0}(st - \varphi(x,t))$, ψ is called the Musielak-Orlicz function complementary to φ (or conjugate of φ).

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0$, this implies convergence for $\sigma(\Pi L_\varphi, \Pi L_\psi)$ (Lemma 4.7 of [12]).

In the space $L_\varphi(\Omega)$ we define the Luxemburg norm by:

$$\|u\|_{\varphi,\Omega} = \inf\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\, dx \le 1\},$$

and the Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \le 1} \int_\Omega |u(x)\, v(x)|\, dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [22]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_\psi(\Omega))^* = L_\varphi(\Omega)$ [22]. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfy the Δ_2-condition (2.1) for large values of t or for all values of t, according to whether Ω has finite measure or not.

We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall |\alpha| \le 1\}$$
$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall |\alpha| \le 1\},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\le 1} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^1_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \le 1\right\} \text{ for } u \in W^1L_\varphi(\Omega).$$

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$ respectively.
The pair $\langle W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [22]:

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \ge c. \tag{2.2}$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\le 1} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

We denote by $\mathfrak{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathfrak{D}(\overline{\Omega})$ the restriction of $\mathfrak{D}(\mathbb{R}^N)$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

For two complementary Musielak-Orlicz functions φ and ψ, we have [22]:

i) The Young inequality:

$$ts \le \varphi(x, t) + \psi(x, s) \quad \text{for all } t, s \ge 0,\ x \in \Omega. \tag{2.3}$$

ii) The Hölder inequality:

$$\left|\int_\Omega u(x)\, v(x)\, dx\right| \le 2\|u\|_{\varphi,\Omega}\, \|v\|_{\psi,\Omega}, \text{ for all } u \in L_\varphi(\Omega), v \in L_\psi(\Omega). \tag{2.4}$$

We say that a sequence of functions u_n converges to u for modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W_0^1L_\varphi(\Omega)$) if, for some $\lambda > 0$, $\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0$.

The following spaces of distributions will also be used:

$$W^{-1}L_\psi(\Omega) = \{f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in L_\psi(\Omega)\}$$

$$W^{-1}E_\psi(\Omega) = \{f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in E_\psi(\Omega)\}.$$

Lemma 2.2. *[11] Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

(i) *There exists a constant $c > 0$ such that* $\inf_{x\in\Omega} \varphi(x,1) \geq c$; [(2.2)]

(ii) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)} \quad \textit{for all} \quad t \geq 1; \tag{2.5}$$

(iii) $$\int_\Omega \varphi(x,1)\,dx < \infty; \tag{2.6}$$

(iv) *There exists a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.* (2.7)

Under these assumptions, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$, $\mathfrak{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ and $\mathfrak{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $\langle S, u\rangle$.

Lemma 2.3. *[12] Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \textit{a.e in} \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & \textit{a.e in} \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.4. *[4] **(The Nemytskii operator)** Let Ω be an open susbset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \to \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$*

$$|f(x,s)| \leq c(x) + \alpha_1\, \psi_x^{-1}\, \varphi(x, \alpha_2|s|)$$

where α_1, α_2 are real positive constants and $c(.) \in E_\psi(\Omega)$.
Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x,u(x))$, is continuous

from $(\mathcal{P}(E_\varphi(\Omega), \frac{1}{\alpha_2}))^p = \Pi\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{\alpha_2}\}$ *into* $(L_\psi(\Omega))^q$ *for the modular convergence.*
Furthermore if $c \in E_\gamma(\Omega)$ *and* $\gamma \prec\prec \psi$ *then* N_f *is strongly continuous from* $(\mathcal{P}(E_\varphi(\Omega), \frac{1}{\alpha_2}))^p$ *into* $(E_\gamma(\Omega))^q$.

Lemma 2.5. *Let* $f_n, f \in L^1(\Omega)$ *such that*

i) $f_n \geq 0$ *a.e in* Ω*;*

ii) $f_n \to f$ *a.e in* Ω*;*

iii) $\int_\Omega f_n(x)\, dx \to \int_\Omega f(x)\, dx.$

Then $f_n \to f$ *strongly in* $L^1(\Omega)$.

The following theorem has already been treated in [5] but we think it is useful to give it again in order to facilitate the reading of this work, it is a Poincaré type inequality in Musielak spaces, for more details see [5].

Theorem 2.6. *[5] Let* Ω *be a bounded Lipschitz domain of* $\mathbb{R}^N$, *and let* φ *be a Musielak-Orlicz function satisfying (1.3) and the conditions (i), (ii), (iii) and (iv) of Lemma 2.2 then there exists a constant* $C(\Omega, \varphi) > 0$ *such that*

$$\|u\|_\varphi \leq C\, \|\nabla u\|_\varphi \qquad \forall u \in W_0^1 L_\varphi(\Omega)$$

Proof:
Suppose, by contradiction, that for every $n \in \mathbb{N}^*$, there exists $w_n \in W_0^1 L_\varphi(\Omega)$ such that

$$\|w_n\|_\varphi > n\, \|\nabla w_n\|_\varphi$$

define the sequence $u_n \in W_0^1 L_\varphi(\Omega)$ by $u_n = \sqrt{n}\, \frac{w_n}{\|w_n\|}$, we have

$$\|u_n\|_\varphi = \sqrt{n} \quad \text{and} \quad \|\nabla u_n\|_\varphi < \frac{1}{\sqrt{n}}$$

then $\nabla u_n \to 0$ strongly in $L_\varphi(\Omega)$, which imply that

$$\nabla u_n \to 0 \text{ in } \mathcal{D}'(\Omega). \tag{2.8}$$

Since (u_n) is bounded in $W_0^1 L_\varphi(\Omega)$, there exists a subsequence, denoted by (u_{n_k}), weakly convergent in $W_0^1 L_\varphi(\Omega)$ for the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$.
By using the compact imbedding $W_0^1 L_\varphi(\Omega) \hookrightarrow\hookrightarrow L_\varphi(\Omega)$ (see Theorem 3 of [10]), there exists a function $v \in L_\varphi(\Omega)$, and a subsequence, still denoted by (u_{n_k}), such that $u_{n_k} \to v$ strongly in $L_\varphi(\Omega)$, thus $u_{n_k} \to v$ in $\mathcal{D}'(\Omega)$, and so

$$\nabla u_{n_k} \to \nabla v \ \text{ in } \ \mathcal{D}'(\Omega). \tag{2.9}$$

By combining (2.8) and (2.9), we obtain $\nabla v = 0$, and this imply that v is a constant function because Ω is connected. Consequently $u_{n_k} \to \alpha$ strongly in $L_\varphi(\Omega)$, where α is a constant.
A contradiction, since $\|u_n\|_\varphi = \sqrt{n}$. □

3. Main result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 2)$, and let φ and γ be two Musielak-Orlicz functions such that $\gamma \prec\prec \varphi$ and φ satisfies the assumption (1.3) and conditions of Lemma 2.2.
Given an obstacle measurable function $\Lambda : \Omega \to \mathbb{R}$ and consider the set

$$K_\Lambda = \{u \in W_0^1 L_\varphi(\Omega) : u \geq \Lambda \text{ a.e in } \Omega\}$$

This convex set is sequentially $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed in $W_0^1 L_\varphi(\Omega)$ (see [12]).
Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ be a mapping (not everywhere defined) given by: $A(u) = -\text{div} a(x, u, \nabla u)$ where ψ is the Musielak function complementary to φ which satisfies the Δ_2-condition and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$ and all ξ ,$\xi_* \in \mathbb{R}^N$, $\xi \neq \xi_*$

$$|a(x, s, \xi)| \leq k_1 \left(c(x) + \psi_x^{-1}(\gamma(x, k_2|s|)) + \psi_x^{-1} (\varphi(x, k_3|\xi|)\right) \tag{3.1}$$

$$(a(x, s, \xi) - a(x, s, \xi_*)) \, (\xi - \xi_*) > 0 \tag{3.2}$$

$$a(x, s, \xi) \, (\xi - \nabla v_0) \geq \alpha \, \varphi(x, |\xi|) - c'(x) \tag{3.3}$$

with $v_0 \in K_\Lambda \cap W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega), c'(.) \in L^1(\Omega), \alpha, k_1, k_2, k_3 > 0$ and $c(.) \in E_\psi(\Omega)$.

Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$g(x, s, \xi) \, s \geq 0 \tag{3.4}$$

$$|g(x, s, \xi)| \leq b(|s|) \, (a_0(x) + \varphi(x, |\xi|)) \tag{3.5}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function and $a_0(.)$ is a given non-negative function in $L^1(\Omega)$.
Now, assume that

$$K_\Lambda \cap W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega) \text{ is dense in } K_\Lambda \cap L^\infty(\Omega) \tag{3.6}$$

for the modular convergence in $W_0^1 L_\varphi(\Omega)$.

Remark 3.1. *[12] If $\Lambda \in W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega)$ or if there exists $\overline{\Lambda} \in K_\Lambda \cap W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega)$ such that $\Lambda - \overline{\Lambda}$ is continuous then (3.6) is satisfied.*

Example 3.2. *Consider the following Dirichlet problem*

$$-div\left(a(x, u) m(x, |\nabla u|) \frac{\nabla u}{|\nabla u|}\right) + g(u) m(x, |\nabla u|) |\nabla u| = f \quad in \ \Omega,$$

where $a(x, u)$ is a Carathéodory function such that $0 \leq \mu \leq a(x, u) \leq \nu$, m is the derivative of the Musielak function φ with respect to t and g is a continuous function satisfying $g(s)s \geq 0$. Then the assumptions (3.1)-(3.5) hold true. (see Remark 3.2 of [16])

Finally, we assume that

$$f \in L^1(\Omega). \tag{3.7}$$

Define $T_0^{1,\varphi}(\Omega)$ to be the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1 L_\varphi(\Omega)$, where $T_k(.)$ is the truncation at height $k > 0$, defined by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

We shall prove the following existence theorem.

Theorem 3.3. *Assume that (3.1)-(3.7) hold true, then there exists at least one solution of the following unilateral problem*

$$(\mathcal{P}_\Lambda) \begin{cases} u \in T_0^{1,\varphi}(\Omega), \quad u \ge \Lambda \ a.e \ in \ \Omega, \quad g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)\; T_k(u-v)\,dx \le \int_\Omega f\; T_k(u-v)\,dx, \\ \text{for all } v \in K_\Lambda \cap L^\infty(\Omega) \text{ and for all } k \ge 0. \end{cases}$$

Proof:
Step 1 : **A priori estimates**.
For $k \ge \|v_0\|_\infty$, let $\delta = (\frac{b(k)}{2\alpha})^2$ and $\phi(s) = s\exp(\delta s^2)$. It is well known that

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \ge \frac{1}{2}, \quad \forall s \in \mathbb{R}. \tag{3.8}$$

Let (f_n) be a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$ and set $g_n(x,s,\xi) = T_n(g(x,s,\xi))$.
Consider the approximate unilateral problems

$$(\mathcal{P}_n) \quad \begin{cases} u_n \in K_\Lambda \cap D(A), \\ \displaystyle\langle A(u_n), u_n - v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \le \int_\Omega f_n(u_n - v)\,dx \\ \text{for all } v \in K_\Lambda. \end{cases}$$

where $\langle .\,,.\rangle$ means the duality between $W_0^1 L_\varphi(\Omega)$ and $W^{-1}L_\psi(\Omega)$.
Note that $g_n(x,s,\xi)\; s \ge 0,\ |g_n(x,s,\xi)| \le |g(x,s,\xi)|$ and $|g_n(x,s,\xi)| \le n$.
Since g_n is bounded for any fixed $n > 0$, there exists at least one solution $u_n \in K_\Lambda \cap D(A)$ of $(\mathcal{P}_n)$. (see Proposition 5 of [20] and Theorem 8 of [12])
Taking $u_n - \beta_1\phi(T_\eta(u_n - v_0))$ as test function in $(\mathcal{P}_n)$, where $\eta = k + \|v_0\|_\infty$ and $\beta_1 = \exp(-\delta\eta^2)$ we obtain

$$\int_{\{|u_n - v_0| < \eta\}} a(x,u_n,\nabla u_n)\,(\nabla u_n - \nabla v_0)\phi'(T_\eta(u_n - v_0))\,dx$$
$$+ \int_\Omega g_n(x,u_n,\nabla u_n)\;\phi(T_\eta(u_n - v_0))\,dx \le \int_\Omega f_n\;\phi(T_\eta(u_n - v_0))\,dx$$

Since $g_n(x, u_n, \nabla u_n)\, \phi(T_\eta(u_n - v_0)) \geq 0$ on the set $\{x \in \Omega : |u_n| \geq k\}$, we have

$$\int\limits_{\{|u_n - v_0| < \eta\}} a(x, u_n, \nabla u_n)\, (\nabla u_n - \nabla v_0)\phi'(T_\eta(u_n - v_0))\, dx + \int\limits_{\{|u_n| < k\}} g_n(x, u_n, \nabla u_n)\, \phi(T_\eta(u_n - v_0))\, dx \leq \int\limits_\Omega f_n\, \phi(T_\eta(u_n - v_0))\, dx$$

and by using (3.5), one easily has

$$\int\limits_{\{|u_n - v_0| < \eta\}} a(x, u_n, \nabla u_n)\, (\nabla u_n - \nabla v_0)\phi'(T_\eta(u_n - v_0))\, dx \leq b(k) \int\limits_{\{|u_n| < k\}} |\phi(T_\eta(u_n - v_0))|(a_0(x) + \varphi(x, |\nabla u_n|))\, dx + \int\limits_\Omega f_n\, \phi(T_\eta(u_n - v_0))\, dx,$$

from (3.3) and by using the fact that $\{x \in \Omega : |u_n| < k\} \subseteq \{x \in \Omega : |u_n - v_0| < \eta\}$ and $a_0(.),\ c'(.),\ f_n \in L^1(\Omega)$ we get

$$\int\limits_{\{|u_n - v_0| < \eta\}} \varphi(x, |\nabla u_n|) \left(\phi'(T_\eta(u_n - v_0)) - \frac{b(k)}{\alpha} |\phi(T_\eta(u_n - v_0))| \right) dx \leq C_\eta$$

where C_η is a positive constant depending on η,
thanks to (3.8), we have

$$\int\limits_{\{|u_n - v_0| < \eta\}} \varphi(x, |\nabla u_n|)\, dx \leq C_\eta, \quad \forall n,$$

consequently

$$\int\limits_{\{|u_n| < k\}} \varphi(x, |\nabla u_n|)\, dx \leq C_\eta, \quad \forall n. \tag{3.9}$$

Now, the use of $v = u_n - T_k(u_n - v_0)$ as test function in $(\mathcal{P}_n)$ yields

$$\int\limits_{\{|u_n - v_0| < k\}} a(x, u_n, \nabla u_n)\, (\nabla u_n - \nabla v_0)\, dx + \int\limits_{\{|u_n| < \|v_0\|_\infty\}} g_n(x, u_n, \nabla u_n) T_k(u_n - v_0)\, dx \leq \int\limits_\Omega f_n T_k(u_n - v_0)\, dx$$

then from (3.5) and (3.9), we get

$$\int_{\{|u_n-v_0|<k\}} a(x,u_n,\nabla u_n)\,(\nabla u_n-\nabla v_0)\,dx \le C\,k, \tag{3.10}$$

where C is independent of k.
Hence, by using (3.3) we obtain

$$\int_{\{|u_n-v_0|<k\}} \varphi(x,|\nabla u_n|)\,dx \le C\,k.$$

Finally, since k is arbitrary we obtain

$$\int_{\{|u_n|<k\}} \varphi(x,|\nabla u_n|)\,dx \le \int_{\{|u_n-v_0|<k+\|v_0\|_\infty\}} \varphi(x,|\nabla u_n|)\,dx \le C\,(k+\|v_0\|_\infty)$$

thus

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \le C\,(k+\|v_0\|_\infty). \tag{3.11}$$

On the other hand, since ψ (the conjugate of φ) satisfies the Δ_2-condition then, from proposition 2.1 of [17], there exists $\nu>0$ and $c>0$ such that

$$\varphi(x,t) \ge c\,t^{1+\nu} \text{ for all } t \ge \text{ some } t_0>0. \tag{3.12}$$

We have

$$meas\{|u_n|\ge k\} = meas\{|T_k(u_n)|\ge k\},$$

then by the Chebyshev, the Poincaré inequality, (3.12) and (3.11) we obtain

$$\begin{aligned} meas\{|u_n|>k\} &\le \int_\Omega \frac{|T_k(u_n)|^{1+\nu}}{k^{1+\nu}}\,dx \\ &\le \frac{C_{\nu,N}}{k^{1+\nu}} \int_\Omega |\nabla T_k(u_n)|^{1+\nu}\,dx \\ &\le \frac{C_{\nu,N}}{k^{1+\nu}} \int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \\ &\le \frac{C_{\nu,N}}{k^{1+\nu}}(k+\|v_0\|_\infty) \qquad \forall n, \quad \forall k>0, \end{aligned}$$

where $C_{\nu,N}$ is a constant from the Poincaré inequality in $W_0^{1,1+\nu}$.
For any $\mu>0$, we have

$$meas\{|u_n-u_m|>\mu\} \le meas\{|u_n|>k\}+meas\{|u_m|>k\}+meas\{|T_k(u_n)-T_k(u_m)|>\mu\}$$

then

$$meas\{|u_n - u_m| > \mu\} \le \frac{2C_{\nu,N}\ (k + \|v_0\|_\infty)}{k^{1+\nu}} + meas\{|T_k(u_n) - T_k(u_m)| > \mu\}. \tag{3.13}$$

From (3.11) and by using Theorem 2.6, we deduce that $(T_k(u_n))_n$ is bounded in $W_0^1L_\varphi(\Omega)$ and then we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then by (3.13) and the fact that $\frac{k+\|v_0\|_\infty}{k^{1+\nu}} \to 0$ as $k \to \infty$, there exists $k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \mu\} \le \varepsilon \quad \text{for all } n, m \ge n_0\ (k(\varepsilon), \mu).$$

This proves that (u_n) is a Cauchy sequence in measure in Ω, and then converges almost everywhere to some measurable function u.
Finally, by Lemma 4.4 of [19], we obtain for all $k > 0$

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi),\\ &\quad \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{3.14}$$

Now, we shall prove that $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $L_\psi(\Omega)^N$ for all $k > 0$.
Let $\vartheta \in E_\varphi(\Omega)^N$ arbitrary. By using (3.2), we have for every $k > 0$,

$$\begin{aligned} \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n)(\frac{\vartheta}{k_3} - \nabla v_0)\, dx &\le \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0)\, dx \\ &+ \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \frac{\vartheta}{k_3})(\frac{\vartheta}{k_3} - \nabla u_n)\, dx \end{aligned}$$

where k_3 is defined in (3.1),
which gives by (3.10)

$$\int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n)(\frac{\vartheta}{k_3} - \nabla v_0)\, dx \le C\ k + \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \frac{\vartheta}{k_3})(\frac{\vartheta}{k_3} - \nabla u_n)\, dx.$$

Since ϑ is arbitrary in $E_\varphi(\Omega)^N$, choose $\omega = \frac{\vartheta}{k_3} - \nabla v_0$ in the last inequality with $\|\omega\|_{L_\varphi(\Omega)^N} = 1$ and we find

$$\int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n)\, \omega\, dx \le C\ k + \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \frac{\vartheta}{k_3})(\frac{\vartheta}{k_3} - \nabla u_n)\, dx$$

On the other hand, for β large enough, we have by using (3.1)

$$\int_{\{|u_n - v_0| \le k\}} \psi(x, \frac{|a(x, u_n, \frac{\vartheta}{k_3})|}{\beta})\, dx \le \frac{k_1}{\beta}\Big(\int_\Omega \psi(x, c(x))\, dx + \int_\Omega \varphi(x, |\vartheta|)\, dx$$
$$+ \int_{\{|u_n - v_0| \le k\}} \gamma(x, k_2(k + \|v_0\|_\infty))\, dx\Big),$$

thanks to Remark 2.1, there exists $\zeta(k) > 0$ such that $\gamma(x, k_2(k + \|v_0\|_\infty)) \le \zeta(k)\varphi(x, 1)$ then

$$\int_{\{|u_n - v_0| \le k\}} \psi(x, \frac{|a(x, u_n, \frac{\vartheta}{k_3})|}{\beta})\, dx \le C_{k,v_0}$$

consequently

$$\int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n)\, \omega\, dx \le C_{k,v_0}$$

where C_{k,v_0} is a constant which depends on k and v_0 but not on n.
Hence, using the dual norm, one has $(a(x, u_n, \nabla u_n)\chi_{\{|u_n - v_0| \le k\}})_n$ is bounded in $L_\psi(\Omega)^N$.
Then, for $k > 0$ we have

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\, \omega\, dx \le \int_\Omega |a(x, u_n, \nabla u_n)|\chi_{\{|u_n - v_0| \le k + \|v_0\|_\infty\}}\, \omega\, dx$$

which gives by Hölder inequality

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\, \omega\, dx \le 2\|a(x, u_n, \nabla u_n)\chi_{\{|u_n - v_0| \le k + \|v_0\|_\infty\}}\|_{L_\psi(\Omega)^N}$$

so that $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $L_\psi(\Omega)^N$,
which implies that, for all $k > 0$ there exists a function $l_k \in L_\psi(\Omega)^N$, such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k \text{ weakly in } L_\psi(\Omega)^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi). \tag{3.15}$$

Step 2 : **Almost everywhere convergence of the gradients**.
For $k > \|v_0\|_\infty$, let $\Omega_r = \{x \in \Omega, |\nabla T_k(u(x))| \le r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $|\Omega \backslash \Omega_r| \to 0$ as $r \to \infty$.

Let $s \geq r$, we have

$$
\begin{aligned}
0 &\leq \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\, dx \\
&\leq \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\, dx \\
&= \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \\
&\leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \qquad (3.16)
\end{aligned}
$$

By assumption (3.6) there exists a sequence $v_j \in K_\Lambda \cap W_0^1 E_\varphi(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_\varphi(\Omega)$.
Let $h > 2k > 0$, and define

$$
\begin{aligned}
\omega_{n,j}^h &= T_{2k}(u_n - v_0 - T_h(u_n - v_o) + T_k(u_n) - T_k(v_j)) \\
\omega_j^h &= T_{2k}(u - v_0 - T_h(u - v_o) + T_k(u) - T_k(v_j)) \\
\omega^h &= T_{2k}(u - v_0 - T_h(u - v_o)).
\end{aligned}
$$

Taking $v_{n,j}^h = u_n - \beta_2\, \phi(\omega_{n,j}^h)$ as test function in $(\mathcal{P}_n)$, where $\beta_2 = \exp(-4\delta k^2)$ we obtain

$$
\langle A(u_n), \phi(\omega_{n,j}^h)\rangle + \int_\Omega g_n(x, u_n, \nabla u_n)\, \phi(\omega_{n,j}^h)\, dx \leq \int_\Omega f_n\, \phi(\omega_{n,j}^h)\, dx,
$$

which implies that

$$
\int_\Omega a(x, u_n, \nabla u_n) \nabla \omega_{n,j}^h\, \phi'(\omega_{n,j}^h)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\, \phi(\omega_{n,j}^h)\, dx \leq \int_\Omega f_n\, \phi(\omega_{n,j}^h)\, dx. \qquad (3.17)
$$

Set $m = h + 5k$, and denote by $\epsilon(n, j, h)$ any quantity such that $\lim_{h\to\infty} \lim_{j\to\infty} \lim_{n\to\infty} \epsilon(n, j, h) = 0$ and by $\epsilon_h(n, j)$ any quantity such that $\lim_{j\to\infty} \lim_{n\to\infty} \epsilon_h(n, j) = 0$, for h fixed.
Observe that $\nabla \omega_{n,j}^h = 0$ on the set $\{x \in \Omega : |u_n| > m\}$, then we have from (3.17)

$$
\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n)) \nabla \omega_{n,j}^h\, \phi'(\omega_{n,j}^h)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\, \phi(\omega_{n,j}^h)\, dx \leq \int_\Omega f_n\, \phi(\omega_{n,j}^h)\, dx,
$$

using (3.14), we have $\phi(\omega_{n,j}^h) \to \phi(\omega_j^h)$ weakly in $L^\infty(\Omega)$ as $n \to +\infty$, and then

$$\int_\Omega f_n\ \phi(\omega_{n,j}^h)\,dx \to \int_\Omega f\ \phi(\omega_j^h)\,dx \text{ as } n \to +\infty,$$

letting j and h to infinity and using Lebesgue theorem we get

$$\int_\Omega f_n\ \phi(\omega_{n,j}^h)\,dx = \epsilon(n,j,h).$$

Since $g_n(x,u_n,\nabla u_n)\phi(\omega_{n,j}^h)\,dx \geq 0$ on the set $\{x \in \Omega : |u_n(x)| > k\}$, we have from (3.17)

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\,dx + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\ \phi(\omega_{n,j}^h)\,dx \leq \epsilon(n,j,h). \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\,dx &= \int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j))\ \phi'(\omega_{n,j}^h)\,dx \\ &\quad + \int_{\{|u_n|>k\}} a(x,T_m(u_n),\nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\,dx. \end{aligned}\tag{3.19}$$

The first term of the right hand side of the last equality can write as

$$\begin{aligned}\int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j))\ \phi'(\omega_{n,j}^h)\,dx &\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j))\ \phi'(\omega_{n,j}^h)\,dx \\ &\quad - \phi'(2k)\int_{\{|u_n|>k\}} |a(x,T_k(u_n),0)||\nabla T_k(v_j)|\,dx. \end{aligned}\tag{3.20}$$

Since $|a(x,T_k(u_n),0)|\chi_{\{|u_n|>k\}}$ converges to $|a(x,T_k(u),0)|\chi_{\{|u|>k\}}$ strongly in $L_\psi(\Omega)$, and $|\nabla T_k(v_j)|$ modular converges to $|\nabla T_k(u)|$, then

$$-\phi'(2k)\int_{\{|u_n|>k\}} |a(x,T_k(u_n),0)||\nabla T_k(v_j)|\,dx = \epsilon(n,j).$$

The second term of the right hand side of (3.19) can write as, using (3.2)

$$\int_{\{|u_n|>k\}} a(x,T_m(u_n),\nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\,dx$$
$$\geq -\phi'(2k)\int_{\{|u_n|>k\}} |a(x,T_m(u_n),\nabla T_m(u))||\nabla T_k(v_j)|\,dx$$
$$-\phi'(2k)\int_{\{|u_n-v_0|>h\}} c'(x)\,dx. \quad (3.21)$$

Using (3.15) and modular convergence of (v_j), it is easy to see that

$$-\phi'(2k)\int_{\{|u_n|>k\}} |a(x,T_m(u_n),\nabla T_m(u))||\nabla T_k(v_j)|\,dx = \epsilon_h(n,j). \quad (3.22)$$

and since $c'(.)\in L^1(\Omega)$ we have

$$-\phi'(2k)\int_{\{|u_n-v_0|>h\}} c'(x)\,dx = \epsilon(n,h). \quad (3.23)$$

Combining (3.19)-(3.23), we deduce

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\,dx$$
$$\geq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))\ \phi'(\omega_{n,j}^h)\,dx$$
$$+\epsilon(n,h)+\epsilon(n,j)+\epsilon_h(n,j),$$

it follows that

$$\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\,dx$$
$$\geq \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\right]$$
$$\times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_{n,j}^h)\,dx$$
$$+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j).\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_{n,j}^h)\,dx$$
$$-\int_{\Omega\setminus\Omega_s^j} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(v_j)\ \phi'(\omega_{n,j}^h)\,dx$$
$$+\epsilon(n,h)+\epsilon(n,j)+\epsilon_h(n,j), \quad (3.24)$$

where χ_s^j is the characteristic function of the set $\Omega_s^j = \{x \in \Omega : |\nabla T_k(v_j)| \le s\}$.
Since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\ \phi'(\omega_{n,j}^h) \to \nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\ \phi'(\omega_j^h)$ strongly in $E_\varphi(\Omega)^N$, we get from (3.15)

$$-\int_{\Omega\setminus\Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(v_j)\ \phi'(\omega_{n,j}^h)\, dx \to -\int_{\Omega\setminus\Omega_s^j} l_k.\nabla T_k(v_j)\ \phi'(\omega_j^h)\, dx$$

as n tends to infinity.
Using the modular convergence of v_j, one has

$$\int_\Omega l_k.\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\ \phi'(\omega_j^h)\, dx \to \int_{\Omega\setminus\Omega_s^j} l_k.\nabla T_k(u)\ \phi'(\omega^h)\, dx \text{ as } j \to \infty,$$

consequently

$$-\int_{\Omega\setminus\Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(v_j)\ \phi'(\omega_{n,j}^h)\, dx = -\int_{\Omega\setminus\Omega_s^j} l_k.\nabla T_k(u)\ \phi'(\omega^h)\, dx + \epsilon_h(n, j). \tag{3.25}$$

For the second term on the right hand side of (3.24) we can write

$$\begin{aligned}\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j).\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_{n,j}^h)\, dx \\ = \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\ \phi'(\omega_{n,j}^h)\, dx \\ - \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\ \phi'(\omega_{n,j}^h)\, dx.\end{aligned}$$

Splitting the first integral on the right hand side of this equality where $|u_n - v_0| > h$ and $|u_n - v_0| \le h$, and remark that $\nabla T_k(u_n) = 0$ on the set $\{x \in \Omega : |u_n - v_0| > h\}$, we get

$$\begin{aligned}\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\ \phi'(\omega_{n,j}^h)\, dx \\ = \int_{\{|u_n - v_0| \le h\}} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\ \phi'(T_k(u_n) - T_k(v_j))\, dx \\ = \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\ \phi'(T_k(u_n) - T_k(v_j))\, dx\end{aligned}$$

then

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j).\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_{n,j}^h)\,dx$$
$$=\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\ \phi'(T_k(u_n)-T_k(v_j))\,dx$$
$$-\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\ \phi'(\omega_{n,j}^h)\,dx. \quad (3.26)$$

Since

$$a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\ \phi'(T_k(u_n)-T_k(v_j))$$
$$\to a(x,T_k(u),\nabla T_k(v_j)\chi_s^j)\ \phi'(T_k(u)-T_k(v_j)),$$

strongly in $E_\psi(\Omega)^N$ by Lemma 2.4, and $\nabla T_k(u_n)\to\nabla T_k(u)$ weakly in $L_\varphi(\Omega)^N$ for $\sigma(\Pi L_\varphi,\Pi E_\psi)$,
then, the first term on the right hand side of (3.26) tends to the quantity

$$\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_s^j)\nabla T_k(u)\ \phi'(T_k(u)-T_k(v_j))\,dx \text{ as } n\to\infty.$$

Concerning the second term on the right hand side of (3.26), it is easy to see that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\ \phi'(\omega_{n,j}^h)\,dx$$
$$\to\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\ \phi'(\omega_j^h)\,dx$$

as $n\to\infty$. Consequently, we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_{n,j}^h)\,dx$$
$$=\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_s^j)\left[\nabla T_k(u)-\nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_j^h)\,dx+\epsilon_{j,h}(n) \quad (3.27)$$

Now, since $\nabla T_k(v_j)\chi_s^j\ \phi'(\omega_j^h)\to\nabla T_k(u)\chi_s\ \phi'(\omega^h)$ strongly in $E_\varphi(\Omega)^N$ as $j\to\infty$, we have

$$\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_s^j)\left[\nabla T_k(u)-\nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_j^h)\,dx$$
$$\to\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\ \phi'(\omega^h)\,dx$$

as $j \to \infty$, then

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j) \left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right] \phi'(\omega_{n,j}^h)\, dx$$
$$= \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\ \phi'(0)\, dx + \epsilon(n, j).$$

Finally, by combining (3.24), (3.25) and (3.27) we get

$$\begin{aligned}\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla\omega_{n,j}^h\ \phi'(\omega_{n,j}^h)\, dx \\ \geq \int_\Omega \left[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\right] \\ \times\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\ \phi'(\omega_{n,j}^h)\, dx + \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\ \phi'(0)\, dx \\ - \int_{\Omega\setminus\Omega_s^j} l_k.\nabla T_k(u)\ \phi'(0)\, dx + \epsilon(n, j, h). \quad (3.28)\end{aligned}$$

We now evaluate the second term on the left hand side of (3.18) by writing

$$\begin{aligned}\Big|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\phi(\omega_{n,j}^h)\, dx\Big| \\ \leq b(k)\int_\Omega (a_0(x) + \varphi(x, |\nabla T_k(u_n)|))|\phi(\omega_{n,j}^h)|\, dx \\ \leq b(k)\int_\Omega a_0(x)|\phi(\omega_{n,j}^h)|\, dx + \frac{b(k)}{\alpha}\int_\Omega c'(x)|\phi(\omega_{n,j}^h)|\, dx \\ + \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\phi(\omega_{n,j}^h)|\, dx \\ - \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla v_0|\phi(\omega_{n,j}^h)|\, dx \\ \leq \epsilon(n, j, h) + \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)|\phi(\omega_{n,j}^h)|\, dx.\end{aligned}$$

As regards the last term on the last side of this inequality, we have

$$\begin{aligned}\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(\omega^h_{n,j})|\,dx \\ = \frac{b(k)}{\alpha}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\right] \\ \times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\right]|\phi(\omega^h_{n,j})|\,dx \\ +\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\right]|\phi(\omega^h_{n,j})|\,dx \\ +\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^j_s|\phi(\omega^h_{n,j})|\,dx,\end{aligned}$$

we argue as above to show that

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\right]|\phi(\omega^h_{n,j})|\,dx = \epsilon(n,j,h)$$

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^j_s|\phi(\omega^h_{n,j})|\,dx = \epsilon(n,j,h).$$

Then

$$\begin{aligned}\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\,\phi(\omega^h_{n,j})\,dx\Big| \\ \le \frac{b(k)}{\alpha}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\right] \\ \times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\right]|\phi(\omega^h_{n,j})|\,dx+\epsilon(n,j,h).\end{aligned} \tag{3.29}$$

Combining (3.18), (3.28) and (3.29), we obtain

$$\begin{aligned}\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\right] \\ \times\left[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\right]\left(\phi'(\omega^h_{n,j})-\frac{b(k)}{\alpha}|\phi(\omega^h_{n,j})|\right)dx \\ \le \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\,\phi'(0)\,dx+\int_{\Omega\setminus\Omega^j_s} l_k.\nabla T_k(u)\,\phi'(0)\,dx+\epsilon(n,j,h)\end{aligned}$$

thanks to (3.8), one has

$$\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\right]\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\,dx$$
$$\le 2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\;\phi'(0)\,dx + 2\int_{\Omega\setminus\Omega_s^j} l_k.\nabla T_k(u)\;\phi'(0)\,dx + \epsilon(n,j,h). \tag{3.30}$$

Now, observe that

$$\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\right]\left[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\right]\,dx$$
$$= \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\right]\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\,dx$$
$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\,dx$$
$$- \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\left[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\right]\,dx$$
$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left[\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s\right]\,dx.$$

Passing to the limit in n and j in the last three terms of the right hand side of the last equality gives

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\,dx$$
$$= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\,dx + \epsilon(n,j),$$

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\left[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\right]\,dx$$
$$= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\,dx + \epsilon(n)$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s)\,dx = \epsilon(n,j).$$

Hence

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\,[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx$$
$$= \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\right] \times$$
$$\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\right]\,dx + \epsilon(n,j). \quad (3.31)$$

Combining (3.16), (3.30) and (3.31) we deduce that

$$\int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx$$
$$\leq \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)]\,[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx$$
$$\leq 2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\;\phi'(0)\,dx + 2\int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\;\phi'(0)\,dx + \epsilon(n,j,h). \quad (3.32)$$

By passing to the lim sup over n, and letting j,h,s tend to infinity, we obtain

$$\lim_{n\to+\infty}\int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx = 0.$$

As in [8], there exists a subsequence, still denoted by u_n, such that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \quad (3.33)$$

Step 3: **Modular convergence of the truncations**.
Since (3.15) and (3.33), we have $l_k = a(x,T_k(u),\nabla T_k(u))$, which implies by using (3.32)

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + c'(x)]\,dx$$
$$= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla v_0) + c'(x)]\,dx$$
$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx$$

$$
\begin{aligned}
\leq \int_\Omega & [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla v_0) + c'(x)]\ dx \\
& + \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\, dx \\
& \quad + 2 \int_{\Omega\setminus\Omega_s} a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\ \phi'(0)\, dx \\
& \qquad + 2 \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0).\nabla T_k(u)\ \phi'(0)\, dx + \epsilon(n, j, h).
\end{aligned}
$$

By using Fatou's Lemma we obtain

$$
\begin{aligned}
\int_\Omega & [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u) - \nabla v_0) + c'(x)]\ dx \\
& \leq \liminf_{n\to+\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + c'(x)]\ dx \\
& \leq \limsup_{n\to+\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + c'(x)]\ dx \\
& \leq \limsup_{n\to+\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla v_0) + c'(x)]\ dx \\
& \quad + \limsup_{n\to+\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\, dx \\
& +2 \int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\ \phi'(0)\, dx + 2 \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0).\nabla T_k(u)\ \phi'(0)\, dx + \epsilon(n, j, h).
\end{aligned}
$$

We proceed as above to get

$$
\begin{aligned}
\limsup_{n\to+\infty} \int_\Omega & [a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla v_0) + c'(x)]\ dx \\
& = \int_\Omega [a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u)\chi_s - \nabla v_0) + c'(x)]\ dx
\end{aligned}
$$

and

$$
\begin{aligned}
\limsup_{n\to+\infty} \int_\Omega & a(x, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\, dx \\
& = \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0).\nabla T_k(u)\, dx.
\end{aligned}
$$

It follows that

$$\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+c'(x)]\,dx$$
$$\le \liminf_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+c'(x)]\,dx$$
$$\le \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+c'(x)]\,dx$$
$$\le \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)\chi_s-\nabla v_0)+c'(x)]\,dx$$
$$+2\int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\ \phi'(0)\,dx+\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0).\nabla T_k(u)\,dx$$
$$+2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\ \phi'(0)\,dx.$$

Taking into account that $[a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)\chi_s-\nabla v_0)+c'(x)]$, $l_k.\nabla T_k(u)\ \phi'(0)$ and $a(x,T_k(u),0).\nabla T_k(u)\ \phi'(0)$ belongs to $L^1(\Omega)$ and letting $s\to+\infty$, we get

$$\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+c'(x)]\,dx$$
$$\le \liminf_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+c'(x)]\,dx$$
$$\le \limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+c'(x)]\,dx$$
$$\le \int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+c'(x)]\,dx,$$

consequently

$$\lim_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+c'(x)]\,dx$$
$$=\int_\Omega [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+c'(x)]\,dx.$$

By Lemma 2.5, we conclude that

$$[a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla v_0)+c'(x)]$$
$$\to [a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u)-\nabla v_0)+c'(x)] \quad (3.34)$$

strongly in $L^1(\Omega)$. The convexity of the Musielak fonction φ and (2.7) allow us to have

$$\varphi\left(x, \frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) \leq \frac{1}{2\alpha}\left[a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla v_0) + c'(x)\right] + \frac{1}{2\alpha}\left[a(x, T_k(u), \nabla T_k(u))(\nabla T_k(u) - \nabla v_0) + c'(x)\right],$$

Then, by (3.34) we get

$$\lim_{|E|\to 0} \sup_n \int_E \varphi\left(x, \frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) dx = 0$$

So that, by Vitali's theorem one has

$$T_k(u_n) \to T_k(u) \text{ in } W_0^1 L_\varphi(\Omega) \text{ for the modular convergence } \forall k > 0. \tag{3.35}$$

Step 4 : **Equi-integrability of the non-linearities.**
As a consequence of (3.14) and (3.33), one has

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a.e in } \Omega,$$

so it suffices to show that $g_n(x, u_n, \nabla u_n)$ is uniformly equi-integrable in Ω.
Let E be a measurable subset of Ω and let $m > 0$. We have

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx = \int_{E\cap\{|u_n - v_0| \leq m\}} |g_n(x, u_n, \nabla u_n)|\, dx + \int_{E\cap\{|u_n - v_0| > m\}} |g_n(x, u_n, \nabla u_n)|\, dx.$$

Taking $u_n - T_1(u_n - v_0 - T_m(u_n - v_0))$ as test function in $(\mathcal{P}_n)$, we obtain

$$\int_{\{m < |u_n - v_0| \leq m+1\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla v_0)\, dx + \int_{\{|u_n - v_0| > m\}} g_n(x, u_n, \nabla u_n)\; T_1(u_n - v_0 - T_m(u_n - v_0))\, dx \leq \int_{\{|u_n - v_0| > m\}} f_n\; T_1(u_n - v_0 - T_m(u_n - v_0))\, dx,$$

Then, assumption (3.3) gives

$$\int_{\{|u_n - v_0| > m\}} |g_n(x, u_n, \nabla u_n)|\, dx \leq \int_{\{|u_n - v_0| > m\}} (|f_n| + c'(x))\, dx.$$

For $\varepsilon > 0$, there exists $m = m(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n - v_0| > m\}} |g_n(x, u_n, \nabla u_n)|\, dx < \frac{\varepsilon}{2}, \quad \forall n.$$

On the other hand, we use (3.3) and (3.5) to get

$$\begin{aligned}\int_{E\cap\{|u_n-v_0|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx &\le \int_E |g_n(x,T_\varrho(u_n),\nabla T_\varrho(u_n))|\,dx\\ &\le b(\varrho)\int_E a_0(x)\,dx + b(\varrho)\int_E \varphi(x,|\nabla T_\varrho(u_n)|)\,dx\\ &\le \frac{b(\varrho)}{\alpha}\int_E \left[a(x,T_\varrho(u_n),\nabla T_\varrho(u_n))(\nabla T_\varrho(u_n)-\nabla v_0)+c'(x)\right]dx\\ &+ b(\varrho)\int_E a_0(x)\,dx,\end{aligned}$$

where $\varrho = m + \|v_0\|_\infty$.
Then, by using (3.34) and the fact that $a_0(.) \in L^1(\Omega)$ we obtain

$$\lim_{|E|\to 0}\sup_n \int_{E\cap\{|u_n-v_0|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx = 0,$$

where $|E|$ denotes the Lebesgue measure of the subset E. Consequently

$$\lim_{|E|\to 0}\sup_n \int_E |g_n(x,u_n,\nabla u_n)|\,dx = 0.$$

Which shows that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable in Ω. By Vitali's theorem, we conclude that $g(x,u,\nabla u) \in L^1(\Omega)$ and $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$.
Step 5 : **Passage to the limit**.
Let $v \in K_\Lambda \cap W_0^1E_\varphi(\Omega) \cap L^\infty(\Omega)$ and taking $u_n - T_k(u_n - v)$ as test function in $(\mathcal{P}_n)$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\,\nabla T_k(u_n-v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\,T_k(u_n-v)\,dx \le \int_\Omega f_n\,T_k(u_n-v)\,dx,$$

which implies that

$$\begin{aligned}&\int_{\{|u_n-v|\le k\}} a(x,u_n,\nabla u_n)\,(\nabla u_n-\nabla v_0)\,dx\\ &\quad+\int_{\{|u_n-v|\le k\}} a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\,(\nabla v_0-\nabla v)\\ &\quad+\int_\Omega g_n(x,u_n,\nabla u_n)\,T_k(u_n-v))\,dx \le \int_\Omega f_n\,T_k(u_n-v)\,dx.\end{aligned}$$

Using Fatou's Lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $L_\psi(\Omega)^N$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$, we get

$$\int_{\{|u-v|\le k\}} a(x,u,\nabla u)\,(\nabla u-\nabla v_0)\,dx + \int_{\{|u-v|\le k\}} a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\,(\nabla v_0-\nabla v)\,dx + \int_\Omega g(x,u,\nabla u)\,T_k(u-v))\,dx \le \int_\Omega f\,T_k(u-v)\,dx.$$

Hence

$$\int_\Omega a(x,u,\nabla u)\,\nabla T_k(u-v)\,dx+\int_\Omega g(x,u,\nabla u)\,T_k(u-v))\,dx \le \int_\Omega f\,T_k(u-v)\,dx. \tag{3.36}$$

Now, let $v\in K_\Lambda\cap L^\infty(\Omega)$, then by using (3.6) there exists $v_j\in K_\Lambda\cap W_0^1E_\varphi(\Omega)\cap L^\infty(\Omega)$ such that v_j converges to v for the modular convergence. Let $h\ge\|v_0\|_\infty$ and taking $v=T_h(v_j)$ in (3.36), we obtain

$$\int_\Omega a(x,u,\nabla u)\,\nabla T_k(u-T_h(v_j))\,dx+\int_\Omega g(x,u,\nabla u)\,T_k(u-T_h(v_j))\,dx \le \int_\Omega f\,T_k(u-T_h(v_j))\,dx$$

letting $j\to+\infty$, we obtain

$$\int_\Omega a(x,u,\nabla u)\,\nabla T_k(u-T_h(v))\,dx+\int_\Omega g(x,u,\nabla u)\,T_k(u-T_h(v))\,dx \le \int_\Omega f\,T_k(u-T_h(v))\,dx \quad \forall v\in K_\Lambda\cap L^\infty(\Omega)$$

Finally, letting h to the infinity we deduce

$$\int_\Omega a(x,u,\nabla u)\,\nabla T_k(u-v)\,dx+\int_\Omega g(x,u,\nabla u)\,T_k(u-v)\,dx \le \int_\Omega f\,T_k(u-v)\,dx \quad \forall v\in K_\Lambda\cap L^\infty(\Omega)\quad \forall k>0.$$

□

Thus the proof of the theorem 3.3 is complete.

4. Bibliography

References

1. Adams, R. A., *Sobolev spaces*, Academic Press, New York, (1975).

2. Aharouch, L. Benkirane, A. and Rhoudaf, M., *Strongly nonlinear elliptic variational unilateral problems in Orlicz spaces*, Abstr. Appl. Anal. Art. ID 46867, 1-20 (2006).

3. Aharouch, L. and Rhoudaf, M., *Strongly nonlinear elliptic unilateral problems in Orlicz spaces and L^1 data*, J. Inequal. Pure and Appl. Math. 6, Issue 2, Art 54, 1-20 (2005).

4. Ait Khellou, M. Benkirane, A. and Douiri, S. M., *Existence of solutions for elliptic equations having natural growth terms in Musielak spaces*, J. Math. Comput. Sci. Vol. 4, No.4, 665-688 (2014).

5. Ait Khellou, M. Benkirane, A. and Douiri, S. M., *An inequality of type Poincaré in Musielak spaces and application to some non-linear elliptic problems with L^1 data*, Complex Variables and Elliptic Equations. Vol. 60, No.9, 1217-1242 (2015).

6. Azroul, E. Redwane, H and Yazough, C., *Strongly Nonlinear Nonhomogeneous Elliptic Unilateral Problems with L^1 data and no sign conditions*, Electronic Journal of Differential Equations. 79, 1-20 (2012).

7. Benkirane, A. Douieb, J and Sidi El Vally (Ould Mohamedhen Val), M., *An approximation theorem in Musielak-Orlicz-Sobolev spaces*,Comm. Math. 51 No.1, 109-120 (2011).

8. Benkirane, A and Elmahi, A., *Almost everywhere convergence of gradients of solutions to elliptic equations in Orlicz spaces and application*, Nonlinear Anal. Theory Methods Appl. 28, 1769-1784 (1997).

9. Benkirane, A. and Elmahi, A., *An existence theorem for a strongly non-linear elliptic problem in Orlicz spaces*, Nonlinear Anal. Theory Methods Appl. 36, 11-24 (1999).

10. Benkirane, A. and Sidi El Vally (Ould Mohamedhen Val), M., *An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin, Vol. 20, No.1, 57-75 (2013).

11. Benkirane, A. and Sidi El Vally (Ould Mohamedhen Val), M., *Some approximation properties in Musielak-Orlicz-Sobolev spaces*, Thai.J. Math, Vol. 10, No.2, 371-381 (2012).

12. Benkirane, A. and Sidi El Vally (Ould Mohamedhen Val), M., *Variational inequalities in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin. Vol. 21, No.5, 787-811 (2014).

13. Boccardo, L. and Gallouët, T., *Problémes unilatéraux avec données dans L^1*, C.R. Acad. Sci. Paris, 311, 617-619 (1990).

14. Boccardo, L. Murat, F. and Puel, J. P., *Existence of bounded solutions for nonlinear ellipìtic unilateral problems*, Ann. Mat. Pura e Appl. 152, 183-196 (1988).

15. Elmahi, A and Meskine, D., *Unilateral elliptic problems in L^1 with natural growth terms*, J. Nonlinear and Convex Analysis, 5(1), 97-112 (2004)

16. Elmahi, A and Meskine, D., *Elliptic inequalities with lower order terms and L1 data in Orlicz spaces*, Journal of Mathematical Analysis and Applications, 328(2), 1417-1434 (2007).

17. Gwiazda, P. Wittbold, P. Wróblewska, A. and Zimmermann, A., *Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces*, Journal of Differential Equations, 253(2), 635-666 (2012).

18. Gossez, J. P., *A strongly non-linear elliptic problem in Orlicz-Sobolev spaces*, Proc. Am. Math. Soc. Symp. Pure Math. 45, 457-461 (1986).

19. Gossez, J. P., *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Am. Math. Soc. 190, 163-205 (1974).

20. Gossez, J. P. and Mustonen, V., *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Anal. Theory Methods Appl. 11, 379-392 (1987).

21. Leray, J. and Lions, J. L., *Quelques résultats de Višik sur les problèmes elliptiques non-linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93, 97-107 (1965).

22. Musielak, J., *Modular spaces and Orlicz spaces*, Lecture Notes in Math. 1034, (1983).

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