Bol. Soc. Paran. Mat. (3s.) v. 35 3 (2017): 241–254.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v35i3.29882

A Takeuchi-Yamada type equation with variable exponents*

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abstract: We prove continuity of the flows and upper semicontinuity of global attractors for a Takeuchi-Yamada type equation with variable exponents.

Key Words: Variable exponents, parabolic problems, global attractors, upper semicontinuity.

Contents

1. Introduction

The study of the continuity with respect to initial conditions and parameters is important to verify the stability of a PDE model. Currently, some researchers investigated in which way the parameter $p(x)$ affects the dynamic of problems involving the $p(x)$-Laplacian, analyzing the continuity properties of the flows and of the global attractors with respect to the parameter $p(x)$. B. Amaziane, L. Pankratov and V. Prytula studied homogenization of $p_\epsilon(x)$-Laplacian elliptic equations (see [2]) and B. Amaziane, L. Pankratov and A. Piatnitski studied nonlinear flow through double porosity media in variable exponent Sobolev spaces (see [1]) where the authors considered the following initial boundary value problem

$$\begin{cases} \omega^\epsilon(x)\frac{\partial u^\epsilon}{\partial t}(t) - \operatorname{div}(k^\epsilon(x)\nabla u^\epsilon|\nabla u^\epsilon|^{p_\epsilon(x)-2}) = g(t,x) \quad \text{in} \quad Q \\ u^\epsilon = 0 \quad \text{on} \quad]0,t[\times\partial\Omega, \\ u^\epsilon(0,x) = u_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2,3)$ is a bounded domain, Q denotes the cylinder $]0,T[\times\Omega$, $T > 0$ is given, $g \in C([0,T]; L^2(\Omega))$ and $u_0 \in H^2(\Omega)$ are given functions. They

* This work was partially supported by the Brazilian research agency FAPEMIG grant CEX-APQ-04098-10 and by Science without Borders-CAPES-PVE-Process 88881.030388/2013-01. The first author was supported with CNPq scholarship - process 202645/2014-2 (Brazil). The second author was supported with CAPES scholarship - process 99999.006893/2014-07 (Brazil).
2000 *Mathematics Subject Classification:* 35B40, 35B41, 35K92
Submitted November 20, 2015. Published March 07, 2016

studied the minimization problem for functionals in the limit of small ϵ and obtained the homogenized functional. We considered in [11] the following nonlinear PDE problem

$$\begin{cases} \frac{\partial u_s}{\partial t}(t) - \frac{\partial}{\partial x}\left(|\frac{\partial u_s}{\partial x}(t)|^{p_s(x)-2}\frac{\partial u_s}{\partial x}(t)\right) = B(u_s(t)), \;\; t > 0 \\ u_s(0) = u_{0s}, \end{cases}$$

under Dirichlet homogeneous boundary conditions, where $u_{0s} \in H := L^2(I)$, $I := (c,d)$, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, $p_s(x) \in C^1(\bar{I})$, $p_s^- :=$ ess inf $p_s > 2$ for all $s \in \mathbb{N}$, and $p_s(\cdot) \to p$ in $L^\infty(I)$ ($p > 2$ constant) as $s \to \infty$. We proved continuity of the flows and upper semicontinuity of the family of global attractors $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ as s goes to infinity.

In this work we consider the nonlinear perturbation $|u|^{p_s(x)-2}u$ of the $p(x)-$Laplacian, i. e., we consider the following nonlinear PDE problem

$$\begin{cases} \frac{\partial u_s}{\partial t}(t) - \mathrm{div}(|\nabla u_s(t)|^{p_s(x)-2}\nabla u_s(t)) + |u_s(t)|^{p_s(x)-2}u_s(t) = B(u_s(t)), \;\; t > 0, \\ u_s(0) = u_{0s}, \end{cases} \tag{1.1}$$

under homogeneous Neumann Boundary conditions, where $u_{0s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a smooth bounded domain, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, $p_s(\cdot) \in C^1(\bar{\Omega})$, $p_s^- :=$ ess inf $p_s \geq p$, $p_s^+ :=$ ess sup $p_s \leq a$, for all $s \in \mathbb{N}$, and $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ as $s \to \infty$ ($p > 2$ and a are constants). We prove continuity of the flows and upper semicontinuity of the family of global attractors $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ as s goes to infinity for the problem (1.1).

In [5], Chafee and Infante considered the equation

$$(P1)\; u_t = \lambda u_{xx} + u - u^3,$$

and Takeuchi and Yamada considered in [14] the following more general equation involving the p-Laplacian operator

$$(P2)\; u_t = \lambda(|u_x|^{p-2}u_x)_x + |u|^{q-2}u(1 - |u|^r),$$

where $p > 2$, $q \geq 2$, $r > 0$ and $\lambda > 0$ are constants. Note that taking $p = q = r = 2$, problem $(P2)$ becomes problem $(P1)$. The authors in [4] proved the continuity of the flows and upper semicontinuity of a family of global attractors for the problem $(P2)$ when $p = q$ and $p \to 2$.

Considering the problem

$$u_t = \lambda \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q-2}u(1 - |u|^{r(x)}),$$

with $q \equiv 2$ and $r(x) := p(x) - 2 > 0$, we obtain

$$(P3)\; u_t = \lambda \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + u(1 - |u|^{p(x)-2}).$$

Note that the problem

$$\begin{cases} u_t = \lambda \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + u(1 - |u|^{p(x)-2}), \;\; t > 0, \\ u(0) = u_0, \end{cases}$$

can be seen as

$$\begin{cases} u_t - \lambda \text{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = u, & t > 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

and $\tilde{B}(u) := u$ is a globally Lipschitz map. So, all the results developed in this paper for an abstract globally Lipschitz external forcing term can be applied to the Takeuchi-Yamada type equation (1.2). The bifurcation studies of solutions to problem (1.2) with respect to the parameter λ remains an open problem.

The study of continuity properties with respect to initial conditions and exponent parameters for the problem $u_t = \lambda(|u_x|^{p(x)-2}u_x)_x + u$ were already contemplated in the papers [10,11].

The paper is organized as follows. In Section 2 we present properties on the operator and we guarantee existence of global solution and global attractor for problem (1.1). In Section 3 we obtain uniform estimates for solutions of (1.1). In Section 4 we prove that the solutions $\{u_s\}$ of (1.1) go to the solution u of the limit problem (4.1) and, after that, we obtain the upper semicontinuity of the global attractors for the problem (1.1).

2. Properties on the operator

The authors in [13] proved that the operator

$$Au := -\text{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u$$

where $p(\cdot)$ is continuous in $\overline{\Omega}$ and $p^- > 2$, is the realization of the operator $A_1 : X \to X^*$, $X := W^{1,p(x)}(\Omega)$,

$$A_1 u(v) := \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)dx,$$

i.e., $A(u) = A_1 u, \ if \ \ u \in \mathcal{D}(A) := \{u \in X;\ A_1 u \in H\}$ and it is a maximal monotone operator in H. Besides, A is the subdifferential of the proper, convex and lower semicontinuos function $\varphi_{p(x)} : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi_{p(x)}(u) := \begin{cases} \Big[\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx\Big], & \text{if } u \in X, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

Moreover, we have the following properties on the operator

Lemma 2.1. *[13]*

$$\langle Au, u\rangle_{X^*,X} \geq \begin{cases} \frac{1}{2^{p^+-1}}||u||_X^{p^+}, & if\ ||u||_{p(x)} \leq 1 \ \ and\ \ ||\nabla u||_{p(x)} \leq 1, \\ \frac{1}{2^{p^--1}}||u||_X^{p^-}, & if\ ||u||_{p(x)} \geq 1 \ \ and\ \ ||\nabla u||_{p(x)} \geq 1, \\ ||\nabla u||_{p(x)}^{p^-} + ||u||_{p(x)}^{p^+}, & if\ ||u||_{p(x)} \leq 1 \ \ and\ \ ||\nabla u||_{p(x)} \geq 1, \\ ||\nabla u||_{p(x)}^{p^+} + ||u||_{p(x)}^{p^-}, & if\ ||u||_{p(x)} \geq 1 \ \ and\ \ ||\nabla u||_{p(x)} \leq 1. \end{cases}$$

By Consequence 3 in [13], it follows that the equation (1.1) determines a continuous semigroup of nonlinear operators $\{T_s(t) : H \to H,\ t \geq 0\}$, where for each $u_{0s} \in H$, $t \mapsto T_s(t)u_{0s}$ is a weak global solution of (1.1) beginning at u_{0s}. This semigroup is such that $\mathbb{R}^+ \times H \ni (t, u_{0s}) \mapsto T_s(t)u_{0s} \in H$ is a continuous map and, if $u_{0s} \in \mathcal{D}(A)$, then $u_s(\cdot) := T_s(\cdot)u_{0s}$ is a Lipschitz continuous strong solution of (1.1).

Considering $h \equiv 0$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz in [7] we get $F = B : H \to H$ globally Lipschitz. So, by Theorem 3.3 in [7] we have that problem (1.1) has a global attractor $\mathcal{A}_s$.

In order to prove the continuity of the flows (in Section 4) for problem (1.1) we need the following result:

Theorem 2.2. *If $p \in C^1(\Omega)$, then $C_0^\infty(\Omega) \subset \mathcal{D}(A)$.*

Proof: If $p \in C^1(\Omega)$ and $u \in C_0^\infty(\Omega)$, then by Theorem 2.6 in [12] we have that $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) \in L^2(\Omega)$. The result follows observing that $|u|^{p(x)-2}u \in L^2(\Omega)$ if $u \in C_0^\infty(\Omega)$. □

3. Uniform estimates

Recall that we are considering $p_s(\cdot) \in C^1(\bar{\Omega})$ such that $2 < p \leq p_s^- \leq p_s^+ \leq a$, for all $s \in \mathbb{N}$, and $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ as $s \to \infty$. From now on, we denote $X_s := W^{1,p_s(x)}(\Omega)$ and $X := W^{1,p}(\Omega)$. It is a known result that $X_s \subset H$ with continuous and dense embedding (see [9]). Moreover,

Lemma 3.1. *There exists a constant $K = K(|\Omega|) > 0$, independent of s, such that if $u_s \in X_s$, $s \in \mathbb{N}$, then*

$$\|u_s\|_H \leq K\|u_s\|_{X_s}, \quad \forall s \in \mathbb{N}.$$

Proof: We know that if $p(x) > q(x)$ then $L^{p(x)}(\Omega) \subset L^{q(x)}(\Omega)$ with $\|u\|_{q(x)} \leq 2(|\Omega|+1)\|u\|_{p(x)}$ for all $u \in L^{p(x)}(\Omega)$ (see [6]). Thus, if $u_s \in X_s \subset X \subset H$ we have

$$\begin{aligned} \|u_s\|_H &\leq 2(|\Omega|+1)\|u_s\|_p \\ &\leq 4(|\Omega|+1)^2\|u_s\|_{p_s(x)} \\ &\leq 4(|\Omega|+1)^2(\|u_s\|_{p_s(x)} + \|\nabla u_s\|_{p_s(x)}) = K\|u_s\|_{X_s}, \end{aligned}$$

where $K = K(|\Omega|) := 4(|\Omega|+1)^2$. □

We have the following uniform estimates on the solutions of (1.1):

Lemma 3.2. *Let u_s be a solution of (1.1) with $u_s(0) = u_{0s} \in H$. Given $T_0 > 0$, there exists a positive number r_0 such that $\|u_s(t)\|_H \leq r_0$, for each $t \geq T_0$ and $s \in \mathbb{N}$. Furthermore, given a bounded set $B \subset H$, there exists $D_1 > 0$ such that $\|u_s(t)\|_H \leq D_1$ for all $t \geq 0$ and $s \in \mathbb{N}$ such that $u_{0s} \in B$.*

Proof: It is enough to consider $u_{0s} \in \mathcal{D}(A)$. Let $\tau > 0$, multiplying the equation in (1.1) by $u_s(\tau)$ we have

$$\left\langle \frac{d}{dt}u_s(\tau), u_s(\tau) \right\rangle - \langle \Delta_{p_s(x)}(u_s(\tau)) + |u_s(\tau)|^{p_s(x)-2}u_s(\tau), u_s(\tau)\rangle = \langle B(u_s(\tau)), u_s(\tau)\rangle.$$

Given $T_0 > 0$, if $||u_s(\tau)||_{p_s(x)} \geq 1$ and $||\nabla u_s(\tau)||_{p_s(x)} \geq 1$ then by Lemma 2.1, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 &\leq -\frac{1}{2^{p_s^- -1}}\|u_s(\tau)\|_{X_s}^{p_s^-} + \|B(u_s(\tau))\|_H \|u_s(\tau)\|_H \\ &\leq -\frac{1}{2^{a-1}}\|u_s(\tau)\|_{X_s}^{p} + L\|u_s(\tau)\|_H^2 + C_0\|u_s(\tau)\|_H \\ &\leq -\frac{1}{2^{a-1}}\|u_s(\tau)\|_{X_s}^{p} + C_1\|u_s(\tau)\|_{X_s}^2 + C_2\|u_s(\tau)\|_{X_s}, \end{aligned}$$

where $C_0 = \|B(0)\|_H \geq 0$, $C_1 = LK^2$ and $C_2 = C_0K$, with K the constant independent of s of Lemma 3.1. We have $C_2 = 0$ if, and only if, $C_0 = 0$.

Now, we consider $\epsilon > 0$ arbitrary, $\alpha := \frac{p}{2}$, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then using Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 \leq \Big(- \frac{1}{2^{a-1}} + \frac{1}{\alpha}\epsilon^{\alpha} + \frac{1}{p}\epsilon^{p}\Big)\|u_s(\tau)\|_{X_s}^{p} + \Big(\frac{1}{\alpha'}(\frac{C_1}{\epsilon})^{\alpha'} + \frac{1}{p'}(\frac{C_2}{\epsilon})^{p'}\Big).$$

Now, choose $\epsilon_0 > 0$ sufficiently small so that $\frac{1}{\alpha}\epsilon_0^{\alpha} + \frac{1}{p}\epsilon_0^{p} < \dfrac{1}{2^a}$ in the case $B(0) \neq 0$ $(C_0 \neq 0)$ and for the case $B(0) = 0$, choose $\epsilon_0 > 0$ sufficiently small so that $\frac{1}{\alpha}\epsilon_0^{\alpha} < \dfrac{1}{2^a}$. So, in both cases, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 \leq -\frac{1}{2^a}\|u_s(\tau)\|_{X_s}^{p} + C_3,$$

where $C_3 = C_3(L, K, \epsilon_0) > 0$ is a constant. So,

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 \leq -\frac{1}{2^a}K^{-p}\|u_s(\tau)\|_H^{p} + C_3.$$

Let $I_s := \{\tau \in (0, \infty); ||u_s(\tau)||_{p_s(x)} \geq 1$ and $||\nabla u_s(\tau)||_{p_s(x)} \geq 1\}$ and $y_s : I_s \to \mathbb{R}$, $y_s(\tau) := \|u_s(\tau)\|_H^2$ satisfies the differential inequality

$$y_s'(\tau) \leq -\frac{K^{-p}}{2^{a-1}}[y_s(\tau)]^{\frac{p}{2}} + 2C_3.$$

Therefore, from Lemma 5.1, p. 163 in [15], we get

$$\|u_s(\tau)\|_H^2 \leq \Big(2^aC_3K^p\Big)^{2/p} + \Big[\frac{1}{2^aK^p}(p-2)T_0\Big]^{\frac{-2}{(p-2)}} := K_1, \forall\, \tau \geq T_0.$$

Similarly for each of the cases: $||u_s(\tau)||_{p_s(x)} \geq 1$ and $||\nabla u_s(\tau)||_{p_s(x)} \leq 1$; $||u_s(\tau)||_{p_s(x)} \leq 1$ and $||\nabla u_s(\tau)||_{p_s(x)} \geq 1$; $||u_s(\tau)||_{p_s(x)} \leq 1$ and $||\nabla u_s(\tau)||_{p_s(x)} \leq 1$, we obtain constants, K_2, K_3 and K_4 such that

$$\|u_s(\tau)\|_H^2 \leq K_i, \forall\, \tau \geq T_0,$$

for $i = 2, 3, 4$ respectively. So, taking $r_0 := \max\{K_1^{1/2}, K_2^{1/2}, K_3^{1/2}, K_4^{1/2}\}$ we obtain

$$\|u_s(\tau)\|_H \leq r_0, \forall\, \tau \geq T_0,\ s \in \mathbb{N},$$

and the first part of the lemma is proved.

The second part of the lemma follows from the Gronwall-Bellman Lemma. □

Remark 3.3. *The constants r_0 and D_1 in the Lemma 3.2 depend neither on the initial data nor on s.*

Corollary 3.4. *There exists a bounded set B_0 in H such that $\mathcal{A}_s \subset B_0$ for all $s \in \mathbb{N}$.*

Lemma 3.5. *Let u_s be a solution of (1.1). Given $T_1 > 0$, there exists a positive constant $r_1 > 0$, independent of s, such that*

$$\|u_s(t)\|_{X_s} < r_1,$$

for every $t \geq T_1$ and $s \in \mathbb{N}$.

Proof: Let u_s be a solution of (1.1) and consider $T_1 > 0$. Take $T_0 \in (0, T_1)$. Considering $\varphi_{p_s(x)}$ as in (2.1), using the definition of subdifferential and Uniform Gronwall Lemma (see [15]), we obtain

$$\varphi_{p_s(x)}(u_s(\tau)) \leq \tilde{r_1},$$

for all $\tau \geq T_1$ and $s \in \mathbb{N}$, where $\tilde{r_1} = \tilde{r_1}(T_1, T_0, L, r_0)$, with r_0 as in Lemma 3.2. Therefore

$$\int_\Omega \frac{1}{p_s(x)} \Big|\nabla u_s(\tau, x)\Big|^{p_s(x)} dx + \int_\Omega \frac{1}{p_s(x)} \Big|u_s(\tau, x)\Big|^{p_s(x)} dx \leq \tilde{r_1},$$

for all $\tau \geq T_1$ and $s \in \mathbb{N}$. So, considering $\rho_s(v) := \int_\Omega |v(x)|^{p_s(x)} dx$, we have

$$\rho_s\left(\nabla u_s(\tau)\right) + \rho_s\left(u_s(\tau)\right) \leq a\tilde{r_1}, \tag{3.1}$$

for all $\tau \geq T_1$ and $s \in \mathbb{N}$. If $\tau \geq T_1$ and $\|u_s(\tau)\|_{X_s} > 1$ then we have four cases to analyze:

Case 1: If $\|\nabla u_s(\tau)\|_{p_s(x)} \geq 1$ and $\|u_s(\tau)\|_{p_s(x)} \geq 1$ then we know that

$$\|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^-} \leq \rho_s(\nabla u_s(\tau)) \leq \|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^+},$$

and

$$\|u_s(\tau)\|_{p_s(x)}^{p_s^-} \le \rho_s(u_s(\tau)) \le \|u_s(\tau)\|_{p_s(x)}^{p_s^+}.$$

Since $p \le p_s^- \le p_s^+ \le a$, we obtain by (3.1)

$$\|u_s(\tau)\|_{X_s} \le (a\tilde{r_1})^{1/p}.$$

Case 2: If $\|\nabla u_s(\tau)\|_{p_s(x)} \ge 1$ and $\|u_s(\tau)\|_{p_s(x)} \le 1$ then we know that

$$\|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^-} \le \rho_s(\nabla u_s(\tau)) \le \|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^+},$$

and

$$\|u_s(\tau)\|_{p_s(x)}^{p_s^+} \le \rho_s(u_s(\tau)) \le \|u_s(\tau)\|_{p_s(x)}^{p_s^-}.$$

Using (3.1) we obtain in this case

$$\|u_s(\tau)\|_{X_s} \le (a\tilde{r_1})^{1/p} + (a\tilde{r_1})^{1/a}.$$

Case 3: If $\|\nabla u_s(\tau)\|_{p_s(x)} \le 1$ and $\|u_s(\tau)\|_{p_s(x)} \ge 1$ then we know that

$$\|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^+} \le \rho_s(\nabla u_s(\tau)) \le \|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^-},$$

and

$$\|u_s(\tau)\|_{p_s(x)}^{p_s^-} \le \rho_s(u_s(\tau)) \le \|u_s(\tau)\|_{p_s(x)}^{p_s^+}.$$

Then, by (3.1) we have that

$$\|u_s(\tau)\|_{X_s} \le (a\tilde{r_1})^{1/a} + (a\tilde{r_1})^{1/p}.$$

Case 4: If $\|\nabla u_s(\tau)\|_{p_s(x)} \le 1$ and $\|u_s(\tau)\|_{p_s(x)} \le 1$, then

$$\|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^+} \le \rho_s(\nabla u_s(\tau)) \le \|\nabla u_s(\tau)\|_{p_s(x)}^{p_s^-},$$

and

$$\|u_s(\tau)\|_{p_s(x)}^{p_s^+} \le \rho_s(u_s(\tau)) \le \|u_s(\tau)\|_{p_s(x)}^{p_s^-}.$$

Using (3.1), we obtain

$$\|u_s(\tau)\|_{X_s} \le (a\tilde{r_1})^{1/a}.$$

So considering $r_1 := \max\{1, (a\tilde{r_1})^{\frac{1}{p}} + (a\tilde{r_1})^{1/a}\}$ we conclude that

$$\|u_s(\tau)\|_{X_s} \le r_1 \text{ for all } \tau \ge T_1 \text{ and } s \in \mathbb{N}.$$

□

Corollary 3.6. *a) There exists a bounded set B_1^s in X_s such that $\mathcal{A}_s \subset B_1^s$.*
b) Let u_s be a solution of problem (1.1). Given $T_1 > 0$ there exists a positive constant r_2, independent of s, such that

$$\|u_s(t)\|_X < r_2$$

for all $t \ge T_1$ and $s \in \mathbb{N}$.
c) $\mathcal{A} := \overline{\bigcup_{s\in\mathbb{N}} \mathcal{A}_s}$ is a compact subset of H.

Proof: a) It follows from Lemma 3.5.

b) By Lemma 3.5 there exists $r_1 > 0$ such that

$$\|u_s(t)\|_{X_s} < r_1 \quad \forall\, t \geq T_1,\ s \in \mathbb{N}.$$

Thus

$$\begin{aligned}\|u_s(t)\|_X &= \|\nabla u_s(t)\|_p + \|u_s(t)\|_p \leq 2(|\Omega|+1)\left(\|\nabla u_s(t)\|_{p_s(x)} + \|u_s(t)\|_{p_s(x)}\right)\\ &= 2(|\Omega|+1)\|u_s(t)\|_{X_s} \leq 2(|\Omega|+1)r_1\end{aligned}$$

for all $t \geq T_1$ and $s \in \mathbb{N}$ and the result follows with $r_2 := 2(|\Omega|+1)r_1$.

c) By b) there exists a bounded set B_1 in X such that $\mathcal{A}_s \subset B_1$ for all $s \in \mathbb{N}$. Since $X \subset H$ with continuous and compact embedding, the result is proved. □

Proposition 3.1. *Let u_s be a solution of (1.1) with initial value u_{0s}. If there is $C > 0$ such that $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$, then given $T_1 > 0$ there exists a positive constant R_1 such that $\|u_s(t)\|_{X_s} \leq R_1$, for all $t \in [0, T_1]$ and $s \in \mathbb{N}$. In this case we can consider $T_1 = 0$ in Lemma 3.5.*

Proof: Given $T_1 > 0$, if u_s is a solution of (1.1) then using the identity

$$\frac{d}{dt}\varphi_{p_s(x)}(u_s(t)) = \langle \partial\varphi_{p_s(x)}(u_s(t)), \frac{\partial u_s}{\partial t}(t)\rangle$$

and Lemma 3.2, we obtain

$$\varphi_{p_s(x)}(u_s(\tau)) \leq \varphi_{p_s(x)}(u_{0s}) + C_1 T_1, \qquad \text{for all } \tau \in [0, T_1], s \in \mathbb{N},$$

where $C_1 > 0$ is a constant. Now, as $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$ we obtain that $\varphi_{p_s(x)}(u_{0s}) \leq \tilde{C}$ for all $s \in \mathbb{N}$. So, the result follows as in the proof of Lemma 3.5. □

Corollary 3.7. *Let u_s be a solution of (1.1) with initial value u_{0s}. If there is $C > 0$ such that $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$, then given $T_1 > 0$ there exists a positive constant $\widetilde{R_1}$ such that*

$$\|u_s(t)\|_X \leq \widetilde{R_1},$$

for all $t \in [0, T_1]$ and $s \in \mathbb{N}$.

Proof: Since $\|u_s(\tau)\|_X \leq 2(|\Omega|+1)\|u_s(\tau)\|_{X_s}$ for all $s \in \mathbb{N}$, the result follows from Proposition 3.1. □

4. Continuity with respect to the initial values and upper semicontinuity of attractors

In this section we prove that, given $T > 0$, the solutions u_s of (1.1) go to the solution u of

$$\begin{cases} \frac{\partial u}{\partial t}(t) - \operatorname{div}(|\nabla u(t)|^{p-2}\nabla u(t)) + |u|^{p-2}u = B(u(t)), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{4.1}$$

in $C([0,T];H)$ and, after that, we obtain the upper semicontinuity on s in H of the family of global attractors $\{\mathcal{A}_s \subset H; s \in \mathbb{N}\}$ of (1.1) at p.

Lemma 4.1. *Given $T > 0$, $M := \{u_s : s \in \mathbb{N}, u_s$ is a solution of (1.1) with $u_s(0) = u_{0s}$ and $u_{0s} \to u_0$ in H, as $s \to +\infty\}$ is relatively compact in $C([0,T];H)$.*

Proof: We observe that it holds:
i) For each $s \in \mathbb{N}$ the function $[0,T] \ni t \mapsto B(u_s(t)) \in H$ is in $L^1(0,T;H)$. Moreover, $\{B(u_s(t))\}_{s\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T;H)$ and consequently uniformly integrable in $L^1(0,T;H)$.
Indeed, as $\int_0^T \|B(u_s(t))\|_H dt \le \int_0^T (L\|u_s(t)\|_H + \|B(0)\|_H)dt$ the result follows from Lemma 3.2.
ii) The operator A^s, $A^s u := -\Delta_{p_s(x)}u + |u|^{p_s(x)-2}u$, is a maximal monotone operator in H, $A^s u = \partial\varphi_{p_s(x)}(u)$ is the subdifferential of the convex, proper and lower semi continuous non negative map $\varphi_{p_s(x)}$ and $\overline{\cap_s D(\varphi_{p_s(x)})} = H$ since $X_a \subset X_s \subset X$, for all s.
iii) For each $u \in \cap_s D(\varphi_{p_s(x)})$ there exists a constant $k = k(u,p,a,|\Omega|) > 0$ such that $\varphi_{p_s(x)}(u) \le k$, $\forall s \in \mathbb{N}$.
In fact, if $u \in \cap_s D(\varphi_{p_s(x)}) = \cap_s X_s$ then for all s

$$\varphi_{p_s(x)}(u) \le \begin{cases} \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{p_s^-} + \|u\|_{p_s(x)}^{p_s^-}\right), \text{ if } \|\nabla u\|_{p_s(x)} \le 1 \text{ and } \|u\|_{p_s(x)} \le 1 \\ \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{p_s^+} + \|u\|_{p_s(x)}^{p_s^-}\right), \text{ if } \|\nabla u\|_{p_s(x)} \ge 1 \text{ and } \|u\|_{p_s(x)} \le 1 \\ \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{p_s^-} + \|u\|_{p_s(x)}^{p_s^+}\right), \text{ if } \|\nabla u\|_{p_s(x)} \le 1 \text{ and } \|u\|_{p_s(x)} \ge 1 \\ \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{p_s^+} + \|u\|_{p_s(x)}^{p_s^+}\right), \text{ if } \|\nabla u\|_{p_s(x)} \ge 1 \text{ and } \|u\|_{p_s(x)} \ge 1 \end{cases}$$

$$\le \begin{cases} \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{p} + \|u\|_{p_s(x)}^{p}\right), \text{ if } \|\nabla u\|_{p_s(x)} \le 1 \text{ and } \|u\|_{p_s(x)} \le 1 \\ \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{a} + \|u\|_{p_s(x)}^{p}\right), \text{ if } \|\nabla u\|_{p_s(x)} \ge 1 \text{ and } \|u\|_{p_s(x)} \le 1 \\ \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{p} + \|u\|_{p_s(x)}^{a}\right), \text{ if } \|\nabla u\|_{p_s(x)} \le 1 \text{ and } \|u\|_{p_s(x)} \ge 1 \\ \frac{1}{2}\left(\|\nabla u\|_{p_s(x)}^{a} + \|u\|_{p_s(x)}^{a}\right), \text{ if } \|\nabla u\|_{p_s(x)} \ge 1 \text{ and } \|u\|_{p_s(x)} \ge 1 \end{cases}$$

$$\le \begin{cases} \frac{1}{2}[2(|\Omega|+1)]^p\left(\|\nabla u\|_p^p + \|u\|_p^p\right) \\ \frac{1}{2}\left\{[2(|\Omega|+1)]^a\|\nabla u\|_p^a + [2(|\Omega|+1)]^p\|u\|_p^p\right\} \\ \frac{1}{2}\left\{[2(|\Omega|+1)]^p\|\nabla u\|_p^p + [2(|\Omega|+1)]^a\|u\|_p^a\right\} \\ \frac{1}{2}[2(|\Omega|+1)]^a\left(\|\nabla u\|_p^a + \|u\|_p^a\right) \end{cases}$$

So $\varphi_{p_s(x)}(u) \leq k$ for all $s \in \mathbb{N}$, where k is the maximum between the values $2^{p-1}(|\Omega|+1)^p \left(\|\nabla u\|_p^p + \|u\|_p^p\right)$, $2^{a-1}(|\Omega|+1)^a\|\nabla u\|_p^a + 2^{p-1}(|\Omega|+1)^p\|u\|_p^p$, $2^{p-1}(|\Omega|+1)^p\|\nabla u\|_p^p + 2^{a-1}(|\Omega|+1)^a\|u\|_p^a$ and $2^{a-1}(|\Omega|+1)^a \left(\|\nabla u\|_p^a + \|u\|_p^a\right)$.
iv) Let $M(t) := \{u_s(t); u_s \in M\}$ and let $\{S^s(t)\}$ be the semigroup generated by A^s in H. For each $t \in (0,T]$ and $h > 0$ such that $t - h \in (0,T]$, the operator $T_h : M(t) \to H$ defined by $T_h u_s(t) = S^s(h)u_s(t-h)$ is compact. Moreover, $M(0)$ is relatively compact in H once $u_{0s} \to u_0$ in H.

Thus, by Theorem 3.2 in [8], M is relatively compact in $C([0,T];H)$. □

Theorem 4.2. *For each $s \in \mathbb{N}$ let u_s be a solution of (1.1) with $u_s(0) = u_{0s}$. Suppose that there exists $C > 0$, independent of s, such that $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$ and $u_{0s} \to u_0$ in H as $s \to \infty$. Then, for each $T > 0$, $u_s \to u$ in $C([0,T];H)$ as $s \to \infty$ where u is a solution of (4.1) with $u(0) = u_0$.*

Proof: By Lemma 4.1 M is relatively compact in $C([0,T];H)$. So, $\{u_s\}$ converges in $C([0,T];H)$ to a function $u : [0,T] \to H$. Proposition 3.6 in [3] implies that

$$\begin{aligned}\frac{1}{2}\|u_s(t) - \phi\|_H^2 \quad &\leq \quad \frac{1}{2}\|u_s(\tau) - \phi\|_H^2 \qquad\qquad (4.2)\\ &\quad + \int_\tau^t \langle B(u_s(t')) + \Delta_{p_s(x)}(\phi) - |\phi|^{p_s(x)-2}\phi, u_s(t') - \phi\rangle dt'\end{aligned}$$

for every $\phi \in \mathcal{D}(A^s)$ and $0 \leq \tau \leq t \leq T$.

Now, the idea is to take the limit as $s \to \infty$ ($p_s \to p$) on the last inequality.

Since $u_s \to u$ in $C([0,T];H)$ and B is globally Lipschitz, we have that $u_s \to u$ and $B \circ u_s \to B \circ u$ in $C([\tau,t];H)$ and, consequently $u_s \to u$ and $B \circ u_s \to B \circ u$ in $L^2(\tau,t;H)$, $\forall\, 0 \leq \tau \leq t \leq T$. Then,

$$\langle B \circ u_s - h, u_s - \theta\rangle_{L^2(\tau,t;H)} \to \langle B \circ u - h, u - \theta\rangle_{L^2(\tau,t;H)}$$

for all $\theta, h \in H$.

Now consider $\overline{\theta} \in C_0^\infty(\Omega) \subset \mathcal{D}(A^s) \subset H$ arbitrarily fixed and let $\overline{h} := -\Delta_p(\overline{\theta}) + |\overline{\theta}|^{p-2}\overline{\theta} \in H$. From (4.2)

$$\begin{aligned}\frac{1}{2} \parallel u_s(t) - \overline{\theta} \parallel_H^2 \leq \frac{1}{2} \parallel u_s(\tau) - \overline{\theta} \parallel_H^2 &\\ &+ \int_\tau^t \langle B(u_s(t')) + \Delta_{p_s(x)}(\overline{\theta}) - |\overline{\theta}|^{p_s(x)-2}\overline{\theta}, u_s(t') - \overline{\theta}\rangle dt'\\ &= \frac{1}{2} \parallel u_s(\tau) - \overline{\theta} \parallel_H^2 + \int_\tau^t \langle B(u_s(t')) - \overline{h}, u_s(t') - \overline{\theta}\rangle dt' \qquad (4.3)\\ &+ \int_\tau^t \langle \overline{h} + \Delta_{p_s(x)}(\overline{\theta}) - |\overline{\theta}|^{p_s(x)-2}\overline{\theta}, u_s(t') - \overline{\theta}\rangle dt'.\end{aligned}$$

We claim that $\int_\tau^t \langle \overline{h} + \Delta_{p_s(x)}(\overline{\theta}) - |\overline{\theta}|^{p_s(x)-2}\overline{\theta}, u_s(t') - \overline{\theta}\rangle dt' \to 0$ as $s \to +\infty$. In fact, for each $t' > 0$

$$\begin{aligned}
& |\langle \overline{h} + \Delta_{p_s(x)}(\overline{\theta}) - |\overline{\theta}|^{p_s(x)-2}\overline{\theta}, u_s(t') - \overline{\theta}\rangle| \\
= \; & |\langle \overline{h}, u_s(t') - \overline{\theta}\rangle - \langle -\Delta_{p_s(x)}(\overline{\theta}) + |\overline{\theta}|^{p_s(x)-2}\overline{\theta}, u_s(t') - \overline{\theta}\rangle| \\
\leq \; & \int_\Omega \left(\left| \left|\nabla\overline{\theta}\right|^{p-1} - \left|\nabla\overline{\theta}\right|^{p_s(x)-1} \right| \right) |\nabla u_s(t')| \, dx + \int_\Omega \left| \left|\nabla\overline{\theta}\right|^{p} - \left|\nabla\overline{\theta}\right|^{p_s(x)} \right| dx \\
& + \int_\Omega \left(\left| \left|\overline{\theta}\right|^{p-1} - \left|\overline{\theta}\right|^{p_s(x)-1} \right| \right) |u_s(t')| \, dx + \int_\Omega \left| \left|\overline{\theta}\right|^{p} - \left|\overline{\theta}\right|^{p_s(x)} \right| dx.
\end{aligned}$$

Since $p_s(x) \to p$ for all $x \in I$ it follows by Dominated Convergence Theorem that

$$\int_\Omega \left| \left|\nabla\overline{\theta}\right|^{p} - \left|\nabla\overline{\theta}\right|^{p_s(x)} \right| dx \to 0 \text{ as } s \to \infty,$$

and

$$\int_\Omega \left| \left|\overline{\theta}\right|^{p} - \left|\overline{\theta}\right|^{p_s(x)} \right| dx \to 0 \text{ as } s \to \infty.$$

On the other hand, considering $\tilde{\Omega} := \{x \in \Omega : \overline{\theta}(x) \neq 0\}$, $\tilde{\Omega}_1 := \{x \in \tilde{\Omega} : |\overline{\theta}(x)| \leq 1\}$, $\tilde{\Omega}_2 := \{x \in \tilde{\Omega} : |\overline{\theta}(x)| > 1\}$, and using the Mean Value Theorem we obtain

$$\begin{aligned}
& \int_\Omega \left(\left| \left|\overline{\theta}\right|^{p-1} - \left|\overline{\theta}\right|^{p_s(x)-1} \right| \right) |u_s(t')| \, dx = \int_{\tilde{\Omega}} \left(\left| \left|\overline{\theta}\right|^{p-1} - \left|\overline{\theta}\right|^{p_s(x)-1} \right| \right) |u_s(t')| \, dx \\
\leq \; & \int_{\tilde{\Omega}} \left| \left|\overline{\theta}\right|^{\tau(s,x)} \ln\left(\left|\overline{\theta}\right|\right) \right| (p_s(x) - p) \, |u_s(t')| \, dx \\
\leq \; & \int_{\tilde{\Omega}_1} \left| \left|\overline{\theta}\right|^{p-1} \ln\left(\left|\overline{\theta}\right|\right) \right| (p_s(x) - p) \, |u_s(t')| \, dx \\
& + \int_{\tilde{\Omega}_2} \left| \left|\overline{\theta}\right|^{a-1} \ln\left(\left|\overline{\theta}\right|\right) \right| (p_s(x) - p) \, |u_s(t')| \, dx
\end{aligned}$$

where $p - 1 < \tau(s, x) < p_s(x) - 1 \leq a - 1$. As $\overline{\theta} \in C_0^\infty(\Omega)$ there exist $K_{\overline{\theta}} > 0$ such that $|\overline{\theta}(x)| \leq K_{\overline{\theta}}$ for all $x \in \Omega$. So by the continuity of the functions $g_\alpha : [0, K_{\overline{\theta}}] \to \mathbb{R}$ given by

$$g_\alpha(w) = \begin{cases} w^\alpha \ln w & \text{if } \; w \in (0, K_{\overline{\theta}}] \\ 0 & \text{if } \; w = 0, \end{cases}$$

for $\alpha = p - 1, a - 1$, we conclude that

$$\begin{aligned}
& \int_\Omega \left(\left| \left|\overline{\theta}\right|^{p-1} - \left|\overline{\theta}\right|^{p_s(x)-1} \right| \right) |u_s(t')| \, dx \leq \|p_s - p\|_\infty \int_\Omega C \, |u_s(t')| \, dx \\
\leq \; & \|p_s - p\|_\infty \left[\int_\Omega \frac{1}{q_s(x)} C^{q_s(x)} dx + \int_\Omega \frac{1}{p_s(x)} \left|u_s(t')\right|^{p_s(x)} dx \right] \\
\leq \; & \|p_s - p\|_\infty \left[\int_\Omega C^{q_s(x)} dx + \frac{1}{2} \int_\Omega \left|u_s(t')\right|^{p_s(x)} dx \right]
\end{aligned}$$

where $q_s(\cdot)$ is such that $\frac{1}{p_s(x)} + \frac{1}{q_s(x)} = 1$, $\forall\, x \in \Omega$. By Proposition 3.1 there exists a constant $C > 0$ such that $\int_\Omega |u_s(t')|^{p_s(x)}\, dx \leq C$ for every $t' \in (\tau, t)$ and $s \in \mathbb{N}$.

As $1 < q_s(x) < 2$ we obtain that

$$\int_\Omega \Big(\Big| \, |\overline{\theta}|^{p-1} - |\overline{\theta}|^{p_s(x)-1} \Big| \Big) \, |u_s(t')|\, dx \leq \|p_s - p\|_\infty \tilde{C} \to 0$$

as $s \to \infty$. Using the same arguments as above it follows that

$$\int_\Omega \Big(\Big| \, |\nabla\overline{\theta}|^{p-1} - |\nabla\overline{\theta}|^{p_s(x)-1} \Big| \Big) \, |\nabla u_s(t')|\, dx \to 0 \quad \text{as } s \to \infty.$$

Thus

$$\int_\tau^t \langle \overline{h} + \Delta_{p_s(x)}(\overline{\theta}) - |\overline{\theta}|^{p_s(x)-2}\overline{\theta}, u_s(t') - \overline{\theta} \rangle dt' \to 0 \text{ as } s \to +\infty.$$

So, taking the limit in (4.3) as $s \to \infty$, we obtain

$$\begin{aligned} \frac{1}{2}\|u(t) - \overline{\theta}\|_H^2 \quad \leq \quad & \frac{1}{2}\|u(\tau) - \overline{\theta}\|_H^2 \\ & + \int_\tau^t \langle B(u(t')) + \Delta_p(\overline{\theta}) - |\overline{\theta}|^{p-2}\overline{\theta}, u(t') - \overline{\theta} \rangle dt' \end{aligned} \tag{4.4}$$

for every $\overline{\theta} \in C_0^\infty(\Omega)$ and $0 \leq \tau \leq t \leq T$.

Now, we use a density argument to conclude that u is a solution of (4.1). Let $\overline{\theta} \in \mathcal{D}(A^p) \subset W^{1,p}(\Omega)$, $A^p u := -\Delta_p u + |u|^{p-2}u$. So, there exists a sequence $\{\overline{\theta_j}\}_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\|\overline{\theta_j} - \overline{\theta}\|_{W^{1,p}(\Omega)} \to 0$ as $j \to \infty$ and consequently $\|\overline{\theta_j} - \overline{\theta}\|_H \to 0$ as $j \to \infty$. By (4.4),

$$\begin{aligned} \frac{1}{2}\|u(t) - \overline{\theta_j}\|_H^2 \quad \leq \quad & \frac{1}{2}\|u(\tau) - \overline{\theta_j}\|_H^2 \\ & + \int_\tau^t \langle B(u(t')) + \Delta_p(\overline{\theta_j}) - |\overline{\theta_j}|^{p-2}\overline{\theta_j}, u(t') - \overline{\theta_j} \rangle dt' \end{aligned}$$

for every $j \in \mathbb{N}$ and $0 \leq \tau \leq t \leq T$. Obviously, $\frac{1}{2}\|u(t) - \overline{\theta_j}\|_H^2 \to \frac{1}{2}\|u(t) - \overline{\theta}\|_H^2$ as $j \to \infty$ and $\frac{1}{2}\|u(\tau) - \overline{\theta_j}\|_H^2 \to \frac{1}{2}\|u(\tau) - \overline{\theta}\|_H^2$ as $j \to \infty$. With some computations and using the Dominated Convergence Theorem we obtain

$$\langle B(u(t')) + \Delta_p(\overline{\theta_j}) - |\overline{\theta_j}|^{p-2}\overline{\theta_j}, u(t') - \overline{\theta_j} \rangle \to \langle B(u(t')) + \Delta_p(\overline{\theta}) - |\overline{\theta}|^{p-2}\overline{\theta}, u(t') - \overline{\theta} \rangle$$

as $j \to \infty$. So, taking the limit with $j \to \infty$, we obtain

$$\begin{aligned} \frac{1}{2}\|u(t) - \overline{\theta}\|_H^2 \quad \leq \quad & \frac{1}{2}\|u(\tau) - \overline{\theta}\|_H^2 \\ & + \int_\tau^t \langle B(u(t')) + \Delta_p(\overline{\theta}) - |\overline{\theta}|^{p-2}\overline{\theta}, u(t') - \overline{\theta} \rangle dt' \end{aligned}$$

for every $\overline{\theta} \in \mathcal{D}(A^p)$ and $0 \leq \tau \leq t \leq T$. Thus, Proposition 3.6 in [3] implies that u is a solution of (4.1). □

Thus, following the same arguments as in Theorem 6 in [11] we conclude:

Theorem 4.3. *The family of global attractors $\{\mathcal{A}_s;\ s \in \mathbb{N}\}$ associated with problem (1.1) is upper semicontinuous on s at infinity, in the topology of H.*

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