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The characterization of generalized Jordan centralizers on algebras *

Quanyuan Chen, Xiaochun Fang and Changjing Li

abstract: In this paper, it is shown that if $\mathcal{A}$ is a CSL subalgebra of a von Neumann algebr and ϕ is a continuous mapping on $\mathcal{A}$ such that $(m+n+k+l)\phi(A^2)-(m\phi(A)A+nA\phi(A)+k\phi(I)A^2+lA^2\phi(I))\in\mathbb{F}I$ for any $A\in\mathcal{A}$, where $\mathbb{F}$ is the real field or the complex field, then ϕ is a centralizer. It is also shown that if ϕ is an additive mapping on $\mathcal{A}$ such that $(m+n+k+l)\phi(A^2)=m\phi(A)A+nA\phi(A)+k\phi(I)A^2+lA^2\phi(I)$ for any $A\in\mathcal{A}$, then ϕ is a centralizer.

Key Words: Jordan centralizers; centralizers, CSL subalgebras of von Neumann algebras

Contents

1. Introduction

Thoughout the paper, $\mathbb{F}$ will denote the real field or the complex field. Let H be a complex Hilbert space and $\mathcal{L}$ be a subspace lattice of H. Denote by $Alg\mathcal{L}$ the algebra of all bounded operators in $B(H)$ which leave every subspace in $\mathcal{L}$ invariant. Dually, for a subalgebra $\mathcal{A}$ of $B(H)$, denote by $Lat\mathcal{A}$ the lattice of all closed subspaces left invariant under every operator in $\mathcal{A}$. For convenience we shall disregard the distinction between a closed subspace of H and the orthogonal projection onto it. A totally ordered subspace lattice is called a nest. If each pair of projections in $\mathcal{L}$ commutate, then the subspace lattice $\mathcal{L}$ is called a commutative subspace lattice, or a CSL. If $\mathcal{L}$ is a CSL, whose projections are contained in a von Neumann algebra $\mathcal{N}$ acting on the Hilbert space H, then $\mathcal{A}=\mathcal{N}\cap Alg\mathcal{L}$ is called a CSL subalgebra of the von Neumann algebra $\mathcal{N}$.

Let $\mathcal{R}$ be a ring or an algebra and ϕ be an additive mapping on $\mathcal{R}$. If $\phi(AB)=\phi(A)B$ (resp. $\phi(AB)=A\phi(B)$) for any $A,B\in\mathcal{R}$, then ϕ is called a left centralizer (resp. a right centralizer). A centralizer of $\mathcal{R}$ is an additive mapping which is a left

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as well as a right centralizer. An additive mapping $\phi : \mathcal{R} \to \mathcal{R}$ is called a left (resp. right) Jordan centralizer, if $\phi(A^2) = \phi(A)A$ (resp. $\phi(A^2) = A\phi(A)$) for any $A \in \mathcal{R}$. A Jordan centralizer of $\mathcal{R}$ is an additive mapping which is a left Jordan as well as a right Jordan centralizer. An $(m, n)-$ Jordan centralizer is defined in ([16]) as follows: An additive mapping $\phi : \mathcal{R} \to \mathcal{R}$ is called an $(m, n)-$ Jordan centralizer if $(m+n)\phi(A^2) = m\phi(A)A + nA\phi(A)$ for any $A \in \mathcal{R}$, where $m, n \in \mathbb{N}$ with $m+n \neq 0$. Obviously, every centralizer is a Jordan centralizer and any Jordan centralizer is an $(m, n)-$ Jordan centralizer, but the converse is not true in general.

The characterization of centralizers on algebras or rings is a subject in various areas. Bresar and Zalar ([2]) have proved that if $\mathcal{R}$ is a prime ring and ϕ is an additive mapping on $\mathcal{R}$ such that $\phi(A^2) = \phi(A)A$ (resp. $\phi(A^2) = A\phi(A)$) for any $A \in \mathcal{R}$, then ϕ is a left (resp. a right) centralizer. Zalar([23]) generalized the result to 2-torsion free semi-prime rings as follows: if $\mathcal{R}$ is a 2-torsion free semi-prime ring and ϕ is an additive mapping on $\mathcal{R}$ such that $\phi(A^2) = \phi(A)A$ (resp. $\phi(A^2) = A\phi(A)$) for any $A \in \mathcal{R}$, then ϕ is a left (resp. a right) centralizer. Vukman([15]) has proved that if $\mathcal{R}$ is a 2-torsion free semi-prime ring and ϕ is an additive mapping on $\mathcal{R}$ such that $2\phi(A^2) = \phi(A)A + A\phi(A)$ for any $A \in \mathcal{R}$, then ϕ is a centralizer. Benkovic and Eremita ([1]) proved that if $\mathcal{R}$ is a prime ring with $Ch(\mathcal{R}) = 0$ or $Ch(\mathcal{R}) \geq n$, where n is a fixed positive integer and $n \geq 2$, and ϕ is an additive mapping on $\mathcal{R}$ such that $\phi(A^n) = \phi(A)A^{n-1}$ for any $A \in \mathcal{R}$, then ϕ is a centralizer. Vukman and Kosi-Ulbl ([17]) proved that if X is a Banach space over the field $\mathbb{F}$, and $\mathcal{A}$ is a standard subalgebra of $B(X)$ and $\phi : \mathcal{A} \to B(X)$ is an additive mapping such that $\phi(A^{m+n+1}) = A^m\phi(A)A^n$ for any $A \in \mathcal{A}$, where $m, n \in \mathbb{Z}^+$ and then ϕ is a centralizer. Qi etc. ([14]) proved that if $\mathcal{A}$ is a standard subalgebra of $B(X)$ with the identity I and $\phi : \mathcal{A} \to B(X)$ is an additive mapping such that $\phi(A^{m+n+1}) - A^m\phi(A)A^n \in \mathbb{F}I$ for any $A \in \mathcal{A}$, where X is a Banach space over the field $\mathbb{F}$ and $m, n \in \mathbb{Z}^+$, then ϕ is a centralizer. Yang and Zhang ([22]) proved that, if $\phi : \tau(\mathcal{N}) \to \tau(\mathcal{N})$ is an additive mapping on a nest algebra $\tau(\mathcal{N})$, such that $(m+n)\phi(A^{p+1}) = m\phi(A)A^p + nA^p\phi(A)$ or $\phi(A^{m+n+1}) = A^m\phi(A)A^n$ for any $A \in \tau(\mathcal{N})$, where $\mathcal{N}$ is a non-trivial nest on H, $\tau(\mathcal{N})$ is the corresponding nest algebra, and $m, n, p \in \mathbb{Z}^+$, then ϕ is a centralizer. J. Vukman ([16]) proved that an $(m, n)-$ Jordan centralizer on a prime ring with $Ch(\mathcal{R}) \neq 6mn(m+n)$ is a centralizer. Li etc. ([12]) proved that a Jordan centralizer on a CSL subalgebra of a von Neumann algebra is a centralizer.

Motivated by these results, we are concerned with an additive mapping ϕ on $\mathcal{A}$, a CSL subalgebra of a von Neumann algebra, which is not a semi-prime ring. It is shown that if ϕ is a continuous mapping on $\mathcal{A}$ such that $(m+n+k+l)\phi(A^2) - (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathbb{F}I$ for any $A \in \mathcal{A}$, then ϕ is a centralizer (Theorem 3.1). It is also shown that if ϕ is an additive mapping on $\mathcal{A}$ such that $(m+n+k+l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)$ for any $A \in \mathcal{A}$, then ϕ is a centralizer (Theorem 3.2). It follows that an $(m, n)-$ Jordan centralizer on $\mathcal{A}$ is a centralizer (Corollary 3.1). Furthermore, it is shown that if ϕ is an additive mapping on $\mathcal{A}$ such that $(m+n)\phi(A^{p+1}) = m\phi(A)A^p + nA^p\phi(A)$ or $\phi(A^{m+n+1}) = A^m\phi(A)A^n$ for any $A \in \mathcal{A}$, then ϕ is a centralizer (Theorem 3.3 and Theorem 3.4).

2. Preliminaries: some lemmas

In this section, let $\mathcal{A}$ be a unital algebra. We discuss an additive mapping ϕ on $\mathcal{A}$ such that

$$(m+n+k+l)\phi(A^2) - (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathbb{F}I, \quad (2.1)$$

that is, for any $A \in \mathcal{A}$, there is $\mu_A \in \mathbb{F}$ (depending on A) such that

$$(m+n+k+l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I) + \mu_A I,$$

where $m > 0, n > 0, k \geq 0, l \geq 0$.

Lemma 2.1. *Suppose that ϕ is an additive mapping on $\mathcal{A}$ as above. Then, for any $A, B \in \mathcal{A}$,*
(1)

$$\begin{aligned}(m+n+k+l)\phi(AB+BA) &= m\phi(A)B + nA\phi(B) + m\phi(B)A \\ &+nB\phi(A) + k\phi(I)AB + k\phi(I)BA + lAB\phi(I) + lBA\phi(I) \\ &+(\mu_{A+B} - \mu_A - \mu_B)I;\end{aligned}$$

(2) $(m+n+2k+2l)\phi(A) = (m+2k)\phi(I)A + (n+2l)A\phi(I) + (\mu_{A+I} - \mu_A)I.$

Proof: For any $A, B \in \mathcal{A}$,

$$\begin{aligned}(m+n+k+l)\phi(A+B)^2 &= m\phi(A+B)(A+B) + n(A+B)\phi(A+B) \\ &\quad + k\phi(I)(A+B)^2 + l(A+B)^2\phi(I) + \mu_{A+B}I \\ &= m\phi(A)A + m\phi(A)B + m\phi(B)A + m\phi(B)B \\ &\quad + nA\phi(A) + nA\phi(B) + nB\phi(A) + nB\phi(B) \\ &\quad + k\phi(I)A^2 + k\phi(I)BA + k\phi(I)AB + k\phi(I)B^2 \\ &\quad + lA^2\phi(I) + lAB\phi(I) + lBA\phi(I) + lB^2\phi(I) + \mu_{A+B}I.\end{aligned}$$

On the other hand,

$$\begin{aligned}(m+n+k+l)\phi(A+B)^2 &= (m+n+k+l)\phi(A^2+AB+BA+B^2) \\ &= m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I) + nB\phi(B) \\ &\quad + m\phi(B)B + k\phi(I)B^2 + lB^2\phi(I) \\ &\quad + (m+n+k+l)\phi(AB+BA) + \mu_A I + \mu_B I.\end{aligned}$$

Comparing above two equalities, we obtain that

$$\begin{aligned}(m+n+k+l)\phi(AB+BA) &= m\phi(A)B + nA\phi(B) + m\phi(B)A + nB\phi(A) \\ &+k\phi(I)AB + k\phi(I)BA + lAB\phi(I) \\ &+lBA\phi(I) + (\mu_{A+B} - \mu_A - \mu_B)I.\end{aligned} \quad (2.2)$$

Putting in (2.2) $B = I$, it follows from $\mu_I = 0$ that

$$(m+n+2k+2l)\phi(A) = (m+2k)\phi(I)A + (n+2l)A\phi(I) + (\mu_{A+I} - \mu_A)I. \quad (2.3)$$

□

Lemma 2.2. *Let ϕ be an additive mapping on $\mathcal{A}$ as above. If $A_0 \in \mathcal{A}$ with $A_0 \notin \mathbb{F}I$ such that $A_0\phi(I) = \phi(I)A_0$, then $\mu_{A_0+I} - \mu_{A_0} = 0$ and $\phi(A_0) = A_0\phi(I) = \phi(I)A_0$.*

Proof: Since $A_0\phi(I) = \phi(I)A_0$, $A_0^2\phi(I) = \phi(I)A_0^2 = A_0\phi(I)A_0$. By (2.3), we have that

$$\phi(A_0) = \phi(I)A_0 + \frac{1}{m+n+2k+2l}(\mu_{A_0+I} - \mu_{A_0})I$$

and

$$\phi(A_0^2) = \phi(I)A_0^2 + \frac{1}{m+n+2k+2l}(\mu_{A_0^2+I} - \mu_{A_0^2})I.$$

Hence

$$(m+n+k+l)\phi(A_0^2) = (m+n+k+l)\phi(I)A_0^2 + \frac{m+n+k+l}{m+n+2k+2l}(\mu_{A_0^2+I} - \mu_{A_0^2})I.$$

On the other hand,

$$\begin{aligned}(m+n+k+l)\phi(A_0^2) &= m\phi(A_0)A_0 + nA_0\phi(A_0) + k\phi(I)A_0^2 + lA_0^2\phi(I) + \mu_{A_0}I \\ &= m(\phi(I)A_0 + \frac{1}{m+n+2k+2l}(\mu_{A_0+I} - \mu_{A_0})I)A_0 \\ &\quad + nA_0(\phi(I)A_0 + \frac{1}{m+n+2k+2l}(\mu_{A_0+I} - \mu_{A_0})I) \\ &\quad + k\phi(I)A_0^2 + lA_0^2\phi(I) + \mu_{A_0}I.\end{aligned}$$

Comparing the two equalities, we have that $\frac{m+n}{m+n+2k+2l}(\mu_{A_0+I} - \mu_{A_0})A_0 \in \mathbb{F}I$. Since $A_0 \notin \mathbb{F}I$ and $m+n > 0$, $\mu_{A_0+I} - \mu_{A_0} = 0$ and $\phi(A_0) = A_0\phi(I) = \phi(I)A_0$. □

Lemma 2.3. *Let ϕ be an additive mapping on $\mathcal{A}$ as above. If $P \in \mathcal{A}$ with $P^2 = P$, then* (1) *$\phi(P) = P\phi(I) = \phi(I)P = \phi(P)P = P\phi(P)$;*
(2) *$\mu_{P+I} = \mu_P = 0$.*

Proof. If $P = 0$ or $P = I$, the result is trivial.

Let P be a non-trivial idempotent, that is, $P \neq 0$ and $P \neq I$. By (2.1),

$$(m+n+k+l)\phi(P) = m\phi(P)P + nP\phi(P) + k\phi(I)P + lP\phi(I) + \mu_P I. \quad (2.4)$$

By (2.3),

$$(m+n+2k+2l)\phi(P) = (m+2k)\phi(I)P + (n+2l)P\phi(I) + (\mu_{P+I} - \mu_P)I, \quad (2.5)$$

Multiplying (2.4) by P from the left and the right sides, gives that

$$(k+l)P\phi(P)P = (k+l)P\phi(I)P + \mu_P P. \quad (2.6)$$

Multiplying (2.5) by P from the left and the right sides, we have that

$$P\phi(P)P = P\phi(I)P + \frac{1}{m+n+2k+2l}(\mu_{P+I} - \mu_P)P. \tag{2.7}$$

By comparing (2.6) with (2.7),

$$(m+n+2k+2l)\mu_P = (k+l)(\mu_{P+I} - \mu_P). \tag{2.8}$$

Multiplying (2.4) by P from the left side gives that

$$(m+n+k+l)P\phi(P) = mP\phi(P)P + nP\phi(P) + kP\phi(I)P + lP\phi(I) + \mu_P P,$$

that is,

$$(m+k+l)P\phi(P) = mP\phi(P)P + kP\phi(I)P + lP\phi(I) + \mu_P P.$$

It follows from (2.7) that

$$(m+k+l)P\phi(P) = (m+k)P\phi(I)P + lP\phi(I) + (\frac{m}{m+n+2k+2l}(\mu_{P+I} - \mu_P) + \mu_P)P. \tag{2.9}$$

Thus

$$\begin{aligned}(m+n+2k+2l)(m+k+l)P\phi(P) &= (m+k)(m+n+2k+2l)P\phi(I)P \\ &+ l(m+n+2k+2l)P\phi(I) + (m(\mu_{P+I} - \mu_P) + (m+n+2k+2l)\mu_P)P.\end{aligned} \tag{2.9$'$}$$

Multiplying (2.5) by P from the left side, yields that

$$(m+n+2k+2l)P\phi(P) = (m+2k)P\phi(I)P + (n+2l)P\phi(I) + (\mu_{P+I} - \mu_P)P, \tag{2.10}$$

Comparing (2.9)$'$ and (2.10), we obtain that

$$\begin{aligned}&(m+k+l)(m+2k)P\phi(I)P + (n+2l)(m+k+l)P\phi(I) + (m+k+l)(\mu_{P+I} - \mu_P)P \\ &= (m+k)(m+n+2k+2l)P\phi(I)P + l(m+n+2k+2l)P\phi(I) \\ &\quad + (m(\mu_{P+I} - \mu_P) + (m+n+2k+2l)\mu_P)P.\end{aligned}$$

It follows from (2.8) that $(m+k+l)(\mu_{P+I} - \mu_P)P = (m(\mu_{P+I} - \mu_P) + (m+n+2k+2l)\mu_P)P$ and

$$P\phi(I) = P\phi(I)P. \tag{2.11}$$

It follows from (2.9) that

$$P\phi(P) = P\phi(I)P + \frac{1}{m+n+2k+2l}(\mu_{P+I} - \mu_P)P. \tag{2.12}$$

Similarly,

$$\phi(I)P = P\phi(I)P \tag{2.13}$$

and

$$\phi(P)P = P\phi(I)P + \frac{1}{m+n+2k+2l}(\mu_{P+I} - \mu_P)P. \tag{2.14}$$

(2.11) and (2.13) yield that $\phi(I)P = P\phi(I)$. And

$$\phi(P)P = P\phi(P) = P\phi(P)P \tag{2.15}$$

by (2.12) and (2.14). By Lemma 2.2 and $\phi(I)P = P\phi(I)$, it follows that $\phi(P) = \phi(I)P = P\phi(I)$ and $\mu_{P+I} - \mu_P = 0$. And by (2.8), $\mu_P = 0$ and $\mu_{P+I} = \mu_P = 0$. Identity (2.4) yields that

$$\phi(P) = \phi(I)P = P\phi(I) = P\phi(I)P = P\phi(P)P = \phi(P)P = P\phi(P).$$

□

Lemma 2.4. *Let ϕ be an additive mapping on $\mathcal{A}$ as above. If $A, P \in \mathcal{A}$ with $P^2 = P$, then (1) $\phi(AP) = \phi(A)P + \mu(AP)I - \mu(A)P$,*
(2) $\phi(PA) = P\phi(A) + \mu(PA)I - \mu(A)P$, where $\mu(A) = \frac{1}{m+n+2k+2l}(\mu_{A+I} - \mu_A)$.

Proof: By (2.3),

$$\begin{aligned}\phi(AP) &= \tfrac{m+2k}{m+n+2k+2l}\phi(I)AP + \tfrac{n+2l}{m+n+2k+2l}AP\phi(I) + \tfrac{1}{m+n+2k+2l}(\mu_{AP+I} - \mu_{AP})I\\ &= (\tfrac{m+2k}{m+n+2k+2l}\phi(I)A + \tfrac{n+2l}{m+n+2k+2l}A\phi(I))P + \tfrac{1}{m+n+2k+2l}(\mu_{AP+I} - \mu_{AP})I\\ &= \phi(A)P + \mu(AP)I - \mu(A)P\end{aligned}$$

Similarly, $\phi(PA) = P\phi(A) + \mu(PA)I - \mu(A)P$. □

Lemma 2.5. *Let ϕ be an additive mapping on $\mathcal{A}$ as above. If $A, P \in \mathcal{A}$ with $P^2 = P$, then*

$$\phi(PAP) = \phi(PAP)P = P\phi(PAP) = P\phi(PAP)P.$$

Proof: If $P = 0$ or $P = I$, the result is trivial.

Let P be a non-trivial idempotent, that is, $P \neq 0$ and $P \neq I$. It follows from Lemma 2.4 that

$$\phi(PAP) = \phi(PAPP) = \phi(PAP)P + \mu(PAP)I - \mu(PAP)P, \tag{2.16}$$

$$\phi(PAP) = \phi(PPAP) = P\phi(PAP) + \mu(PAP)I - \mu(PAP)P. \tag{2.17}$$

Comparing (2.16) and (2.17), we have that

$$P\phi(PAP) = \phi(PAP)P, \tag{2.18}$$

It follows from Lemma 2.1(2) that

$$(m+n+2k+2l)\phi(PAP) = (m+2k)\phi(I)PAP + (n+2l)PAP\phi(I) + (\mu_{PAP+I} - \mu_{PAP})I. \tag{2.19}$$

By Lemma 2.3, we have that $\phi(I)PAP = \phi(P)PAP$, $PAP\phi(I) = PAP\phi(P)$ and $\mu_P = 0$. Putting PAP for A and P for B in (2.2), we have that

$$\begin{aligned}
&2(m+n+k+l)\phi(PAP) = (m+n+k+l)\phi((PAP)P + P(PAP)) \\
&= m\phi(PAP)P + nP\phi(PAP) + nPAP\phi(P) + m\phi(P)PAP \\
&\quad +2k\phi(I)PAP + 2lPAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP} - \mu_P)I \\
&= (m+n)P\phi(PAP) + (m+2k)\phi(I)PAP \\
&\quad +(n+2l)PAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP})I.
\end{aligned} \tag{2.20}$$

By (2.19) with (2.20),

$$\begin{aligned}
&2(m+n+k+l)(m+n+2k+2l)\phi(PAP) \\
&= (m+n)P((m+2k)\phi(I)PAP + (n+2l)PAP\phi(I) + (\mu_{PAP+I} - \mu_{PAP})I) \\
&\quad +(m+2k)(m+n+2k+2l)\phi(I)PAP + (n+2l)(m+n+2k+2l)PAP\phi(I) \\
&\quad +(m+n+2k+2l)(\mu_{PAP+P} - \mu_{PAP})I \\
&= 2(m+2k)(m+n+k+l)\phi(I)PAP + 2(n+2l)(m+n+k+l)PAP\phi(I) \\
&\quad +(m+n+2k+2l)(\mu_{PAP+P} - \mu_{PAP})I + (m+n)(\mu_{PAP+I} - \mu_{PAP})P.
\end{aligned} \tag{2.21}$$

It follows from (2.19) that

$$\begin{aligned}
&2(m+n+k+l)(m+n+2k+2l)\phi(PAP) \\
&= 2(m+2k)(m+n+k+l)\phi(I)PAP + 2(n+2l)(m+n+k+l)PAP\phi(I) \\
&\quad +2(m+n+k+l)(\mu_{PAP+I} - \mu_{PAP})I.
\end{aligned} \tag{2.22}$$

Comparing (2.21) and (2.22), we have that

$$\begin{aligned}
&(m+n+2k+2l)(\mu_{PAP+P} - \mu_{PAP})I + (m+n)(\mu_{PAP+I} - \mu_{PAP})P \\
&\quad = 2(m+n+k+l)(\mu_{PAP+I} - \mu_{PAP})I.
\end{aligned} \tag{2.23}$$

Multiplying (2.19) by P from the left and the right sides gives that

$$(m+n+2k+2l)P\phi(PAP)P = (m+2k)\phi(I)PAP + (n+2l)PAP\phi(I) + (\mu_{PAP+I} - \mu_{PAP})P. \tag{2.24}$$

Multiplying (2.20) by P from the left side yields that

$$\begin{aligned}
2(m+n+k+l)P\phi(PAP)P &= (m+n)P\phi(PAP)P + (m+2k)\phi(I)PAP \\
&\quad +(n+2l)PAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP})P.
\end{aligned}$$

It follows that

$$\begin{aligned}
(m+n+2k+2l)P\phi(PAP)P &= (m+2k)\phi(I)PAP \\
&\quad +(n+2l)PAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP})P.
\end{aligned} \tag{2.25}$$

Comparing (2.24) and (2.25), we have that

$$\mu_{PAP+P} - \mu_{PAP} = \mu_{PAP+I} - \mu_{PAP}. \tag{2.26}$$

It follows from (2.23) and (2.26) that $(m+n)(\mu_{PAP+P}-\mu_{PAP})=0$. Since $m+n>0$, $\mu_{PAP+P}-\mu_{PAP}=0$ and

$$\mu_{PAP+P}-\mu_{PAP}=\mu_{PAP+I}-\mu_{PAP}=0. \tag{2.27}$$

By (2.20) and (2.27),

$$\begin{aligned}2(m+n+k+l)\phi(PAP) &= (m+n)P\phi(PAP)+(m+2k)\phi(I)PAP\\ &\quad +(n+2l)PAP\phi(I).\end{aligned} \tag{2.28}$$

By (2.19) and (2.27),

$$(m+n+2k+2l)\phi(PAP)=(m+2k)\phi(I)PAP+(n+2l)PAP\phi(I). \tag{2.29}$$

Combating it with (2.28), we have that $(m+n)\phi(PAP)=(m+n)P\phi(PAP)$ and

$$\phi(PAP)=P\phi(PAP)=\phi(PAP)P=P\phi(PAP)P. \tag{2.30}$$

□

3. Generalized Jordan centralizers on CSL subalgebras of von Neumann algebras

In this section, we discuss an additive mapping ϕ on $\mathcal{A}$, a CSL subalgebra of a von Neumann algebra, such that $(m+n+k+l)\phi(A^2)-(m\phi(A)A+nA\phi(A)+k\phi(I)A^2+lA^2\phi(I))\in\mathbb{F}I$ for any $A\in\mathcal{A}$, where $\mathbb{F}$ is the real field or the complex field. The main result is as follows:

Theorem 3.1. *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space H, and let $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$, and $\mathcal{A}=\mathcal{N}\cap Alg\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. If $\phi:\mathcal{A}\to\mathcal{A}$ is a continuous mapping on $\mathcal{A}$ such that*

$$(m+n+k+l)\phi(A^2)-(m\phi(A)A+nA\phi(A)+k\phi(I)A^2+lA^2\phi(I))\in\mathbb{F}I$$

for any $A\in\mathcal{A}$, where $m,n,k,l\geq 0$ with $mn\neq 0$, then ϕ is a centralizer. That is, $\phi(A)=\phi(I)A=A\phi(I)$ for any $A\in\mathcal{A}$.

The proof of Theorem 3.1 will proceed through several lemmas, in each of which we maintain the same notation.

Proposition 3.1 ([12]). *Suppose that $\mathcal{A}=\mathcal{N}\cap Alg\mathcal{L}$ is a CSL subalgebra of the von Neumann algebra $\mathcal{N}$. Let $Q_1(H)$, or Q_1 simply, be the orthogonal projection onto the linear span of the set $\{PAP^\perp x:\ \ P\in\mathcal{L},\ \ A\in\mathcal{A},\ \ x\in H\}$; and let $Q_2(H)$, or Q_2 simply, be the orthogonal projection onto the linear span of the set $\{P^\perp A^*Px:\ \ P\in\mathcal{L},\ \ A\in\mathcal{A},\ \ x\in H\}$, and $Q=Q_1(H)\vee Q_2(H)$. Then*
(1) Q_1, Q_2 and $Q\in\mathcal{L}'\cap\mathcal{N}\subseteq\mathcal{A}$, where $\mathcal{L}'$ is the commutant of $\mathcal{L}$. And Q_1 commutes with Q_2, and Q, $Q_1\in Lat\mathcal{A}$. Furthermore, $Q^\perp AQ=QAQ^\perp=0$ for any $A\in\mathcal{A}$, so that $\mathcal{A}=Q\mathcal{A}Q\oplus Q^\perp\mathcal{A}Q^\perp$.
(2) If $Q\neq I$, then $Q^\perp\mathcal{A}Q^\perp$ is a von Neumann algebra on $Q^\perp H$.

In the sequel of this section, let $\mathcal{A}$ be a CSL subalgebra of a von Neumann algebra $\mathcal{N}$. We choose an arbitrary non-trivial projection P in $(\mathcal{A}\cap Lat(\mathcal{A}))(\supseteq \mathcal{L}\cup\{Q,Q_1\})$. And let $P_1=P,\ \ P_2=P^{\perp}$, then $P_1,\ \ P_2\in\mathcal{A}$ and $P_2AP_1=0$ for any $A\in\mathcal{A}$. So $A=P_1AP_1+P_1AP_2+P_2AP_2$. Let $\mathcal{A}_{11}=P_1\mathcal{A}P_1$, $\mathcal{A}_{12}=P_1\mathcal{A}P_2$, $\mathcal{A}_{22}=P_2\mathcal{A}P_2$. Then $\mathcal{A}=\mathcal{A}_{11}\oplus\mathcal{A}_{12}\oplus\mathcal{A}_{22}$ is the Pierce decomposition of $\mathcal{A}$. Let ϕ be an additive mapping on $\mathcal{A}$ such that

$$(m+n+k+l)\phi(A^2)-(m\phi(A)A+nA\phi(A)+k\phi(I)A^2+lA^2\phi(I))\in\mathbb{F}I,$$

that is, for any $A\in\mathcal{A}$, there exists $\mu_A\in\mathbb{F}$, depending on A, such that

$$(m+n+k+l)\phi(A^2)=m\phi(A)A+nA\phi(A)+k\phi(I)A^2+lA^2\phi(I)+\mu_A I,$$

where $m>0,n>0,k\geq 0,l\geq 0$.

Lemma 3.1. (1) *If $A_{ij}\in\mathcal{A}_{ij}$, then $\phi(A_{ij})\in\mathcal{A}_{ij}$, where $1\leq i\leq j\leq 2$;*
(2) $\phi(A_{12})=A_{12}\phi(I)=\phi(I)A_{12}=A_{12}\phi(P_2)=\phi(P_1)A_{12}$.

Proof: By (2.30),

$$\phi(A_{ii})=\phi(P_iA_{ii}P_i)=P_i\phi(P_iA_{ii}P_i)P_i\in\mathcal{A}_{ii}.(i=1,2)$$

Let $A_{12}=PAP^{\perp}$. Since $A_{12}=P-(P-PAP^{\perp})$ is the difference of two idempotents, it follows from Lemma 2.3 that

$$\phi(A_{12})=\phi(I)A_{12}=\phi(I)P_1A_{12}=\phi(P_1)A_{12}\in\mathcal{A}_{12},$$

$$\phi(A_{12})=A_{12}\phi(I)=A_{12}P_2\phi(I)=A_{12}\phi(P_2)\in\mathcal{A}_{12}.$$

□

Lemma 3.2. *For any $A\in\mathcal{A}$, $B\in\mathcal{A}$, $A_{ij}\in\mathcal{A}_{ij}$, $B_{ij}\in\mathcal{A}_{ij}$ $(1\leq i\leq j\leq 2)$,*
(1) $\phi(A_{11}B_{12})=\phi(A_{11})B_{12}=A_{11}\phi(B_{12})$.
(2) $\phi(A_{12}B_{22})=\phi(A_{12})B_{22}=A_{12}\phi(B_{22})$.
(3) $\phi(AB_{12})=\phi(A)B_{12}=A\phi(B_{12})=\phi(I)AB_{12}=AB_{12}\phi(I)=A\phi(I)B_{12}$.
(4) $\phi(A_{12}B)=\phi(A_{12})B=A_{12}\phi(B)=\phi(I)A_{12}B=A_{12}B\phi(I)=A_{12}\phi(I)B$.

Proof: (1) By (2.2),

$$\begin{aligned}&(m+n+k+l)\phi(A_{11}B_{12}+B_{12}A_{11})=m\phi(A_{11})B_{12}+nA_{11}\phi(B_{12})\\&\quad+m\phi(B_{12})A_{11}+nB_{12}\phi(A_{11})+k\phi(I)A_{11}B_{12}+k\phi(I)B_{12}A_{11}\\&\quad+lA_{11}B_{12}\phi(I)+lB_{12}A_{11}\phi(I)+(\mu_{A_{11}+B_{12}}-\mu_{A_{11}}-\mu_{B_{12}})I.\end{aligned}\tag{3.1}$$

Since $B_{12}A_{11}=0$, $\phi(B_{12})A_{11}\in\mathcal{A}_{12}A_{11}=0$ and $B_{12}\phi(A_{11})\in B_{12}\mathcal{A}_{11}=0$, it follows that

$$\begin{aligned}&(m+n+k+l)\phi(A_{11}B_{12})=m\phi(A_{11})B_{12}+nA_{11}\phi(B_{12})\\&\quad+k\phi(I)A_{11}B_{12}+lA_{11}B_{12}\phi(I)+(\mu_{A_{11}+B_{12}}-\mu_{A_{11}}-\mu_{B_{12}})I.\end{aligned}\tag{3.2}$$

Multiplying (3.2) by P_1 from the right sides, using the fact that $\phi(B_{12}), \phi(A_{11}B_{12}) \in \mathcal{A}_{12}$. yields $(\mu_{A_{11}+B_{12}} - \mu_{A_{11}} - \mu_{B_{12}})P_1 = 0$ and $\mu_{A_{11}+B_{12}} - \mu_{A_{11}} - \mu_{B_{12}} = 0$. Since $B_{12} \in \mathcal{A}_{12}$, it follows from Lemma 3.1(2) that

$$\phi(B_{12}) = \phi(I)B_{12} = B_{12}\phi(I). \tag{3.3}$$

Since $A_{11}B_{12} \in \mathcal{A}_{12}$,

$$\phi(A_{11}B_{12}) = \phi(I)A_{11}B_{12} = A_{11}B_{12}\phi(I) = A_{11}\phi(B_{12}), \tag{3.4}$$

Combining it with (3.2), we have that $m\phi(A_{11}B_{12}) = m\phi(A_{11})B_{12}$. Since $m \neq 0$,

$$\phi(A_{11})B_{12} = A_{11}B_{12}\phi(I) = \phi(A_{11}B_{12}) = A_{11}\phi(B_{12}) = \phi(I)A_{11}B_{12}. \tag{3.5}$$

(2) By (2.1),

$$\begin{aligned}(m+n+k+l)\phi(A_{12}B_{22} + B_{22}A_{12}) &= m\phi(A_{12})B_{22} + nA_{12}\phi(B_{22}) \\ +m\phi(B_{22})A_{12} + nB_{22}\phi(A_{12}) &+ k\phi(I)A_{12}B_{22} + k\phi(I)B_{22}A_{12} \\ +lA_{12}B_{22}\phi(I) + lB_{22}A_{12}\phi(I) &+ (\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}})I,\end{aligned}$$

Using the fact that $\phi(B_{22})A_{12}, B_{22}\phi(A_{12}) \in \mathcal{A}_{22}\mathcal{A}_{12} = 0$, yields that

$$\begin{aligned}(m+n+k+l)\phi(A_{12}B_{22}) &= m\phi(A_{12})B_{22} + nA_{12}\phi(B_{22}) \\ +k\phi(I)A_{12}B_{22} &+ lA_{12}B_{22}\phi(I) + (\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}})I.\end{aligned} \tag{3.6}$$

Multiplying (3.6) by P_1 from the right side, we have that $(\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}})P_1 = 0$ and $\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}} = 0$. It follows from Lemma 3.1(2) that $\phi(A_{12}) = \phi(I)A_{12} = A_{12}\phi(I)$ and

$$\phi(A_{12}B_{22}) = A_{12}B_{22}\phi(I) = \phi(I)A_{12}B_{22} = \phi(A_{12})B_{22}.$$

Combining it with (3.6), we have that $n\phi(A_{12}B_{22}) = nA_{12}\phi(B_{22})$. Since $n \neq 0$,

$$\phi(A_{12}B_{22}) = \phi(A_{12})B_{22} = A_{12}\phi(B_{22}) = \phi(I)A_{12}B_{22} = A_{12}B_{22}\phi(I). \tag{3.7}$$

(3) Let $B_{12} = PBP^{\perp}$. Then $AB_{12} = PAPBP^{\perp} \in \mathcal{A}_{12}$. It follows from Lemma 3.1(2) that $\phi(AB_{12}) = \phi(I)AB_{12} = AB_{12}\phi(I) = A\phi(B_{12})$. It follows from (1) that

$$\phi(AB_{12}) = \phi(PAPPBP^{\perp}) = \phi(PAP)PBP^{\perp}.$$

By Lemma 3.1, $\phi(PBP^{\perp}) \in \mathcal{A}_{12}$, $\phi(PAP) \in \mathcal{A}_{11}$, $\phi(PAP^{\perp}) \in \mathcal{A}_{12}$, and $\phi(P^{\perp}AP^{\perp}) \in \mathcal{A}_{22}$. Therefore,

$$\begin{aligned}\phi(A)B_{12} = \phi(A)PBP^{\perp} &= \phi(PAP)PBP^{\perp} + \phi(PAP^{\perp})PBP^{\perp} + \phi(P^{\perp}AP^{\perp})PBP^{\perp} \\ &= \phi(PAP)PBP^{\perp} = \phi(APBP^{\perp}) = \phi(AB_{12})\end{aligned}$$

It follows that

$$\phi(AB_{12}) = \phi(A)B_{12} = A\phi(B_{12}) = \phi(I)AB_{12} = AB_{12}\phi(I) \tag{3.8}$$

for any $A, B \in \mathcal{A}$.
(4) The proof is similar to the proof of (3). □

Lemma 3.3. *For any $A, B \in \mathcal{A}$,*
(1) $(\phi(AB) - A\phi(B))Q_1(H) = 0$, $(\phi(AB) - \phi(A)B)Q_1(H) = 0$.
(2) $Q_2(H)(\phi(AB) - A\phi(B)) = 0$, $Q_2(H)(\phi(AB) - \phi(A)B) = 0$.

Proof: (1) Let $T \in \mathcal{A}$, $P \in \mathcal{L}$. It follows from Lemma 3.2 that

$$\begin{aligned}\phi(AB)PTP^{\perp} &= \phi(APBPTP^{\perp}) = \phi(A)PBPTP^{\perp} = \phi(A)BPTP^{\perp} \\ &= A\phi(PBPTP^{\perp}) = A\phi(BPTP^{\perp}) = A\phi(B)PTP^{\perp}\end{aligned} \tag{3.9}$$

for any A, $B \in \mathcal{A}$. So that $(\phi(AB) - A\phi(B))PTP^{\perp} = 0$ and $(\phi(AB) - \phi(A)B)PTP^{\perp} = 0$. It follows that

$$(\phi(AB) - A\phi(B))Q_1(H) = 0, \quad (\phi(AB) - \phi(A)B)Q_1(H) = 0. \tag{3.10}$$

(2) Similarly, for any $T \in \mathcal{A}$, $P \in \mathcal{L}$,

$$\begin{aligned}PTP^{\perp}\phi(AB) &= \phi(PTP^{\perp}AB) = \phi(PTP^{\perp}AP^{\perp}B) = PTP^{\perp}AP^{\perp}\phi(B) \\ &= PTP^{\perp}A\phi(B) = \phi(PTP^{\perp}A)B = PTP^{\perp}\phi(A)B.\end{aligned} \tag{3.11}$$

Thus $PTP^{\perp}(\phi(AB) - A\phi(B)) = 0$ and $PTP^{\perp}(\phi(AB) - \phi(A)B) = 0$. Thus

$$Q_2(H)(\phi(AB) - A\phi(B)) = 0, \quad Q_2(H)(\phi(AB) - \phi(A)B) = 0. \tag{3.12}$$

□

Lemma 3.4. *If $Q_1(H) \vee Q_2(H) = I$, then ϕ is a centralizer, that is, $\phi(A) = A\phi(I) = \phi(I)A$ for any $A \in \mathcal{A}$.*

Proof: Let $Q_1 = Q_1(H)$, $Q_2 = Q_2(H)$ for simplicity. By lemma 3.2(3), for any $A \in \mathcal{A}$, $T \in \mathcal{A}$, $P \in \mathcal{L}$,

$$\phi(APTP^{\perp}) = \phi(I)APTP^{\perp} = \phi(A)PTP^{\perp} = APTP^{\perp}\phi(I) = A\phi(I)PTP^{\perp}.$$

It follows that $\phi(I)AQ_1 = \phi(A)Q_1 = A\phi(I)Q_1$. Since $Q_1 \in \mathcal{A}$ is an idempotent, we have that $A\phi(I)Q_1 = AQ_1\phi(I)$ and $\phi(I)AQ_1 = AQ_1\phi(I)$. It follows from Lemma 2.2 that $\phi(AQ_1) = \phi(I)AQ_1 = AQ_1\phi(I)$ for any $A \in \mathcal{A}$. And

$$\phi(Q_1AQ_1) = \phi(I)Q_1AQ_1 = Q_1AQ_1\phi(I). \tag{3.13}$$

If $Q_1(H) \vee Q_2(H) = I$, then $Q_1^{\perp}Q_2 = Q_1^{\perp}$ and

$$Q_1^{\perp}(\phi(AB) - \phi(A)B) = Q_1^{\perp}Q_2(\phi(AB) - \phi(A)B) = 0$$

and

$$Q_1^{\perp}(\phi(AB) - A\phi(B)) = Q_1^{\perp}Q_2(\phi(AB) - A(\phi(B))) = 0$$

for any $A, B \in \mathcal{A}$. In particular, $Q_1^{\perp}\phi(A) = Q_1^{\perp}\phi(AI) = Q_1^{\perp}A\phi(I)$, $Q_1^{\perp}\phi(A) = Q_1^{\perp}\phi(I)A = \phi(I)Q_1^{\perp}A$, so $Q_1^{\perp}A\phi(I) = \phi(I)Q_1^{\perp}A$. By Lemma 2.2, $\phi(Q_1^{\perp}A) = Q_1^{\perp}A\phi(I) = \phi(I)Q_1^{\perp}A$ and

$$\phi(Q_1^{\perp}AQ_1^{\perp}) = Q_1^{\perp}AQ_1^{\perp}\phi(I) = \phi(I)Q_1^{\perp}AQ_1^{\perp}. \tag{3.14}$$

Since $Q_1AQ_1^\perp = Q_1-(Q_1-Q_1AQ_1^\perp)$ is the difference of two idempotents, it follows from Lemma 2.3 that

$$\phi(Q_1AQ_1^\perp) = Q_1AQ_1^\perp\phi(I) = \phi(I)Q_1AQ_1^\perp. \tag{3.15}$$

By (3.13), (3.14) and (3.15),

$$\begin{aligned}\phi(A) &= \phi(Q_1AQ_1 + Q_1AQ_1^\perp + Q_1^\perp AQ_1^\perp)\\ &= \phi(Q_1AQ_1) + \phi(Q_1AQ_1^\perp) + \phi(Q_1^\perp AQ_1^\perp)\\ &= Q_1AQ_1\phi(I) + Q_1AQ_1^\perp\phi(I) + Q_1^\perp AQ_1^\perp\phi(I)\\ &= \phi(I)Q_1AQ_1 + \phi(I)Q_1AQ_1^\perp + \phi(I)Q_1^\perp AQ_1^\perp\\ &= A\phi(I) = \phi(I)A.\end{aligned}$$

□

Lemma 3.5. *Let $\mathcal{A}_1$ be a von Neumann algebra and $\phi : \mathcal{A}_1 \to \mathcal{A}_1$ a continuous mapping such that*

$$(m+n+k+l)\phi(A^2) - (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathbb{F}I$$

for any $A \in \mathcal{A}_1$,where $m, n, k, l \geq 0$ with $mn \neq 0$. Then ϕ is a centralizer, that is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in \mathcal{A}_1$.

Proof: Since a von Neumann algebra is the norm-closure of the subalgebra generated by the idempotents in it, the result follows from lemma 2.3. □

Proof of Theorem 3.1 By Proposition 3.1(1), we have that $\mathcal{A} = Q\mathcal{A}Q \oplus Q^\perp\mathcal{A}Q^\perp$. Let ϕ_1, ϕ_2 be the restriction of ϕ on $Q\mathcal{A}Q$, $Q^\perp\mathcal{A}Q^\perp$ respectively. By Lemma 2.5, we have that $\phi(Q\mathcal{A}Q) = Q\phi(Q\mathcal{A}Q)Q$ and $\phi(Q^\perp\mathcal{A}Q^\perp) = Q^\perp\phi(Q^\perp\mathcal{A}Q^\perp)Q^\perp$. So that ϕ_1 is an additive mapping from $Q\mathcal{A}Q$ to itself, and ϕ_2 is an additive mapping from $Q^\perp\mathcal{A}Q^\perp$ to $Q^\perp\mathcal{A}Q^\perp$. Since $QA^2Q = QAQQAQ$ and $Q^\perp A^2Q^\perp = Q^\perp AQ^\perp Q^\perp AQ^\perp$, ϕ_1, ϕ_2 both satisfy the equality: $(m+n+k+l)\phi_i(A_i^2) - (m\phi_i(A_i)A_i + nA_i\phi_i(A_i) + k\phi_i(I_i)A_i^2 + lA_i^2\phi_i(I_i)) \in \mathbb{F}I (i = 1, 2)$ for any $A_1 \in Q\mathcal{A}Q$ and $A_2 \in Q^\perp\mathcal{A}Q^\perp$, where $I_1 = Q$ is the identity element of $Q\mathcal{A}Q$ and $I_2 = Q^\perp$ is the identity element of $Q^\perp\mathcal{A}Q^\perp$. Since

$$Q\mathcal{A}Q = \{T \in Q\mathcal{N}Q : (Q-QP)TQP = 0 \text{ for any } P \in \mathcal{L}\} = Q\mathcal{N}Q \cap Alg(Q\mathcal{L}),$$

we have that $Q\mathcal{A}Q$ is a CSL subalgebra of the von Neumann algebra $Q\mathcal{N}Q$. For any $P \in \mathcal{L}$, $A \in \mathcal{A}$ and $x \in H$, we have that $QAQ^\perp = 0$ and

$$PAP^\perp x = QPAP^\perp x = PQAP^\perp x = PQA(Q-QP)x = QPA(Q-QP)x.$$

Since

$$Q_1(H) = \overline{span}\left\{PAP^\perp x : \quad P \in \mathcal{L}, \quad A \in \mathcal{A}, \quad x \in H\right\}$$

and

$$Q_1(QH) = \overline{span}\left\{QPA(Q-QP)x : \quad P \in \mathcal{L}, \quad A \in \mathcal{A}, \quad x \in H\right\},$$

we have that $Q_1(H) = Q_1(QH)$ and $Q_2(H) = Q_2(QH)$. It follows that $Q_1(QH) \vee Q_2(QH) = Q$ is the identity element of $Q\mathcal{A}Q$. All the conditions for Lemma 3.4 are satisfied, so we have that ϕ_1 is a centralizer on $Q\mathcal{A}Q$.

Since ϕ_2 is a continuous mapping on the von Neumann algebra $Q^\perp\mathcal{A}Q^\perp$ such that $(m+n+k+l)\phi_2(A^2) - (m\phi_2(A)A + nA\phi_2(A) + k\phi_2(I_2)A^2 + lA^2\phi_2(I_2)) \in \mathbb{F}I$ for any $A \in Q^\perp\mathcal{A}Q^\perp$, ϕ_2 is a centralizer by Lemma 3.5. It follows that ϕ is a centralizer, that is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in \mathcal{A}$. □

Theorem 3.2. *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space H, and $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$. And let $\mathcal{A} = \mathcal{N} \cap Alg\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. If ϕ is an additive mapping on $\mathcal{A}$ such that*

$$(m+n+k+l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)$$

for any $A \in \mathcal{A}$, where $m, n, k, l \geq 0$ with $mn \neq 0$, then ϕ is a centralizer. That is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in \mathcal{A}$.

In order to prove Theorem 3.2, we need a Lemma.

Lemma 3.6. *Let $\mathcal{A}$ be a unital C^*-algebra with the unity element I. If $\phi : \mathcal{A} \to \mathcal{A}$ is an additive mapping on $\mathcal{A}$ such that $(m+n+k+l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)$ for any $A \in \mathcal{A}$, where $m, n, k, l \geq 0$ with $mn \neq 0$, then ϕ is a centralizer. That is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in \mathcal{A}$.*

Proof: By the condition of the Lemma,

$$(m+n+k+l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I) \tag{3.16}$$

for any $A \in \mathcal{A}$. Putting $A + I$ for A in (3.16), we have that

$$(m+n+2k+2l)\phi(A) = (m+2k)\phi(I)A + (n+2l)A\phi(I). \tag{3.17}$$

By (3.16),

$$\begin{aligned}&(m+n+k+l)(m+n+2k+2l)\phi(A^2)\\ &\quad = m(m+n+2k+2l)\phi(A)A + nA(m+n+2k+2l)\phi(A)\\ &\quad +k(m+n+2k+2l)\phi(I)A^2 + l(m+n+2k+2l)A^2\phi(I).\end{aligned} \tag{3.18}$$

By (3.17) and (3.18),

$$\begin{aligned}&(m+n+k+l)(m+n+2k+2l)\phi(A^2)\\ &= m((m+2k)\phi(I)A+(n+2l)A\phi(I))A+nA((m+2k)\phi(I)A+(n+2l)A\phi(I))\\ &+k(m+n+2k+2l)\phi(I)A^2 + l(m+n+2k+2l)A^2\phi(I)\\ &= (k(m+n+k+l) + m(m+2k))\phi(I)A^2 + (l(m+n+2k+2l)\\ &\quad +n(n+2l))A^2\phi(I) + (m(n+2l) + n(m+2k))A\phi(I)A.\end{aligned} \tag{3.19}$$

On the other hand, putting A^2 for A in (3.17), we have that

$$\begin{aligned}&(m+n+k+l)(m+n+2k+2l)\phi(A^2)\\&=(m+n+k+l)(m+2k)\phi(I)A^2+(m+n+k+l)(n+2l)A^2\phi(I).\end{aligned}\tag{3.20}$$

Comparing (3.19) with (3.20), we have that $(mn+ml+nk)\phi(I)A^2+(mn+ml+nk)A^2\phi(I)=2(mn+ml+nk)A\phi(I)A$. Since $(mn+ml+nk)\neq 0$, $\phi(I)A^2+A^2\phi(I)=2A\phi(I)A$, that is, $[[\phi(I),A],A]=0$. Then we have that $\phi(I)A=A\phi(I)$.

Indeed, let $\Delta(A)=[\phi(I),A]$ $(A\in\mathcal{A})$, where $[A,B]=AB-BA$ is the commutator. Then Δ is an inner derivation on $\mathcal{A}$, and $[\Delta(A),A]=0$ for any $A\in\mathcal{A}$. In particular, $[\Delta(A+B),A+B]=0$ for any $A,B\in\mathcal{A}$. It follows that $[\Delta(A),B]+[\Delta(B),A]=0$. In the identity, putting $\phi(I)$ for B, we get that $[\Delta(A),\phi(I)]=0$, that is, $\Delta^2(A)=0$ for any $A\in\mathcal{A}$. For any $A,B\in\mathcal{A}$, $\Delta^2(AB)=\Delta^2(A)B+2\Delta(A)\Delta(B)+A\Delta^2(B)$ and $\Delta^2(AB)=\Delta^2(A)=\Delta^2(B)=0$. So we have that $\Delta(A)\Delta(B)=0$ for any $A,B\in\mathcal{A}$. Thus $\Delta(A)\Delta(DA)=0$ for any $D\in\mathcal{A}$, that is, $\Delta(A)\Delta(D)A+\Delta(A)D\Delta(A)=0$. So that $\Delta(A)D\Delta(A)=0$. Since D is arbitrary, we have that $\Delta(A)\mathcal{A}\Delta(A)=0$. By the truth that every unital C^*-algebra is a semi-prime ring, we have that $\Delta=0$, that is, $\phi(I)A=A\phi(I)$ for any $A\in\mathcal{A}$. By (3.17), $\phi(A)=\phi(I)A=A\phi(I)$ for any $A\in\mathcal{A}$. □

Proof of Theorem 3.2 By Propositon 3.1(1), $\mathcal{A}=Q\mathcal{A}Q\oplus Q^\perp\mathcal{A}Q^\perp$. Let ϕ_1,ϕ_2 be the restrictions of ϕ on $Q\mathcal{A}Q$, $Q^\perp\mathcal{A}Q^\perp$ respectively. By Lemma 2.5, $\phi(QAQ)=Q\phi(QAQ)Q$ and $\phi(Q^\perp AQ^\perp)=Q^\perp\phi(Q^\perp AQ^\perp)Q^\perp$. So ϕ_1 is an additive mapping from $Q\mathcal{A}Q$ to $Q\mathcal{A}Q$, and ϕ_2 is an additive mapping from $Q^\perp\mathcal{A}Q^\perp$ to $Q^\perp\mathcal{A}Q^\perp$. Since $QA^2Q=QAQQAQ$ and $Q^\perp A^2Q^\perp=Q^\perp AQ^\perp Q^\perp AQ^\perp$, ϕ_1,ϕ_2 both satisfy the equality: $(m+n+k+l)\phi_i(A_i^2)=m\phi_i(A_i)A_i+nA_i\phi_i(A_i)+k\phi_i(I_i)A_i^2+lA_i^2\phi_i(I_i)$ for any $A_1\in Q\mathcal{A}Q$ and $A_2\in Q^\perp\mathcal{A}Q^\perp$, where $I_1=Q$ is the identity element of $Q\mathcal{A}Q$ and $I_2=Q^\perp$ is the identity element of $Q^\perp\mathcal{A}Q^\perp$. Similar to the proof of Theorem 3.1, ϕ_1 is a centralizer on $Q\mathcal{A}Q$.

Since ϕ_2 is an additive mapping on the von Neumann algebra $Q^\perp\mathcal{A}Q^\perp$ such that $(m+n+k+l)\phi_2(A^2)=m\phi_2(A)A+nA\phi_2(A)+k\phi_2(I_2)A^2+lA^2\phi_2(I_2)$ for any $A\in Q^\perp\mathcal{A}Q^\perp$, it follows from Lemma 3.6 that ϕ_2 is a centralizer. Therefore, ϕ is a centralizer, that is, $\phi(A)=\phi(I)A=A\phi(I)$ for any $A\in\mathcal{A}$. □

Corollary 3.1. *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space H, and $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$, and $\mathcal{A}=\mathcal{N}\cap Alg\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. If $\phi:\mathcal{A}\to\mathcal{A}$ is an additive mapping on $\mathcal{A}$ such that*

$$(m+n)\phi(A^2)=m\phi(A)A+nA\phi(A)$$

for any $A\in\mathcal{A}$, where $m,n>0$, then ϕ is a centralizer. That is, $\phi(A)=\phi(I)A=A\phi(I)$ for any $A\in\mathcal{A}$.

The following theorems characterize the generalized Jordan centralizer. Zhang etc. ([22]) have proved them for the nest algebra. They are also true for the CSL subalgebra of a von Neumann algebra.

Theorem 3.3. *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space H, and $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$, and $\mathcal{A} = \mathcal{N} \cap Alg\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. If $\phi : \mathcal{A} \to \mathcal{A}$ is an additive mapping $\mathcal{A}$ such that*

$$(m+n)\phi(A^{p+1}) = m\phi(A)A^p + nA^p\phi(A)$$

for any $A \in \mathcal{A}$, where $m, n > 0$, $p \in \mathbb{Z}^+$, then ϕ is a centralizer. That is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in \mathcal{A}$.

Theorem 3.4. *Let $\mathcal{N}$ be a von Neumann algebra on a Hilbert space H, and $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$, and $\mathcal{A} = \mathcal{N} \cap Alg\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. If $\phi : \mathcal{A} \to \mathcal{A}$ is an additive mapping such that*

$$\phi(A^{m+n+1}) = A^m\phi(A)A^n$$

for any $A \in \mathcal{A}$, where m, n are two positive integers, then ϕ is a centralizer. That is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in \mathcal{A}$.

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References

1. D. Benkovic and D. Eremita, *Characterizing left centralizers by their action on polynomial*. Publl. Math. Debrecen.. 64, 343–351,(2004).
2. M. Bresar and B. Zalar, *On the structure of Jordan -derivations*. Colloquium Math., 63 163–171,(1992).
3. Q. Chen and X. Fang, *Spatiality of Derivations of Operator Algebras in Banach Spaces*. Abstract and Applied Analysis, Volume 2011, Article ID 813723, 13 pages, doi:10.1155/2011/813723,(2011).
4. Q. Chen and X. Fang, *Strictly cyclic functionals, reflexivity and hereditary reflexivity of operator algebras*. Abstract and Applied Analysis, Volume 2012, Article ID 434308, 12 pages, doi:10.1155/2012/434308,(2012).
5. W. S. Cheung, *Mappings on triangular algebras*. PhD Dissertation, U. Victoria, (2000).
6. W. S. Cheung,*Commuting maps of triangular algebras*. J. London Math. Soc., 63, 117-127,(2001).
7. X. Fang, *The classification of certain non-simple C^*-algebras of tracial rank zero*. J. Funct. Anal., 256, 3861–3891,(2009).
8. X. Fang, *The real rank zero property of crossed product*. Proc. Amer. Math. Soc., 134, 3015–3024,(2006).
9. X. Fang, *Graph C^*-algebras and their ideals defined by Cuntz-Krieger family of possibly row-infinite directed graphs*. Integral Equations and Operator Theory, 54, 301–316,(2006).
10. X. Fang, *Amenability of group in the C^*-dynamic system*. Chinese Sci. Bull., 37, 1150–1152,(1992).

11. X. Fang. *The Arveson spectrum of coaction*. Chinese J. of contemp Math., 15, 49–58,(1994).

12. P. Li, D. Han and W. Tang. *Centralizers and Jordan derivations for CSL subalgebras of von Neumann algebras*. J. Oper. Theory. 69 , 117-133, (2013).

13. F. Lu, *The Jordan structure of CSL algebras*. Studia Math., 190(3), 283–299,(2009).

14. X. Qi, S. Du, J. Hou, *Characterazation of centralizers*. Acta. Math. Sinica, Series A, 51, 509–516, (2008) (in Chinese).

15. J. Vukman, *An identity related to centralizers in semiprime rings*. Comment. Math. Univ. Carolin., 40, 447–456,(1999).

16. J. Vukman, *On* $(m, n)-$*Jordan centralizers in rings and algebras*. Glasnik Mateematicki, 45, 43–53,(2010).

17. J. Vukman, I. Kosi-Ulbl, *Centralizers on rings and algebras*. Bull. Austral. Math. Soc., 71, 225–234,(2005).

18. J.Vukman, *An identity related to centralizers in semiprime rings*, Math. Univ. Carol., 40, 447–456,(1999).

19. J.Vukman, *Centralizers on semiprime rings*, Math. Univ. Carol., 42, 237–245,(2001).

20. J. Vukman, I. Kosi-Ulbl and D. Eremita, *On certain equations in rings*, Bull. Austral. Math. Soc., 71, 53-60,(2005).

21. Y. Wang, *Local generalized derivations in prime rings with idempotents*. Algebra Colloquium, 17, 295–300,(2010).

22. C. Yang, J. Zhang, *Generalized Jordan centralizers on nest algebras*. Acta. Math. Sinica, Series A, 53, 975–980,(2010) (in Chinese).

23. B. Zalar, *On centralizers of semiprime rings*. Comment. Math. Univ. Carolin., 32, 609–614,(1991).

Quanyuan Chen
Department of Mathematics,
Jingdezhen Ceramic Institute,
China
E-mail address: `cqy0798@163.com`

and

Xiaochun Fang
Department of Mathematics,
Tongji University,
China
E-mail address: `xfang@tongji.edu.cn`

and

Changjing Li
Department of Mathematics,
Shandong Normal University,
China
E-mail address: `lcjbxh@163.com`