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Existence of Entropy solutions for some nonlinear elliptic problems involving variable exponent and measure data

Taghi Ahmedatt, Elhoussine Azroul, Hassane Hjiaj and Abdelfattah Touzani

ABSTRACT: In this paper, we examine the existence of entropy solutions for some nonlinear $p(x)-$elliptic equation of the type:

$$Au - \text{div}\ \phi(u) + H(x,u,\nabla u) = \mu,$$

where A is an operator of Leray-Lions type acting from $W_0^{1,p(x)}(\Omega)$ into its dual. The strongly nonlinear term H is assumed only to satisfy some nonstandard growth condition with respect to $|\nabla u|$. We assume that $\phi(\cdot) \in C^0(I\!R, I\!R^N)$ and μ belongs to $\mathcal{M}_0^b(\Omega)$.

Key Words: Sobolev spaces with variable exponents, nonlinear elliptic problem, entropy solutions, renormalized solutions.

Contents

1. Introduction

Let Ω be a bounded open subset of $I\!R^N, (N > 2)$. Boccardo has studied in [13] the existence of entropy solutions for the problem:

$$\begin{cases} -\text{div}\ a(x,u,\nabla u) = f - \text{div}\ \phi(u) & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega)$ and $\phi(\cdot) \in (\mathcal{C}^0(I\!R, I\!R^N)$, no growth hypothesis is assumed on $\phi(\cdot)$, hence the term div $\phi(u)$ may be meaningless, even as a distribution for $u \in W_0^{1,r}(\Omega)$, $r > 1$. He proved the existences of solutions $T_k(u) \in W_0^{1,p}(\Omega)$ such that $u \in W_0^{1,q}(\Omega)$ for $1 \le q < \overline{p} = \frac{N(p-1)}{N-1}$.

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Boccardo, Giachetti, Diaz and Murat have studied in [14] the following nonlinear elliptic problem

$$\begin{cases} -\text{div } a(x,u,\nabla u) - \text{div } \phi(u) + g(x,u) = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $g(x,\cdot)$ satisfying some growth and sign conditions and $f \in W^{-1,p'}(\Omega)$. they proved the existence of renormalized solutions and some regularity results, (see also. [6,12,16]).

Aharouch and Bennouna [3] have studied the quasilinear unilateral problems of the type:

$$\begin{cases} -\text{div } (a(x,\nabla u)) = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of solutions in the framework of Orlicz spaces $W_0^1 L_M(\Omega)$ without any restriction on the $N-$function M of the Orlicz spaces, (see also [10,11]).

In [20], Porretta has proved the existence of solutions for the strongly nonlinear elliptic problem:

$$\begin{cases} -\text{div } a(x,u,\nabla u) + H(x,u,\nabla u) = \mu \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

with $H(x,u,\nabla u)$ satisfying some growth condition without any sign condition, and the data μ is a nonnegative bounded Radon measure on Ω.

Recently, variable exponent Sobolev spaces have attracted an increasing attention of many researchers, the impulse for this mainly comes from their applications in various fields, as in image processing (underline the borders and eliminate the noise) and electro-rheological fluids (see for example: [15,21]). In [9], Bendahmane and Wittbold have proved the existence and uniqueness of a renormalized solution to the nonlinear elliptic equation:

$$\begin{cases} -\text{div } \left(|\nabla u|^{p(x)-2}\nabla u\right) = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where the right-hand side $f \in L^1(\Omega)$, the exponent $p(\cdot) : \overline{\Omega} \mapsto (1,+\infty)$ is continuous. Sanchón and Urbano have studied in [22] the quasilinear elliptic problem:

$$\begin{cases} -\text{div}(a(x,\nabla u)\,) \;=\; f & \text{in} \quad \Omega, \\ u \;=\; 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$. They have proved the existence and uniqueness of renormalized solution. Moreover, they have shown some regularity results. For other interesting existence and regularity results, we refer to [1,2,4].

In this paper, we consider the strongly nonlinear $p(x)-$elliptic problem:

$$\begin{cases} Au - \text{div } \phi(u) + H(x,u,\nabla u) = f - \text{div } F \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

where $\phi(\cdot) \in (\mathcal{C}^0(\mathbb{R}))^N$, the right-hand side is assumed to satisfy $f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$.

This paper is organized as follows: We introduce in the section 2 some assumptions on $a(x,s,\xi)$ and $H(x,s,\xi)$ and some important lemmas useful to prove our main result. The section 3 will be devoted to show the existence of entropy solutions for our problem (1.6). In the section 4, we prove that entropy solutions are also renormalized solutions, and we give an example.

2. Essential assumptions and some technical Lemmas

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|log|x-y||} \qquad \forall x, y \in \overline{\Omega} \quad \text{such that} \quad |x-y| < \frac{1}{2},$$

with possible different constant C. We denote

$$C_+(\overline{\Omega}) = \{\text{log-Hölder continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R} \quad \text{such that}$$

$$1 < p_- \leq p_+ < N\},$$

where

$$p_- = \min\{p(x) \ / \ x \in \overline{\Omega}\} \qquad \text{and} \qquad p_+ = \max\{p(x) \ / \ x \in \overline{\Omega}\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ for which the convex modular:

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite, then

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$ called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable and reflexive Banach space, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. The generalized Hölder type inequality:

$$\left| \int_\Omega uv \;\; dx \right| \leq (\frac{1}{p_-} + \frac{1}{(p')_-}) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by:

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the following norm:

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We define $W_0^{1,p(x)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [5,9] and [17].

Remark 2.1. *Recall that the definition of these spaces requires only the measurability of $p(\cdot)$, while the Poincaré and the Sobolev-Poincaré inequality are proved for $p(\cdot)$ log-Hölder continuous, (see [17,19]).*

Let $p(\cdot) \in C_+(\overline{\Omega})$. We consider a Leray-Lions operator A from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(\cdot)}(\Omega)$, defined by the formula

$$Au = -\text{div }a(x,u,\nabla u), \tag{2.2}$$

where $a : \Omega \times I\!R \times I\!R^N \mapsto I\!R$ is a *Carathéodory* function satisfying the following conditions:

$$|a(x,s,\xi)| \leq \beta\,(K(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{2.3}$$

$$a(x,s,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)}, \tag{2.4}$$

$$\big(a(x,s,\xi) - a(x,s,\overline{\xi})\big)\cdot(\xi - \overline{\xi}) > 0 \quad \text{for all} \quad \xi \neq \overline{\xi} \quad \text{in} \quad I\!R^N, \tag{2.5}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in I\!R \times I\!R^N$, where $K(x)$ is a nonnegative function lying in $L^{p'(\cdot)}(\Omega)$ and $\alpha, \beta > 0$.
The nonlinear term $H(x,s,\xi)$ is *Carathéodory* function which satisfies:

$$|H(x,s,\xi)| \leq f_0(x) + b(|s|)|\xi|^{p(x)}, \tag{2.6}$$

with $b(\cdot) : I\!R \mapsto I\!R^+$ is a continuous positive function that belongs to $L^1(I\!R) \cap L^\infty(I\!R)$, while $f_0(\cdot) \in L^1(\Omega)$.
The aim of this paper is to study the existence of solutions for the strongly nonlinear $p(x)-$elliptic problem:

$$\begin{cases} Au - \text{div }\phi(u) + H(x,u,\nabla u) = f - \text{div }F & \text{in } \Omega, \\ u = \ 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where $f \in L^1(\Omega)$, $\phi(\cdot) \in C^0(I\!R, I\!R^N)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$.

In order to prove our main result, we will require the following lemmas :

Lemma 2.2. *(see [7]). Let $g \in L^{p(\cdot)}(\Omega)$ and $g_n \in L^{p(\cdot)}(\Omega)$ with $\|g_n\|_{p(\cdot)} \leq C$ for $1 < p(x) < \infty$.*
If $g_n \to g$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{p(\cdot)}(\Omega)$.

Lemma 2.3. *(see [23]). Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 1)$. If $u \in (W_0^{1,p(x)}(\Omega))^N$, then*

$$\int_\Omega div\, u\, dx = 0.$$

Lemma 2.4. *(see [5]) Assuming that (2.3) – (2.5) hold and let $(u_n)_{n\in I\!N}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx \longrightarrow 0, \tag{2.8}$$

then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

3. Main results

Definition 3.1. *A function u is called an entropy solution of the strongly nonlinear $p(x)$-elliptic problem (2.6) if $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$, $H(x,u,\nabla u) \in L^1(\Omega)$ and*

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)\ dx + \int_\Omega \phi(u)\cdot\nabla T_k(u-v)\ dx \\ + \int_\Omega H(x,u,\nabla u)T_k(u-v)\ dx \le \int_\Omega f T_k(u-v)\ dx + \int_\Omega F\cdot\nabla T_k(u-v)\ dx,\end{aligned} \tag{3.1}$$

for every $v \in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$.

Theorem 3.2. *Assuming that (2.3) – (2.6) hold, let $f \in L^1(\Omega)$ and $\phi(\cdot) \in C^0(I\!R, I\!R^N)$, then the problem (2.7) has at least one entropy solution.*

3.1. Proof of Theorem 3.2

Step 1: Approximate problems Let $(f_n)_{n\in I\!N}$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$. We consider the approximate problem

$$\begin{cases} Au_n - \text{div}\ \phi_n(u_n) + H_n(x,u_n,\nabla u_n) = f_n - \text{div}\ F \\ u_n \in W_0^{1,p(x)}(\Omega), \end{cases} \tag{3.2}$$

with $\phi_n(s) = \phi(T_n(s))$ and $H_n(x,s,\xi) = T_n(H(x,s,\xi))$, In view of [5] there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega)$ of the problem (3.2).

Step 2: A priori estimates Let n be large enough ($n \ge k$), we define

$$B(s) = \frac{2}{\alpha}\int_0^s b(|\tau|)\,d\tau.$$

Note that, since the function $b(\cdot)$ is integrable on $I\!R$, then $0 \le B(\infty) := \dfrac{2}{\alpha}\displaystyle\int_0^{+\infty} b(|t|)dt$ is a finite real number and since $b(\cdot) \in L^\infty(I\!R)$, then $T_k(u_n)e^{B(|u_n|)} \in W_0^{1,p(x)}(\Omega)$.

Taking $T_k(u_n)e^{B(|u_n|)}$ as a test function in (3.2), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)e^{B(|u_n|)}\,dx \tag{3.3}$$
$$+\frac{2}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}\,dx$$
$$+\int_\Omega \phi_n(u_n)\cdot\nabla(T_k(u_n)e^{B(|u_n|)})\,dx+\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)e^{B(|u_n|)}\,dx$$
$$=\int_\Omega f_nT_k(u_n)e^{B(|u_n|)}\,dx$$
$$+\int_\Omega F\cdot\nabla(T_k(u_n)e^{B(|u_n|)})\,dx. \tag{3.4}$$

Using (2.6), we have

$$\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)e^{B(|u_n|)}\,dx\le ke^{B(\infty)}\int_\Omega|f_0(x)|\,dx$$
$$+\int_\Omega|\nabla u_n|^{p(x)}b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}\,dx,$$

and using of Young's inequality, we have

$$\begin{aligned}\int_\Omega F\cdot\nabla(T_k(u_n)e^{B(|u_n|)})\,dx &= \int_\Omega F\cdot\nabla T_k(u_n)e^{B(|u_n|)}\,dx\\ &+\int_\Omega F\cdot\nabla u_n b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}\,dx\\ &\le C_1\int_\Omega |F|^{p'(x)}e^{B(|u_n|)}\,dx\\ &+C_2\int_\Omega |F|^{p'(x)}b(|u_n|)\ |T_k(u_n)|\ e^{B(|u_n|)}\,dx\\ &+\frac{\alpha}{2}\int_\Omega|\nabla T_k(u_n)|^{p(x)}e^{B(|u_n|)}\,dx\\ &+\int_\Omega|\nabla u_n|^{p(x)}b(|u_n|)\ |T_k(u_n)|\ e^{B(|u_n|)}\,dx.\end{aligned}$$

Therefore, thanks to (2.4) we obtain

$$\alpha\int_\Omega|\nabla T_k(u_n)|^{p(x)}e^{B(|u_n|)}\,dx+2\int_\Omega|\nabla u_n|^{p(x)}b(|u_n|)\ |T_k(u_n)|\ e^{B(|u_n|)}\,dx$$
$$+\int_\Omega\phi_n(u_n)\cdot\nabla(T_k(u_n)e^{B(|u_n|)})\,dx\le ke^{B(\infty)}(\|f\|_1+\|f_0\|_1)+C_1\int_\Omega|F|^{p'(x)}e^{B(|u_n|)}\,dx$$
$$+C_2\int_\Omega|F|^{p'(x)}b(|u_n|)\ |T_k(u_n)|\ e^{B(|u_n|)}\,dx+\frac{\alpha}{2}\int_\Omega|\nabla T_k(u_n)|^{p(x)}e^{B(|u_n|)}\,dx$$
$$+2\int_\Omega|\nabla u_n|^{p(x)}b(|u_n|)\ |T_k(u_n)|\ e^{B(|u_n|)}\,dx,$$

since $e^{B(|u_n|)}\ge 1$, we conclude that

$$\frac{\alpha}{2}\int_\Omega|\nabla T_k(u_n)|^{p(x)}\,dx+\int_\Omega\phi_n(u_n)\cdot\nabla(T_k(u_n)e^{B(|u_n|)})\,dx\le kC_3. \tag{3.5}$$

We set

$$\Phi_n(t) := \int_0^t \phi_n(\tau) b(|\tau|)|T_k(\tau)|e^{B(|\tau|)}\, d\tau \qquad \text{and} \qquad \Psi_n(t) := \int_0^t \phi_n(\tau) e^{B(|\tau|)}\, d\tau,$$

then $\Phi_n(u_n), \Psi_n(T_k(u_n)) \in (W_0^{1,p(x)}(\Omega))^N$. In view of the Lemma 2.3, we obtain

$$\begin{aligned} \int_\Omega \phi_n(u_n)\cdot\nabla(T_k(u_n)e^{B(|u_n|)})\, dx &= \int_\Omega \phi_n(u_n) b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}\cdot\nabla u_n\, dx \\ &\quad + \int_\Omega \phi_n(T_k(u_n)) e^{B(|T_k(u_n)|)}\cdot\nabla T_k(u_n)\, dx \\ &= \int_\Omega \text{div } \Phi_n(u_n)\, dx + \int_\Omega \text{div } \Psi_n(T_k(u_n))\, dx \\ &= 0. \end{aligned} \tag{3.6}$$

Thus, in view of (3.5), we obtain

$$\|\nabla T_k(u_n)\|_{p(\cdot)}^{p_-} \le \int_\Omega |\nabla T_k(u_n)|^{p(x)}\, dx + 1 \le kC_4,$$

then

$$\|\nabla T_k(u_n)\|_{p(\cdot)} \le C_5 k^{\frac{1}{p_-}} \qquad \text{for all} \quad k \ge 1, \tag{3.7}$$

with C_5 is a constant that does not depend on n and k. According to the Poincaré's inequality and (3.7), we have

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \le \int_\Omega |T_k(u_n)|\, dx \\ &\le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|1\|_{p'(\cdot)}\|T_k(u_n)\|_{p(\cdot)} \\ &\le 2\ C\ (meas(\Omega) + 1)^{\frac{1}{(p')_-}}\|\nabla T_k(u_n)\|_{p(\cdot)} \\ &\le C_6 k^{\frac{1}{p_-}}. \end{aligned} \tag{3.8}$$

Using the same argument in ([22], Proposition 5.1), we can prove that $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n \to u \quad \text{a.e. in } \Omega,$$

we conclude that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in} \quad W_0^{1,p(x)}(\Omega) \\ T_k(u_n) \to T_k(u) & \text{in} \quad L^{p(\cdot)}(\Omega) \quad \text{and} \quad \text{a.e. in } \Omega. \end{cases} \tag{3.9}$$

Step 3 : Strong convergence of truncations. In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots$ various real-valued functions of real variables that converge to 0 as n tends to infinity.

Let $h > k > 0$ and $b_k := \max\{b(s) : |s| \leq k\}$, we define $\varphi_k(s) = s\exp(\gamma s^2)$ with $\gamma = \left(\frac{2b_k}{\alpha}\right)^2$, it's clear that

$$\varphi_k'(s) - \frac{4b_k}{\alpha}|\varphi_k(s)| \geq \frac{1}{2} \quad \forall s \in I\!R. \tag{3.10}$$

We set

$$M = 4k + h, \qquad z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u) \qquad \text{and} \qquad \omega_n := T_{2k}(z_n).$$

Taking $\varphi_k(\omega_n)e^{B(|u_n|)}$ as a test function in (3.2), we obtain

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \omega_n \varphi_k'(\omega_n) e^{B(|u_n|)}\, dx \\
&+\frac{2}{\alpha}\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n\ b(|u_n|)\varphi_k(\omega_n)\mathrm{sign}(u_n)e^{B(|u_n|)}\, dx \\
&+\int_\Omega \phi_n(u_n) \cdot \nabla\big(\varphi_k(\omega_n)e^{B(|u_n|)}\big)\, dx + \int_\Omega H_n(x, u_n, \nabla u_n)\varphi_k(\omega_n)e^{B(|u_n|)}\, dx \\
&\leq \int_\Omega f_n\varphi_k(\omega_n)e^{B(|u_n|)}\, dx + \int_\Omega F \cdot \nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\, dx + \\
&\quad \frac{2}{\alpha}\int_\Omega |F|\ |\nabla u_n|\ b(|u_n|)\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\, dx.
\end{aligned} \tag{3.11}$$

Since $\varphi_k(\omega_n)$ has the same sign as u_n on the set $\{|u_n| > k\}$ and using (2.4), we get

$$\begin{aligned}
&\frac{2}{\alpha}\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n\ b(|u_n|)\ \varphi_k(\omega_n)\mathrm{sign}(u_n)e^{B(|u_n|)}\, dx \\
&\geq -\frac{2}{\alpha}\int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n b(|u_n|)\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\, dx \\
&+2\int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)} b(|u_n|)\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\, dx.
\end{aligned} \tag{3.12}$$

Also, using (2.4), (2.6) and Young's inequality, we obtain

$$\Big|\int_\Omega H_n(x, u_n, \nabla u_n)\varphi_k(\omega_n)e^{B(|u_n|)}\, dx\Big| \tag{3.13}$$

$$\begin{aligned}
&\leq \int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \frac{b(|u_n|)}{\alpha}|\varphi_k(\omega_n)|e^{B(|u_n|)}\, dx \\
&\quad + \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)}\ b(|u_n|)\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\, dx \\
&\quad + \int_\Omega |f_0|\ |\varphi_k(\omega_n)|e^{B(|u_n|)}\, dx.
\end{aligned} \tag{3.14}$$

In addition,

$$\frac{2}{\alpha}\int_\Omega |F|\ |\nabla u_n|\ b(|u_n|)\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx$$
$$\leq \int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\frac{b(|u_n|)}{\alpha}|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx$$
$$+\int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)}\ b(|u_n|)|\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx$$
$$+C_7\int_\Omega |F|^{p'(x)}\ b(|u_n|)|\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx. \tag{3.15}$$

We have $\omega_n = T_k(u_n) - T_k(u)$ on $\{|u_n|\leq k\}$, and $\nabla\omega_n = 0$ on $\{|u_n|>M\}$. Thus, using (3.1) − (3.1) we obtain

$$\int_{\{|u_n|\leq k\}} a(y_n)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx$$
$$+\int_{\{k<|u_n|\leq M\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx$$
$$-\frac{4b_k}{\alpha}\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx$$
$$\leq e^{B(\infty)}\int_\Omega\Big(|f_n|+|f_0|\Big)|\varphi_k(\omega_n)|\,dx+\int_\Omega F\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx$$
$$-\int_\Omega \phi_n(u_n)\cdot\nabla\big(\varphi_k(\omega_n)e^{B(|u_n|)}\big)\,dx$$
$$+C_7\int_\Omega |F|^{p'(x)}\ b(|u_n|)|\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx. \tag{3.16}$$

Concerning the second term on the left-hand side of (3.1) and denoting $y_n = (x,T_M(u_n),\nabla T_M(u_n))$ we have

$$\int_{\{k<|u_n|\leq M\}} a(y_n)\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx$$
$$=\int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a(y_n)\cdot\nabla(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx$$
$$\geq -e^{B(\infty)}\varphi_k'(2k)\int_{\{k<|u_n|\leq M\}}|a(y_n)|\,|\nabla T_k(u)|\,dx, \tag{3.17}$$

since $(|a(x,T_M(u_n),\nabla T_M(u_n))|)_n$ is bounded in $L^{p'(\cdot)}(\Omega)$, then there exists $\psi\in L^{p'(\cdot)}(\Omega)$ such that $|a(x,T_M(u_n),\nabla T_M(u_n))|\rightharpoonup\psi$ in $L^{p'(\cdot)}(\Omega)$. Therefore,

$$\int_{\{k<|u_n|\leq M\}}|a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|\,dx\longrightarrow\int_{\{k<|u|\leq M\}}\psi\,|\nabla T_k(u)|\,dx=0, \tag{3.18}$$

as a result,

$$\int_{\{k<|u_n|\leq M\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\geq\varepsilon_1(n). \tag{3.19}$$

Then, in view of (3.1), we obtain

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(y_n)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&-\frac{4b_k}{\alpha}\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx\\
&\leq e^{B(\infty)}\int_\Omega\Big(|f_n|+|f_0|\Big)|\varphi_k(\omega_n)|\,dx+\int_\Omega F\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&-\int_\Omega \phi_n(u_n)\cdot\nabla\big(\varphi_k(\omega_n)e^{B(|u_n|)}\big)\,dx\\
&+C_7\int_\Omega |F|^{p'(x)}\ b(|u_n|)|\ |\varphi_k(\omega_n)|\ e^{B(|u_n|)}\,dx+\varepsilon_2(n).
\end{aligned}\tag{3.20}$$

Now, we will study each term on the left-hand side of (3.20).

<u>First estimate :</u>
Denoting $y_n=(x,T_k(u_n),\nabla T_k(u_n))$, we have

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(y_n)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&=\int_\Omega\big(a(y_n)-a(x,T_k(u_n),\nabla T_k(u))\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&+\int_{\{|u_n|>k\}} a(y_n)\cdot\nabla T_k(u)\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx.
\end{aligned}\tag{3.21}$$

For the second term on the right-hand side of (3.21), we have

$$\begin{aligned}
&\Big|\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\Big|\\
&\leq\varphi_k'(2k)e^{B(\infty)}\int_\Omega|a(x,T_k(u_n),\nabla T_k(u))|\ |\nabla T_k(u_n)-\nabla T_k(u)|\,dx.
\end{aligned}$$

Applying the Lebesgue dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ in $L^{p(\cdot)}(\Omega)$, then $a(x,T_k(u_n),\nabla T_k(u))\to a(x,T_k(u),\nabla T_k(u))$ in $(L^{p'(\cdot)}(\Omega))^N$, and since $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ in $(L^{p(\cdot)}(\Omega))^N$, we deduce that

$$\varepsilon_3(n)=\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\longrightarrow 0, \tag{3.22}$$

as $n \to \infty$. Concerning the last term on the right-hand side of (3.21), similarly to (3.18), it is easy to see

$$\varepsilon_4(n) = \int_{\{|u_n|>k\}} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u)\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx \longrightarrow 0, \quad (3.23)$$

as $n \to \infty$. It follows that

$$\begin{aligned}&\int_\Omega \big(a(y_n) - a(x, T_k(u_n), \nabla T_k(u))\big) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx \\ &= \int_{\{|u_n|\le k\}} a(y_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx + \varepsilon_5(n).\end{aligned} \quad (3.24)$$

Second estimate :

For the second term on the right-hand side of (3.20), we have

$$\begin{aligned}&\frac{4b_k}{\alpha}\int_{\{|u_n|\le k\}} a(y_n) \cdot \nabla T_k(u_n)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx \\ &= \frac{4b_k}{\alpha}\int_\Omega \big(a(y_n) - a(x, T_k(u_n), \nabla T_k(u))\big) \cdot \big(\nabla T_k(u_n) - \nabla T_k(u)\big)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx \\ &+ \frac{4b_k}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \big(\nabla T_k(u_n) - \nabla T_k(u)\big)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx \\ &+ \frac{4b_k}{\alpha}\int_\Omega a(y_n) \cdot \nabla T_k(u)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx.\end{aligned} \quad (3.25)$$

Similarly to (3.22), we prove that

$$\varepsilon_6(n) = \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \big(\nabla T_k(u_n) - \nabla T_k(u)\big)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx \longrightarrow 0. \quad (3.26)$$

For the last term on right-hand side of (3.25), the sequence $(a(x, T_k(u_n), \nabla T_k(u_n)))_{n\in\mathbb{N}}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, then there exists $\xi \in (L^{p'(\cdot)}(\Omega))^N$ such that $a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \xi$ in $(L^{p'(\cdot)}(\Omega))^N$, and using the fact that

$$\nabla T_k(u)\,|\varphi_k(\omega_n)| \to \nabla T_k(u)\,|\varphi_k(T_{2k}(u - T_h(u)))| \quad \text{in} \quad (L^{p(\cdot)}(\Omega))^N,$$

and since $0 < k \le h$, then

$$\begin{aligned}\varepsilon_7(n) &= \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u)|\varphi_k(\omega_n)|\,dx \\ &\longrightarrow \int_\Omega \xi \cdot \nabla T_k(u)|\varphi_k(T_{2k}(u - T_h(u)))|\,dx = 0.\end{aligned} \quad (3.27)$$

By combining (3.25) – (3.27), we deduce that

$$\begin{aligned}&\frac{4b_k}{\alpha}\int_\Omega \big(a(y_n) - a(x, T_k(u_n), \nabla T_k(u))\big) \cdot \big(\nabla T_k(u_n) - \nabla T_k(u)\big)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx \\ &= \frac{4b_k}{\alpha}\int_{\{|u_n|\le k\}} a(y_n) \cdot \nabla T_k(u_n)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx + \varepsilon_8(n),\end{aligned} \quad (3.28)$$

where $y_n = (x, T_k(u_n), \nabla T_k(u_n))$. Using the fact $e^{B(|u_n|)} \geq 1$, the relations (3.20), (3.24) and (3.28) allows us to write

$$\begin{aligned}
&\int_\Omega \big(a(y_n) - a(x, T_k(u_n), \nabla T_k(u))\big) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\big(\varphi_k'(\omega_n) - \frac{4b_k}{\alpha}|\varphi_k(\omega_n)|\big)\, dx \\
&\quad \leq e^{B(\infty)} \int_\Omega \Big(|f_n| + |f_0|\Big)|\varphi_k(\omega_n)|\, dx + \int_\Omega F \cdot \nabla\omega_n \varphi_k'(\omega_n) e^{B(|u_n|)}\, dx \\
&\qquad - \int_\Omega \phi_n(u_n) \cdot \nabla\big(\varphi_k(\omega_n) e^{B(|u_n|)}\big)\, dx \\
&\qquad + C_7 \int_\Omega |F|^{p'(x)}\, b(|u_n|)\, |\varphi_k(\omega_n)|\, e^{B(|u_n|)}\, dx + \varepsilon_9(n). \qquad (3.29)
\end{aligned}$$

For the third term of equation (3.29), we have

$$\begin{aligned}
&\int_\Omega \phi_n(u_n) \cdot \nabla\big(\varphi_k(\omega_n) e^{B(|u_n|)}\big)\, dx = \int_\Omega \phi_n(T_M(u_n)) \cdot \nabla\omega_n \varphi_k'(\omega_n) e^{B(|u_n|)}\, dx \\
&+ \frac{2}{\alpha} \int_\Omega \phi_n(u_n) \cdot \nabla u_n\, b(|u_n|)\, \varphi_k(\omega_n) \text{sign}(u_n) e^{B(|u_n|)}\, dx. \qquad (3.30)
\end{aligned}$$

For the first term on the right-hand side of (3.30), since $\phi_n(T_M(u_n)) = \phi(T_M(u_n))$ for n large enough $(n \geq M)$, then

$$\begin{aligned}
&\int_\Omega \phi_n(T_M(u_n)) \cdot \nabla\omega_n \varphi_k'(\omega_n) e^{B(|u_n|)}\, dx \\
&\to \int_\Omega \phi(T_M(u)) \cdot \nabla T_{2k}(u - T_h(u)) \varphi_k'(T_{2k}(u - T_h(u))) e^{B(|u|)}\, dx \qquad (3.31)
\end{aligned}$$

as n tend to infinity.

Taking $\Psi_1(t) = \displaystyle\int_0^t \phi(\tau)\varphi_k'(\tau - T_h(\tau))\chi_{\{h<|\tau|\leq 2k+h\}}\, e^{B(|\tau|)}\, d\tau$, then $\Psi_1(u) \in (W_0^{1,p(\cdot)}(\Omega))^N$. In view of Lemma 2.3, we obtain

$$\begin{aligned}
&\int_\Omega \phi(T_M(u)) \cdot \varphi_k'(T_{2k}(u - T_h(u))) \cdot \nabla T_{2k}(u - T_h(u)) e^{B(|u|)}\, dx \\
&= \int_\Omega \phi(T_M(u))\varphi_k'(u - T_h(u)) \cdot \nabla u \chi_{\{h<|u|\leq 2k+h\}} e^{B(|u|)}\, dx \\
&= \int_\Omega \text{div}\ \Psi_1(u)\, dx = 0.
\end{aligned}$$

Concerning the second term on the right-hand side of (3.30), we have

$$\begin{aligned}
&\int_\Omega \phi_n(u_n) \cdot \nabla u_n\, b(|u_n|)\varphi_k(\omega_n)\text{sign}(u_n) e^{B(|u_n|)}\, dx \\
&= \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n)\, b(|u_n|)\, \varphi_k(T_k(u_n) - T_k(u))\, \text{sign}(u_n)\, e^{B(|u_n|)}\, dx \qquad (3.32) \\
&\quad + \int_{\{|u_n|>k\}} \phi_n(u_n) \cdot \nabla u_n b(|u_n|)\, |\varphi_k(\omega_n)|\, e^{B(|u_n|)}\, dx.
\end{aligned}$$

It's clear that

$$\lim_{n\to\infty}\int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)\; b(|u_n|)\;\varphi_k(T_k(u_n)-T_k(u))\;\mathrm{sign}(u_n)\; e^{B(|u_n|)}\,dx=0. \tag{3.33}$$

Concerning the second term of the right-hand side of (3.32), we set

$$\Psi_2(s)=\int_0^s \phi_n(t)b(|t|)|\varphi_k(T_{2k}(t-T_h(t)+T_k(t)-T_k(u)))|e^{B(|t|)}\cdot\chi_{\{|t|>k\}}\,dt, \tag{3.34}$$

then $\Psi_2(u_n)\in(W_0^{1,p(x)}(\Omega))^N$, and in view of Lemma 2.3, we find

$$\int_{\{|u_n|>k\}}\phi_n(u_n)\cdot\nabla u_n b(|u_n|)\;|\varphi_k(\omega_n)|\;e^{B(|u_n|)}\,dx=\int_\Omega \mathrm{div}\;\Psi_2(u_n)\,dx=0. \tag{3.35}$$

As a result

$$\varepsilon_{10}(n)=\int_\Omega \phi_n(u_n)\cdot\nabla\big(\varphi_k(\omega_n)e^{B(|u_n|)}\big)\,dx. \tag{3.36}$$

Now, since $f_n\to f$ in $L^1(\Omega)$ and $\varphi_k(\omega_n)\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as n and h tend to infinity, then

$$\int_\Omega\Big(|f_n|+|f_0|\Big)|\varphi_k(\omega_n)|\,dx\to 0 \text{ and } \int_\Omega |F|^{p'(x)}\;b(|u_n|)\;|\varphi_k(\omega_n)|\;e^{B(|u_n|)}\,dx\to 0. \tag{3.37}$$

Also, we show (see Appendix) that

$$\int_\Omega F\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\longrightarrow 0 \qquad \text{as}\quad n,h\to\infty. \tag{3.38}$$

By combining (3.29) and (3.36) – (3.38) and letting h tend to infinity, we get

$$\lim_{n\to\infty}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))dx=0.$$

In view of Lemma 2.4, we deduce that

$$\begin{cases} T_k(u_n)\longrightarrow T_k(u) & \text{in}\quad W_0^{1,p(x)}(\Omega),\\ \nabla u_n\longrightarrow\nabla u & \text{a.e. in }\Omega.\end{cases} \tag{3.39}$$

Step 4 : The equi-integrability of $H_n(x,u_n,\nabla u_n)$ In order to pass to the limit in the approximate problems, we shall show that $H_n(x,u_n,\nabla u_n)$ tends to $H(x,u,\nabla u)$ strongly in $L^1(\Omega)$. By using Vitali's theorem, it suffices to prove that $H_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable.
By taking $T_1(u_n-T_h(u_n))e^{B(|u_n|)}$ as a test function in (3.2), according to (2.4)

and (2.6), we obtain

$$\begin{aligned}
&\alpha \int_{\{h<|u_n|\le h+1\}} |\nabla u_n|^{p(x)} e^{B(|u_n|)}\, dx \\
&+2 \int_{\{h<|u_n|\}} |\nabla u_n|^{p(x)} b(|u_n|)|T_1(u_n - T_h(u_n))| e^{B(|u_n|)}\, dx \\
&+ \int_{\{h<|u_n|\}} \phi_n(u_n)\cdot \nabla\big(T_1(u_n - T_h(u_n)) e^{B(|u_n|)}\big)\, dx \\
&\le e^{B(\infty)} \int_{\{h<|u_n|\}} (|f_n| + |f_0|)\, dx \\
&+ \int_{\{h<|u_n|\}} |\nabla u_n|^{p(x)} b(|u_n|)|T_1(u_n - T_h(u_n))| e^{B(|u_n|)}\, dx + \\
&\int_\Omega F\cdot \nabla\big(T_1(u_n - T_h(u_n)) e^{B(|u_n|)}\big)\, dx. \qquad (3.40)
\end{aligned}$$

Taking

$$\Psi_3(t) = \int_0^t \phi_n(\tau) e^{B(|\tau|)} d\tau \text{ and } \Psi_4(t) = \int_0^t \phi_n(\tau) b(|\tau|)|T_1(u_n - T_h(u_n))|\chi_{\{h<|u_n|\}} e^{B(|\tau|)} d\tau.$$

We have $\Psi_3(u_n), \Psi_4(u_n) \in (W_0^{1,p(x)}(\Omega))^N$. In view of Lemma 2.3, we have

$$\begin{aligned}
&\int_{\{h<|u_n|\}} \phi_n(u_n)\cdot \nabla\big(T_1(u_n - T_h(u_n)) e^{B(|u_n|)}\big)\, dx \\
&= \int_{\{h<|u_n|\le h+1\}} \phi_n(u_n)\cdot \nabla u_n e^{B(|u_n|)}\, dx \\
&+\frac{2}{\alpha} \int_{\{h<|u_n|\}} \phi_n(u_n)\cdot \nabla u_n b(|u_n|)|T_1(u_n - T_h(u_n))| e^{B(|u_n|)}\, dx \\
&= \int_\Omega \phi_n(u_n)\cdot \nabla T_{h+1}(u_n) e^{B(|u_n|)}\, dx - \int_\Omega \phi_n(u_n)\cdot \nabla T_h(u_n) e^{B(|u_n|)}\, dx \\
&\quad +\frac{2}{\alpha} \int_{\{h<|u_n|\}} \phi_n(u_n)\cdot \nabla u_n b(|u_n|)|T_1(u_n - T_h(u_n))| e^{B(|u_n|)}\, dx \\
&= \int_\Omega \text{div } \Psi_3(T_{h+1}(u_n))\, dx - \int_\Omega \text{div } \Psi_3(T_h(u_n))\, dx \qquad (3.41) \\
&+\frac{2}{\alpha} \int_\Omega \text{div } \Psi_4(u_n)\, dx = 0. \qquad (3.42)
\end{aligned}$$

Moreover, using Young's inequality, we have

$$\begin{aligned}
&\int_\Omega F\cdot\nabla\big(T_1(u_n-T_h(u_n))e^{B(|u_n|)}\big)\,dx\\
&=\int_{\{h<|u_n|\le h+1\}} F\cdot\nabla u_n e^{B(|u_n|)}\,dx+\frac{2}{\alpha}\int_{\{h<|u_n|\}} F\cdot\nabla u_n b(|u_n|)|T_1(u_n-T_h(u_n))|e^{B(|u_n|)}\,dx\\
&\le C_8\int_{\{h<|u_n|\le h+1\}} |F|^{p'(x)}e^{B(|u_n|)}\,dx+C_9\int_{\{h<|u_n|\}}|F|^{p'(x)}b(|u_n|)|T_1(u_n-T_h(u_n))|e^{B(|u_n|)}\,dx\\
&+\frac{\alpha}{2}\int_{\{h<|u_n|\le h+1\}} |\nabla u_n|^{p(x)}e^{B(|u_n|)}\,dx+\frac{1}{2}\int_{\{h<|u_n|\}}|\nabla u_n|^{p(x)}b(|u_n|)|T_1(u_n-T_h(u_n))|e^{B(|u_n|)}\,dx.
\end{aligned} \tag{3.43}$$

Thus, we conclude that

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{h<|u_n|\le h+1\}}|\nabla u_n|^{p(x)}\,dx+\frac{1}{2}\int_{\{h+1<|u_n|\}}|\nabla u_n|^{p(x)}b(|u_n|)\,dx\\
&\le e^{B(\infty)}\int_{\{h<|u_n|\}}(|f_n|+|f_0|)\,dx+C_8\int_{\{h<|u_n|\le h+1\}}|F|^{p'(x)}e^{B(|u_n|)}\,dx\\
&+C_9\int_{\{h<|u_n|\}}|F|^{p'(x)}b(|u_n|)e^{B(|u_n|)}\,dx,
\end{aligned} \tag{3.44}$$

with $b(\cdot)\in L^\infty(I\!R)$. Therefore, for any $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)<|u_n|\}} b(|u_n|)|\nabla u_n|^{p(x)}\,dx\le\frac{\eta}{2}. \tag{3.45}$$

For any measurable subset $E\subseteq\Omega$, we have

$$\begin{aligned}
&\int_E b(|u_n|)|\nabla u_n|^{p(x)}\,dx\le\int_E b(|T_{h(\eta)}(u_n)|)|\nabla T_{h(\eta)}(u_n)|^{p(x)}\,dx\\
&+\int_{\{h(\eta)<|u_n|\}} b(u_n)|\nabla u_n|^{p(x)}\,dx
\end{aligned}$$

(3.46)

In view of (3.39), there exists $\lambda(\eta)>0$ such that

$$\int_E b(|T_{h(\eta)}(u_n)|)|\nabla T_{h(\eta)}(u_n)|^{p(x)}\,dx\le\frac{\eta}{2}\qquad\text{for}\quad \text{meas}(E)\le\lambda(\eta). \tag{3.47}$$

Finally, by combining (3.45), (3.46) and (3.47), one easily has

$$\int_E b(|u_n|)|\nabla u_n|^{p(x)}\,dx\le\eta\qquad\text{for}\quad\text{meas}(E)\le\lambda(\eta). \tag{3.48}$$

Using (2.6), we deduce that $(H_n(x,u_n,\nabla u_n))_n$ is equi-integrable, and in view of Vitali's theorem, we conclude that

$$H_n(x,u_n,\nabla u_n)\longrightarrow H(x,u,\nabla u)\qquad\text{in}\quad L^1(\Omega). \tag{3.49}$$

Step 5 : Passage to the limit Let $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, by using $T_k(u_n - v)$ as a test function in (3.2), we obtain

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v)\, dx &+ \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n - v)\, dx \\ + \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - v)\, dx &= \int_\Omega f_n T_k(u_n - v)\, dx \\ &+ \int_\Omega F \cdot \nabla T_k(u_n - v)\, dx. \end{aligned} \tag{3.50}$$

Choosing $M = k + \|v\|_\infty$, then $\{|u_n - v| \le k\} \subseteq \{|u_n| \le M\}$. For the first term on the left-hand side of the above relation, we have

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v)\, dx = \\ &= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot (\nabla T_M(u_n) - \nabla v) \chi_{\{|u_n - v| \le k\}}\, dx \\ &= \int_\Omega (a(x, T_M(u_n), \nabla T_M(u_n)) - a(x, T_M(u_n), \nabla v)) \cdot (\nabla T_M(u_n) - \nabla v) \chi_{\{|u_n - v| \le k\}}\, dx \\ &\quad + \int_\Omega a(x, T_M(u_n), \nabla v) \cdot (\nabla T_M(u_n) - \nabla v) \chi_{\{|u_n - v| \le k\}}\, dx, \end{aligned} \tag{3.51}$$

since $a(x, T_M(u_n), \nabla v) \to a(x, T_M(u), \nabla v)$ in $(L^{p'(\cdot)}(\Omega))^N$, then

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla v) \cdot (\nabla T_M(u_n) - \nabla v) \chi_{\{|u_n - v| \le k\}}\, dx \\ &= \int_\Omega a(x, T_M(u), \nabla v) \cdot (\nabla T_M(u) - \nabla v) \chi_{\{|u - v| \le k\}}\, dx, \end{aligned}$$

and according to Fatou's lemma, we get

$$\begin{aligned} &\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v)\, dx \\ &\ge \int_\Omega (a(x, T_M(u), \nabla T_M(u)) - a(x, T_M(u), \nabla v)) \cdot (\nabla T_M(u) - \nabla v) \chi_{\{|u - v| \le k\}}\, dx \\ &\quad + \int_\Omega a(x, T_M(u), \nabla v) \cdot (\nabla T_M(u) - \nabla v) \chi_{\{|u - v| \le k\}}\, dx \\ &= \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot (\nabla T_M(u) - \nabla v) \chi_{\{|u - v| \le k\}}\, dx \\ &= \int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v)\, dx. \end{aligned} \tag{3.52}$$

Being $T_k(u_n - v) \rightharpoonup T_k(u - v)$ weak-$\star$ in $L^\infty(\Omega)$ and thanks to (3.49), we deduce that

$$\int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - v)\, dx \longrightarrow \int_\Omega H(x, u, \nabla u) T_k(u - v)\, dx. \tag{3.53}$$

Also, since $T_k(u_n - v) \rightharpoonup T_k(u - v)$ weakly in $W_0^{1,p(x)}(\Omega)$ and $\phi_n(u_n) = \phi(T_M(u_n))$ in $\{|u_n - v| \leq k\}$ for $\{n \geq M\}$, then

$$\lim_{n\to\infty} \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n - v)\, dx = \int_\Omega \phi(u) \cdot \nabla T_k(u - v)\, dx. \tag{3.54}$$

Concerning the two terms on the right-hand side of (3.50), we have

$$\int_\Omega f_n T_k(u_n - v)\, dx \longrightarrow \int_\Omega f T_k(u - v)\, dx, \tag{3.55}$$

$$\int_\Omega F \cdot \nabla T_k(u_n - v)\, dx \longrightarrow \int_\Omega F \cdot \nabla T_k(u - v)\, dx. \tag{3.56}$$

Hence, putting all the terms together, we conclude the proof of Theorem 3.2.

4. Renormalized solutions

In this section, we show that entropy solutions are also renormalized solutions for our problem.

Definition 4.1. *A measurable function $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$ is a renormalized solution of problem (2.7), if*

$$H(x, u, \nabla u) \in L^1(\Omega) \qquad \textit{and} \qquad \lim_{h\to\infty} \int_{\{h<|u|\leq h+1\}} |\nabla u|^{p(x)}\, dx = 0,$$

and

$$\begin{aligned}&\int_\Omega a(x, u, \nabla u) \cdot (S'(u)\varphi\nabla u + S(u)\nabla\varphi)\, dx + \int_\Omega \phi(u) \cdot (S'(u)\varphi\nabla u + S(u)\nabla\varphi)\, dx\\ &+\int_\Omega H(x, u, \nabla u)S(u)\ \varphi\, dx = \int_\Omega f\ S(u)\ \varphi\, dx + \int_\Omega F \cdot (S'(u)\varphi\nabla u + S(u)\nabla\varphi)\, dx,\end{aligned} \tag{4.1}$$

for every function $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and smooth function $S(\cdot) \in W^{1,\infty}(I\!R)$ with compact support.

Theorem 4.2. *Assuming that (2.3) − (2.6) hold, then the entropy solution $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$ is also a renormalized solution of the problem (2.7).*

Proof of Theorem 4.2 We observe that the entropy solution u in theorem 3.2 satisfies the requirements that $u \in T_0^{1,p(x)}(\Omega)$, and there exists a sequence of weak solutions $(u_n)_n$ for approximate problems (3.2), such that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(x)}(\Omega)$ for any $k > 0$, and

$$H_n(x, u_n, \nabla u_n) \longrightarrow H(x, u, \nabla u) \quad \text{ in } \quad L^1(\Omega).$$

Thanks to (3.44), we have

$$\frac{\alpha}{2}\int_{\{h<|u_n|\leq h+1\}}|\nabla u_n|^{p(x)}\,dx \leq e^{B(\infty)}\int_{\{h<|u_n|\}}(|f_n|+|f_0|)\,dx$$
$$+C_8\int_{\{h<|u_n|\leq h+1\}}|F|^{p'(x)}e^{B(|u_n|)}\,dx$$
$$+C_9\int_{\{h<|u_n|\}}|F|^{p'(x)}b(|u_n|)e^{B(|u_n|)}\,dx, \tag{4.2}$$

by letting $n\to\infty$ and in view of Fatou's lemma, we obtain

$$\frac{\alpha}{2}\int_{\{h<|u|\leq h+1\}}|\nabla u|^{p(x)}\,dx \leq e^{B(\infty)}\int_{\{h<|u|\}}(|f|+|f_0|)\,dx$$
$$+C_6\int_{\{h<|u|\leq h+1\}}|F|^{p'(x)}e^{B(|u|)}\,dx$$
$$+C_7\int_{\{h<|u|\}}|F|^{p'(x)}b(|u|)e^{B(|u|)}\,dx, \tag{4.3}$$

we conclude that

$$\lim_{h\mapsto\infty}\int_{\{h<|u|\leq h+1\}}|\nabla u|^{p(x)}\,dx = 0. \tag{4.4}$$

Let $S(\cdot)\in W^{1,\infty}(I\!R)$ be a function such that $\operatorname{supp} S(\cdot)\in[-M,M]$ for some $M>0$. For $\varphi\in C_0^\infty(\Omega)$, taking $S(u_n)\varphi\in W_0^{1,p(x)}(\Omega)$ as a test function in (3.2), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot(S'(u_n)\varphi\nabla u_n+S(u_n)\nabla\varphi)\,dx$$
$$+\int_\Omega \phi_n(u_n)\cdot(S'(u_n)\varphi\nabla u_n+S(u_n)\nabla\varphi)\,dx$$
$$+\int_\Omega H_n(x,u_n,\nabla u_n)\;S(u_n)\;\varphi\,dx = \int_\Omega f_n\;S(u_n)\;\varphi\,dx$$
$$+\int_\Omega F\cdot(S'(u_n)\varphi\nabla u_n+S(u_n)\nabla\varphi)\,dx. \tag{4.5}$$

Concerning the first integral on the left-hand side of (4.5), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot(S'(u_n)\varphi\nabla u_n+S(u_n)\nabla\varphi)\,dx$$
$$=\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot(S'(u_n)\varphi\nabla T_M(u_n)+S(u_n)\nabla\varphi)\,dx.$$

We have $a(x,T_M(u_n),\nabla T_M(u_n))\to a(x,T_M(u),\nabla T_M(u))$ a.e. in Ω and $(a(x,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, using Lemma 2.2, we get

$$a(x,T_M(u_n),\nabla T_M(u_n))\rightharpoonup a(x,T_M(u),\nabla T_M(u)) \quad \text{in} \quad (L^{p'(\cdot)}(\Omega))^N,$$

and since

$$S'(u_n)\varphi\nabla T_M(u_n)+S(u_n)\nabla\varphi \longrightarrow S'(u)\varphi\nabla T_M(u)+S(u)\nabla\varphi \quad \text{in} \quad (L^{p(\cdot)}(\Omega))^N.$$

Then, we deduce that

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot(S'(u_n)\varphi\nabla T_M(u_n)+S(u_n)\nabla\varphi)\,dx\\&=\int_\Omega a(x,u,\nabla u)\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,dx.\end{aligned}\tag{4.6}$$

For the second term on the left-hand side of (4.5), since $\phi_n(u_n)=\phi(T_M(u_n))\in(L^{p'(\cdot)}(\Omega))^N$ in $\{|u_n|\le M\}$ for n large enough, then

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \phi_n(u_n)\cdot(S'(u_n)\varphi\nabla u_n+S(u_n)\nabla\varphi)\,dx\\&=\lim_{n\to\infty}\int_{\{|u_n|\le M\}}\phi(T_M(u_n))\cdot(S'(u_n)\varphi\nabla T_M(u_n)+S(u_n)\nabla\varphi)\,dx\\&=\int_{\{|u|\le M\}}\phi(T_M(u))\cdot(S'(u)\varphi\nabla T_M(u)+S(u)\nabla\varphi)\,dx\\&=\int_\Omega\phi(u)\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,dx.\end{aligned}\tag{4.7}$$

Concerning the other terms, we have $S(u_n)\,\varphi\rightharpoonup S(u)\,\varphi$ in $W_0^{1,p(x)}(\Omega)$ and weak$-*$ in $L^\infty(\Omega)$, then

$$\int_\Omega H_n(x,u_n,\nabla u_n)\;S(u_n)\;\varphi\,dx\longrightarrow\int_\Omega H(x,u,\nabla u)\;S(u)\;\varphi\,dx,\tag{4.8}$$

$$\int_\Omega f_n\;S(u_n)\;\varphi\,dx\longrightarrow\int_\Omega f\;S(u)\;\varphi\,dx,\tag{4.9}$$

and

$$\int_\Omega F\cdot(S'(u_n)\varphi\nabla u_n+S(u_n)\nabla\varphi)\,dx\longrightarrow\int_\Omega F\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,dx.\tag{4.10}$$

By combining (4.5) – (4.10), we deduce that

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u)\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,dx+\int_\Omega\phi(u)\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,dx\\&+\int_\Omega H(x,u,\nabla u)\;S(u)\;\varphi\,dx=\int_\Omega f\;S(u)\;\varphi\,dx+\int_\Omega F\cdot(S'(u)\varphi\nabla u+S(u)\nabla\varphi)\,dx,\end{aligned}\tag{4.11}$$

which is (4.1) in definition 4.1. Therefore, u is a renormalized solution to problem (2.7).

Example 4.3. *Let* $\phi(\cdot)\equiv 0$ *and* $F\equiv 0$, *we consider the Carathéodory functions*

$$a(x,u,\nabla u)=|\nabla u|^{p(x)-2}\nabla u \qquad \textit{and} \qquad H(x,u,\nabla u)=-e^{-|u|^2}|\nabla u|^{p(x)}.$$

We have $\int_{-\infty}^{+\infty} e^{-s^2}\, ds = \sqrt{\pi}$, *then the conditions* (2.3) − (2.6) *are satisfied. In view of Theorem* 3.2, *the problem:*

$$\begin{cases} -div\big(|\nabla u|^{p(x)-2}\nabla u\big) = f + e^{-|u|^2}|\nabla u|^{p(x)} & in \;\; \Omega, \\ u \;= 0 & on \;\; \partial\Omega, \end{cases} \tag{4.12}$$

has at least one entropy solution $u \in T_0^{1,p(x)}(\Omega)$ *for any* $f \in L^1(\Omega)$. *Moreover, the solution* u *is also a renormalized solution for our problem.*

5. Appendix

We will prove that

$$\int_\Omega F \cdot \nabla\omega_n \varphi_k'(\omega_n) e^{B(|u_n|)}\, dx \longrightarrow 0 \qquad \text{as} \quad n, h \to \infty. \tag{5.1}$$

we have

$$\lim_{n\to\infty}\int_\Omega F\cdot\nabla\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}dx = \int_\Omega F\cdot\nabla T_{2k}(u - T_h(u))\varphi_k'(T_{2k}(u - T_h(u)))e^{B(|u|)}dx. \tag{5.2}$$

By taking $\varphi_k(T_{2k}(u_n - T_h(u_n)))e^{B(|u_n|)}$ as a test function in (3.2) and in view of (2.4) and (2.6), we obtain

$$\begin{aligned} &\alpha \int_{\{h<|u_n|\le 2k+h\}} |\nabla u_n|^{p(x)} \varphi_k'(T_{2k}(u_n - T_h(u_n)))e^{B(|u_n|)}\, dx \\ &+2\int_{\{h<|u_n|\}} |\nabla u_n|^{p(x)} b(|u_n|)\; |\varphi_k(T_{2k}(u_n - T_h(u_n)))| e^{B(|u_n|)}\, dx \\ &+\int_\Omega \phi_n(u_n)\cdot\nabla\big(\varphi_k(T_{2k}(u_n - T_h(u_n)))e^{B(|u_n|)}\big)\, dx \\ &\le e^{B(\infty)}\varphi_k(2k)\int_{\{h<|u_n|\}} (|f_n| + |f_0|)\, dx \\ &+\int_{\{h<|u_n|\}} |\nabla u_n|^{p(x)} b(|u_n|)\; |\varphi_k(T_{2k}(u_n - T_h(u_n)))| e^{B(|u_n|)}\, dx \\ &+\int_\Omega F\cdot\nabla\varphi_k(T_{2k}(u_n - T_h(u_n)))e^{B(|u_n|)}\, dx \\ &+\frac{2}{\alpha}\int_\Omega F\cdot\nabla u_n b(|u_n|)|\varphi_k(T_{2k}(u_n - T_h(u_n)))| e^{B(|u_n|)}\, dx \end{aligned} \tag{5.3}$$

Using the Lemma 2.3, we can prove that

$$\int_\Omega \phi_n(u_n)\cdot\nabla\big(\varphi_k(T_{2k}(u_n - T_h(u_n)))e^{B(|u_n|)}\big)\, dx = 0,$$

and in view of Young's inequality, we obtain

$$\begin{aligned}
&\int_\Omega F\cdot\nabla\varphi_k(T_{2k}(u_n-T_h(u_n)))e^{B(|u_n|)}\,dx\\
&+\frac{2}{\alpha}\int_\Omega F\cdot\nabla u_n b(|u_n|)|\varphi_k(T_{2k}(u_n-T_h(u_n)))|e^{B(|u_n|)}\,dx\\
&\le C_1\int_{\{h<|u_n|\le 2k+h\}}|F|^{p'(x)}\,dx\\
&+\frac{\alpha}{2}\int_{\{h<|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi_k'(T_{2k}(u_n-T_h(u_n)))e^{B(|u_n|)}\,dx\\
&+C_2\int_{\{h<|u_n|\}}|F|^{p'(x)}\,dx\\
&+\int_{\{h<|u_n|\}}|\nabla u_n|^{p(x)}b(|u_n|)|\varphi_k(T_{2k}(u_n-T_h(u_n)))|e^{B(|u_n|)}\,dx.
\end{aligned}$$

As a result

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{h<|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi_k'(T_{2k}(u_n-T_h(u_n)))e^{B(|u_n|)}\,dx\\
&\le e^{B(\infty)}\varphi_k(2k)\int_{\{h<|u_n|\}}(|f_n|+|f_0|)\,dx+C_1\int_{\{h<|u_n|\le 2k+h\}}|F|^{p'(x)}\,dx\\
&+C_2\int_{\{h<|u_n|\}}|F|^{p'(x)}\,dx.
\end{aligned}\tag{5.4}$$

Moreover, since $\varphi_k'(s)\ge 1$ and the modular $\rho_{p(x)}(\cdot)$ is weakly (sequentially) lower semicontinuous (see. Theorem 3.2.9 [17]), we get

$$\begin{aligned}
&\frac{\alpha}{2}\int_{\{h<|u|\le 2k+h\}}|\nabla u|^{p(x)}\varphi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx\\
&\le C_3\int_\Omega|\nabla T_{2k}(u-T_h(u))|^{p(x)}\,dx\\
&\le C_3\liminf_{n\to\infty}\int_\Omega|\nabla T_{2k}(u_n-T_h(u_n))|^{p(x)}\,dx\\
&\le C_3\liminf_{n\to\infty}\int_{\{h<|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi_k'(T_{2k}(u_n-T_h(u_n)))e^{B(|u_n|)}\,dx\\
&\le C_4\lim_{n\to\infty}\Big(\int_{\{h<|u_n|\}}(|f_n|+|f_0|)\,dx+\int_{\{h<|u_n|\le 2k+h\}}|F|^{p'(x)}\,dx+\int_{\{h<|u_n|\}}|F|^{p'(x)}\,dx\Big)\\
&\le C_4\Big(\int_{\{h<|u|\}}(|f|+|f_0|)\,dx+\int_{\{h<|u|\le 2k+h\}}|F|^{p'(x)}\,dx+\int_{\{h<|u|\}}|F|^{p'(x)}\,dx\Big).
\end{aligned}\tag{5.5}$$

Hence, by letting h tend to infinity in the previous identity, we find that

$$\limsup_{h\longrightarrow\infty}\int_{\{h<|u|\le 2k+h\}}|\nabla u|^{p(x)}\varphi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx=0.$$

Thus,

$$\lim_{h\to\infty}\int_\Omega F\nabla T_{2k}(u-T_h(u))\varphi_k'(T_{2k}(u-T_h(u)))e^{B(|u|)}\,dx=0.\tag{5.6}$$

By combining (5.2) and (5.6), we can conclude (5.1).

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T. Ahmedatt, Laboratoire d'Analyse Mathématique et Applications (LAMA),
Faculté des Sciences Dhar El Mahraz, Université Sidi Mohamed Ben Abdellah,
BP 1796 Atlas Fès, Maroc
E-mail address: taghi-med@hotmail.fr

and

E. Azroul, Laboratoire d'Analyse Mathématique et Applications (LAMA),
Faculté des Sciences Dhar El Mahraz, Université Sidi Mohamed Ben Abdellah,
BP 1796 Atlas Fès, Maroc
E-mail address: azroulelhoussine@gmail.com

and

H. Hjiaj, Université Abdelmalek Essaadi, Faculté des Sciences Tétouan,,
B.P. 2121 M'Hannech II. Tétouan, Maroc
E-mail address: hassane.hjiaj@yahoo.fr

and

A. Touzani, Laboratoire d'Analyse Mathématique et Applications (LAMA),
Faculté des Sciences Dhar El Mahraz, Université Sidi Mohamed Ben Abdellah,
BP 1796 Atlas Fès, Maroc
E-mail address: atouzani07@gmail.com