Bol. Soc. Paran. Mat. (3s.) v. 36 3 (2018): 173–192.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i3.30799

Quadratic Ideals, Indefinite Quadratic Forms and Some Specific Diophantine Equations

Ahmet Tekcan and Seyma Kutlu

abstract: Let $k \geq 1$ be an integer and let $P = k+2, Q = k$ and $D = k^2 + 4$. In this paper, we derived some algebraic properties of quadratic ideals I_γ and indefinite quadratic forms F_γ for quadratic irrationals γ, and then we determine the set of all integer solutions of the Diophantine equation $F_\gamma^{\pm k}(x, y) = \pm Q$.

Key Words: Quadratic ideals, indefinite quadratic forms, Diophantine equations.

Contents

1. Preliminaries

A real **binary quadratic form** (or just a form) F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2,$$

where $a, b, c \in \mathbb{R}$. We denote F briefly by $F = (a, b, c)$. The **discriminant** of F is $\Delta = \Delta(F) = b^2 - 4ac$. F is an **integral form** if and only if $a, b, c \in \mathbb{Z}$, and is indefinite if and only if $\Delta > 0$.

Gauss defined the **group action** of GL(2,$\mathbb{Z}$) which is the multiplicative group of 2×2 matrices $g = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ such that $r, s, t, u \in \mathbb{Z}$ with $\det(g) = \pm 1$, on the set of forms as

$$gF(x, y) = F(rx + ty, sx + uy). \tag{1.1}$$

2010 *Mathematics Subject Classification:* 11E12, 11E18, 11D09, 11D41, 11D45.
Submitted February 01, 2016. Published October 25, 2016

If there exists a $g \in$GL$(2,\mathbb{Z})$ such that $gF = G$, then F and G are called **equivalent**. If $\det(g) = 1$, then F and G are called **properly equivalent** and if $\det(g) = -1$, then F and G are called **improperly equivalent**. An element $g \in$GL$(2,\mathbb{Z})$ is called an **automorphism** of F if $gF = F$. If $\det g = 1$, then g is called a **proper automorphism**, and if $\det g = -1$, then g is called an **improper automorphism** of F. The set of proper automorphisms of F is denoted by $Aut(F)^+$, and the set of improper automorphisms is denoted by $Aut(F)^-$. Also we set $Aut(F)^* = \{g \in \mathrm{GL}(2,\mathbb{Z}) : gF = -F \text{ with } \det(g) = -1\}$.

The **right neighbor** $R(F)$ of an integral indefinite form $F = (a, b, c)$ of discriminant Δ is the form (A, B, C) determined by $A = c, b+B \equiv 0 \pmod{2A}$, $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$ and $B^2 - 4AC = \Delta$. It is clear that

$$R(F) = \begin{bmatrix} 0 & -1 \\ 1 & -\delta \end{bmatrix} (a, b, c), \tag{1.2}$$

where

$$\delta = \frac{b + B}{2c}. \tag{1.3}$$

The **left neighbor** $L(F)$ of F is defined as

$$L(F) = \chi\tau R(c, b, a), \tag{1.4}$$

where $\tau(F) = (-a, b, -c)$ and $\chi(F) = (-c, b, -a)$ (see [3], [4] and [6]).

Mollin considered the arithmetic of ideals in his book [9]. Let $D \neq 1$ be a square–free integer and let $\Delta = \frac{4D}{r^2}$, where $r = 2$ if $D \equiv 1 (\mathrm{mod}\ 4)$ or $r = 1$ otherwise. The value Δ is congruent to either 1 or 0 modulo 4 and is called a **fundamental discriminant** with **fundamental radicand** D. If we set $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, then $\mathbb{K}$ is called a **real quadratic number field** of discriminant Δ.

A real number γ is called a **quadratic irrational** associated with the radicand D, if γ can be written as $\gamma = \frac{P+\sqrt{D}}{Q}$, where $P, Q, D \in \mathbb{Z}$, $D > 0, Q \neq 0$ and $P^2 \equiv D (\mathrm{mod}\ Q)$. We denote the **continued fraction expansion** of γ by $\gamma = [m_0; m_1, m_2, \cdots, \gamma_i]$, where (for $i \geq 0$ and $\gamma = \gamma_0, P_0 = P, Q_0 = Q$) we recursively define $\gamma_i = \frac{P_i+\sqrt{D}}{Q_i}$,

$$m_i = \left\lfloor \frac{P_i + \sqrt{D}}{Q_i} \right\rfloor, \ P_{i+1} = m_i Q_i - P_i \ \text{ and } \ Q_{i+1} = \frac{D - P_{i+1}^2}{Q_i}. \tag{1.5}$$

An infinite simple continued fraction γ is called **periodic** if $\gamma = [m_0; m_1, m_2, \cdots]$, where $m_n = m_{n+l}$ for all $n \geq k$ with $k, l \in \mathbb{N}$. In this case we use the notation

$$[m_0; m_1, \cdots, m_{k-1}; \overline{m_k, m_{k+1}, \cdots, m_{l+k-1}}].$$

An infinite simple continued fraction γ is called **purely periodic** if $\gamma = [\overline{m_0; m_1, \cdots, m_{l-1}}]$ with **period length** l. If $\gamma = \frac{P+\sqrt{D}}{Q}$ is a quadratic irrational, then $I_\gamma = [Q, P + \sqrt{D}]$ is a quadratic ideal and $F_\gamma(x, y) = Q(x + \gamma y)(x + \overline{\gamma} y)$ is an indefinite quadratic form of discriminant $\Delta = 4D$ (see also [10], [13] and [14]).

2. Quadratics.

Let $k \in \mathbb{Z}^+$ and let $D = k^2+4, P = k+2, Q = k$. Then $\gamma = \frac{k+2+\sqrt{k^2+4}}{k}$ is a quadratic irrational. So

$$I_\gamma = [k,\, k+2+\sqrt{k^2+4}]$$

is a quadratic ideal and

$$F_\gamma = (k, 2k+4, 4)$$

is an indefinite quadratic form of discriminant $\Delta = 4D$.

2.1. Case 1) Let $k \geq 1$ be odd.

Then we can give the following results.

Theorem 2.1. *If $k = 1$, then*

1. *the continued fraction expansion of $\gamma = 3+\sqrt{5}$ is $\gamma = [5; \overline{4}]$ with period length 1.*
2. *the cycle of $I_\gamma = [1, 3+\sqrt{5}]$ is $I_{\gamma_0} = [1, 3+\sqrt{5}] \sim I_{\gamma_1} = [1, 2+\sqrt{5}]$ of length 2.*
3. *the right neighbors of $F_\gamma = (1, 6, 4)$ are*
$$R^1(F_\gamma) = (4, 2, -1), R^2(F_\gamma) = (-1, 4, 1), R^3(F_\gamma) = (1, 4, -1)$$
and the left neighbors are
$$L^1(F_\gamma) = (-1, 4, 1), L^2(F_\gamma) = (1, 4, -1).$$
4. *$Aut^+(F_\gamma) = \{\pm(g_{F_\gamma,4} g_{F_\gamma,2}^{-1})^t : t \in \mathbb{Z}\}$, where $g_{F_\gamma,4} g_{F_\gamma,2}^{-1} = \begin{bmatrix} -21 & 4 \\ -16 & 3 \end{bmatrix}$.*

Proof. (1) Let $\gamma = 3+\sqrt{5}$. Then we easily get

$$\gamma = 5 + (-2+\sqrt{5}) = 5 + \frac{1}{4+(-2+\sqrt{5})}.$$

So $\gamma = [5; \overline{4}]$.

(2) Let $I_\gamma = [1, 3+\sqrt{5}]$. Then from (1.5) we get $m_0 = 5$ and hence $P_1 = 2$, $Q_1 = 1$. For $i = 1$, we get $m_1 = 4$ and $P_2 = 2 = P_1$ and $Q_2 = 1 = Q_1$. So the cycle of I_γ is $I_{\gamma_0} = [1, 3+\sqrt{5}] \sim I_{\gamma_1} = [1,\, 2+\sqrt{5}]$.

(3) For the form $F_\gamma = (1, 6, 4)$, we have Table 1.

So the result is obvious since $R^4(F_\gamma) = R^2(F_\gamma)$. For the left neighbors, we have from (1.4) that

$$\begin{aligned} L^1(F_\gamma) &= \chi\tau R(4, 6, 1) = \chi\tau(1, 4, -1) = (-1, 4, 1) \\ L^2(F_\gamma) &= \chi\tau R(1, 4, -1) = \chi\tau(-1, 4, 1) = (1, 4, -1) \\ L^3(F_\gamma) &= \chi\tau R(-1, 4, 1) = \chi\tau(1, 4, -1) = (-1, 4, 1) = L^1(F_\gamma). \end{aligned}$$

i	0	1	**2**	3	**4**
A_i	1	4	$\mathbf{-1}$	1	$\mathbf{-1}$
B_i	6	2	**4**	4	**4**
C_i	4	-1	**1**	-1	**1**
δ_i	1	-3	4	-4	

Table 1

So the left neighbors of F_γ are $L^1(F_\gamma) = (-1, 4, 1), L^2(F_\gamma) = (1, 4, -1)$.

(4) For the matrix $\begin{bmatrix} 0 & -1 \\ 1 & -\delta \end{bmatrix}$ defined in (1.2), we set $T(\delta) = \begin{bmatrix} 0 & -1 \\ 1 & -\delta \end{bmatrix}^{-1} = \begin{bmatrix} -\delta & 1 \\ -1 & 0 \end{bmatrix}$ and define $g_{F,n} = T(\delta_0)T(\delta_1)\cdots T(\delta_{n-1})$, where δ is defined in (1.3). Then

$$g_{F_\gamma,4} = \begin{bmatrix} 72 & 17 \\ 55 & 13 \end{bmatrix} \text{ and } g_{F_\gamma,2} = \begin{bmatrix} -4 & -1 \\ -3 & -1 \end{bmatrix}.$$

Thus $g_{F_\gamma,4}g^{-1}_{F_\gamma,2} = \begin{bmatrix} -21 & 4 \\ -16 & 3 \end{bmatrix}$ and hence the result is clear from [6, Corollary 9.5].

Theorem 2.2. *If $k \geq 3$, then*

1. *the continued fraction expansion of γ is $\gamma = [2; \overline{\frac{k-1}{2}, 2k, \frac{k-1}{2}, 1, 1}]$ with period length 5.*
2. *the cycle of I_γ is*

$$\begin{aligned} I_{\gamma_0} &= [k, k+2+\sqrt{k^2+4}] \sim I_{\gamma_1} = [4, k-2+\sqrt{k^2+4}] \sim \\ I_{\gamma_2} &= [1, k+\sqrt{k^2+4}] \sim I_{\gamma_3} = [4, k+\sqrt{k^2+4}] \sim \\ I_{\gamma_4} &= [k, k-2+\sqrt{k^2+4}] \sim I_{\gamma_5} = [k, 2+\sqrt{k^2+4}] \end{aligned}$$

of length 6.

3. *the right neighbors of F_γ are*

$$\begin{aligned} &R^1(F_\gamma) = (4, 2k, -1), R^2(F_\gamma) = (-1, 2k, 4), R^3(F_\gamma) = (4, 2k-4, -k), \\ &R^4(F_\gamma) = (-k, 4, k), R^5(F_\gamma) = (k, 2k-4, -4), R^6(F_\gamma) = (-4, 2k, 1), \\ &R^7(F_\gamma) = (1, 2k, -4), R^8(F_\gamma) = (-4, 2k-4, k), R^9(F_\gamma) = (k, 4, -k), \\ &R^{10}(F_\gamma) = (-k, 2k-4, 4) \end{aligned}$$

and the left neighbors are

$$\begin{aligned} &L^1(F_\gamma) = (-4, 2k-4, k), L^2(F_\gamma) = (1, 2k, -4), L^3(F_\gamma) = (-4, 2k, 1), \\ &L^4(F_\gamma) = (k, 2k-4, -4), L^5(F_\gamma) = (-k, 4, k), L^6(F_\gamma) = (4, 2k-4, -k), \\ &L^7(F_\gamma) = (-1, 2k, 4), L^8(F_\gamma) = (4, 2k, -1), L^9(F_\gamma) = (-k, 2k-4, 4), \\ &L^{10}(F_\gamma) = (k, 4, -k). \end{aligned}$$

4. $Aut^+(F_\gamma) = \{\pm(g_{F_\gamma,11}g_{F_\gamma,1}^{-1})^t : t \in \mathbb{Z}\}$ *for* $g_{F_\gamma,11}g_{F_\gamma,1}^{-1} = \begin{bmatrix} R & S \\ T & U \end{bmatrix}$, *where* $R = -k^6-k^5-5k^4-4k^3-6k^2-3k-1$, $S = \frac{k^6+4k^4+3k^2}{2}$, $T = -2k^5-8k^3-6k$ *and* $U = k^5-k^4+4k^3-3k^2+3k-1$.

Proof. **(1)** Let $\gamma = \frac{k+2+\sqrt{k^2+4}}{k}$. Then we easily get

$$\gamma = 2 + (\frac{k+2+\sqrt{k^2+4}}{k} - 2) = 2 + \cfrac{1}{\frac{k-1}{2} + \cfrac{1}{2k + \cfrac{1}{\frac{k-1}{2} + \cfrac{1}{1 + \cfrac{1}{1 + (\frac{k+2+\sqrt{k^2+4}}{k} - 2)}}}}}.$$

So $\gamma = [2; \overline{\frac{k-1}{2}, 2k, \frac{k-1}{2}, 1, 1}]$.

(2) For the ideal $I_{\gamma_0} = [k, k+2+\sqrt{k^2+4}]$, we get Table 2:

i	0	**1**	2	3	4	5	**6**
P_i	$k+2$	$\mathbf{k-2}$	k	k	$k-2$	2	$\mathbf{k-2}$
Q_i	k	**4**	1	4	k	k	**4**
m_i	2	$\mathbf{\frac{k-1}{2}}$	$2k$	$\frac{k-1}{2}$	1	1	

Table 2

So the cycle of I_γ is $I_{\gamma_0} = [k, k+2+\sqrt{k^2+4}] \sim I_{\gamma_1} = [4, k-2+\sqrt{k^2+4}] \sim I_{\gamma_2} = [1, k+\sqrt{k^2+4}] \sim I_{\gamma_3} = [4, k+\sqrt{k^2+4}] \sim I_{\gamma_4} = [k,\ k-2+\sqrt{k^2+4}] \sim I_{\gamma_5} = [k, 2+\sqrt{k^2+4}]$.

(3) For the form $F_\gamma = (k, 2k+4, 4)$, we get Table 3:

i	A_i	B_i	C_i	δ_i
0	k	$2k+4$	4	$\frac{k+1}{2}$
1	**4**	**2k**	**−1**	**−2k**
2	-1	$2k$	4	$\frac{k-1}{2}$
3	4	$2k-4$	$-k$	-1
4	$-k$	4	k	1
5	k	$2k-4$	-4	$\frac{1-k}{2}$
6	-4	$2k$	1	$2k$
7	1	$2k$	-4	$\frac{1-k}{2}$
8	-4	$2k-4$	k	1
9	k	4	$-k$	-1
10	$-k$	$2k-4$	4	$\frac{k-1}{2}$
11	**4**	**2k**	**−1**	

Table 3

So the result is obvious since $R^{11}(F_\gamma)=R^1(F_\gamma)$. For the left neighbors, we get

$$\begin{aligned}
L^1(F_\gamma)&=\chi\tau R(4,2k+4,k)=(-4,2k-4,k)\\
L^2(F_\gamma)&=\chi\tau R(k,2k-4,-4)=(1,2k,-4)\\
L^3(F_\gamma)&=\chi\tau R(-4,2k,1)=(-4,2k,1)\\
L^4(F_\gamma)&=\chi\tau R(1,2k,-4)=(k,2k-4,-4)\\
L^5(F_\gamma)&=\chi\tau R(-4,2k-4,k)=(-k,4,k)\\
L^6(F_\gamma)&=\chi\tau R(k,4,-k)=(4,2k-4,-k)\\
L^7(F_\gamma)&=\chi\tau R(-k,2k-4,4)=(-1,2k,4)\\
L^8(F_\gamma)&=\chi\tau R(4,2k,-1)=(4,2k,-1)\\
L^9(F_\gamma)&=\chi\tau R(-1,2k,4)=(-k,2k-4,4)\\
L^{10}(F_\gamma)&=\chi\tau R(4,2k-4,-k)=(k,4,-k)\\
L^{11}(F_\gamma)&=\chi\tau R(-k,4,k)=(-4,2k-4,k)=L^1(F_\gamma).
\end{aligned}$$

(4) We see as above that $R^{11}(F_\gamma)=R^1(F_\gamma)$. So $n=11$ and $m=1$. Thus

$$g_{F_\gamma,11}=\begin{bmatrix}\frac{k^7+k^6+6k^5+5k^4+10k^3+6k^2+4k+1}{2} & -k^6-k^5-5k^4-4k^3-6k^2-3k-1\\ k^6+5k^4+6k^2+1 & -2k^5-8k^3-6k\end{bmatrix}$$

and $g_{F_\gamma,1}=\begin{bmatrix}-\frac{k+1}{2} & 1\\ -1 & 0\end{bmatrix}$. So $g_{F_\gamma,11}g^{-1}_{F_\gamma,1}=\begin{bmatrix}R & S\\ T & U\end{bmatrix}$, where $R=-k^6-k^5-5k^4-4k^3-6k^2-3k-1$, $S=\frac{k^6+4k^4+3k^2}{2}$, $T=-2k^5-8k^3-6k$ and $U=k^5-k^4+4k^3-3k^2+3k-1$. Thus the set of proper automorphisms of F_γ is $Aut^+(F_\gamma)=\{\pm(g_{F_\gamma,11}g^{-1}_{F_\gamma,1})^t:t\in\mathbb{Z}\}$.

2.2. Case 2) Let $k\geq 2$ be even.

We can give the following results without giving their proofs since they can be proved as in the same way that Theorems in Case 1) were proved.

Theorem 2.3. *If $k=2$, then*

1. *the continued fraction expansion of γ is $\gamma=[3;\overline{2}]$ with period length* 1.
2. *the cycle of $I_\gamma=[2,4+\sqrt{8}]$ is $I_{\gamma_0}=[2,4+\sqrt{8}]\sim I_{\gamma_1}=[2,2+\sqrt{8}]$ of length* 2.
3. *the right neighbors of $F_\gamma=(2,8,4)$ are*

$$R^1(F_\gamma)=(4,0,-2),R^2(F_\gamma)=(-2,4,2),R^3(F_\gamma)=(2,4,-2)$$

and the left neighbors are

$$L^1(F_\gamma)=(-2,4,2),L^2(F_\gamma)=(2,4,-2).$$

4. *$Aut^+(F_\gamma)=\{\pm(g_{F_\gamma,4}g^{-1}_{F_\gamma,2})^t:t\in\mathbb{Z}\}$, where*

$$g_{F_\gamma,4}g^{-1}_{F_\gamma,2}=\begin{bmatrix}-7 & 2\\ -4 & 1\end{bmatrix}.$$

Theorem 2.4. *If $k = 4$, then*

1. *the continued fraction expansion of γ is $\gamma = [2;\overline{1}]$ with period length* 1.
2. *the cycle of $I_\gamma = [4, 6+\sqrt{20}]$ is $I_{\gamma_0} = [4, 6+\sqrt{20}] \sim I_{\gamma_1} = [4, 2+\sqrt{20}]$ of length* 2.
3. *the right neighbors of $F_\gamma = (4, 12, 4)$ are*
$$R^1(F_\gamma) = (4, 4, -4), R^2(F_\gamma) = (-4, 4, 4)$$
and the left neighbors are
$$L^1(F_\gamma) = (-4, 4, 4), L^2(F_\gamma) = (4, 4, -4).$$
4. *$Aut^+(F_\gamma) = \{\pm(g_{F_\gamma,3}g_{F_\gamma,1}^{-1})^t : t \in \mathbb{Z}\}$, where*
$$g_{F_\gamma,3}g_{F_\gamma,1}^{-1} = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Theorem 2.5. *If $k \geq 6$ is even, then*

1. *the continued fraction expansion of γ is $\gamma = [2;\overline{\frac{k-2}{2}, 1, 1}]$ with period length* 3.
2. *the cycle of I_γ is $I_{\gamma_0} = [k, k+2+\sqrt{k^2+4}] \sim I_{\gamma_1} = [4, k-2+\sqrt{k^2+4}] \sim I_{\gamma_2} = [k,\ k-2+\sqrt{k^2+4}] \sim I_{\gamma_3} = [k, 2+\sqrt{k^2+4}]$ of length* 4.
3. *the right neighbors of F_γ are*
$$\begin{aligned} &R^1(F_\gamma) = (4, 2k-4, -k), R^2(F_\gamma) = (-k, 4, k), \\ &R^3(F_\gamma) = (k, 2k-4, -4), R^4(F_\gamma) = (-4, 2k-4, k), \\ &R^5(F_\gamma) = (k, 4, -k), R^6(F_\gamma) = (-k, 2k-4, 4) \end{aligned}$$
and the left neighbors of F_γ are
$$\begin{aligned} &L^1(F_\gamma) = (-4, 2k-4, k), L^2(F_\gamma) = (k, 2k-4, -4), \\ &L^3(F_\gamma) = (-k, 4, k), L^4(F_\gamma) = (4, 2k-4, -k), \\ &L^5(F_\gamma) = (-k, 2k-4, 4), L^6(F_\gamma) = (k, 4, -k). \end{aligned}$$
4. *$Aut^+(F_\gamma) = \{\pm(g_{F_\gamma,7}g_{F_\gamma,1}^{-1})^t : t \in \mathbb{Z}\}$, where*
$$g_{F_\gamma,7}g_{F_\gamma,1}^{-1} = \begin{bmatrix} k^2-k-1 & \frac{k^2}{2} \\ -2k & k-1 \end{bmatrix}.$$

3. Diophantine Equation.

Recall that the equation

$$x^2 - Dy^2 = \pm n \tag{3.1}$$

is called a **norm–form equation** since $N(x+y\sqrt{D}) = x^2 - Dy^2$ is called the **norm** of $x+y\sqrt{D}$, where D is any positive non–square integer and n is any fixed integer. When $n=1$, (3.1) is known as the **Pell equation** after John Pell (1611–1685), who actually had little to do with its solution. The Pell equation $x^2 - Dy^2 = \pm 1$ has infinitely many integer solutions. (In particular, $x^2 - Dy^2 = -1$ has infinitely many solutions when the length of the continued fraction expansion of $\sqrt{D}$ is odd). The first non–trivial positive integer solutions (x_1, y_1) is called the **fundamental solution** from which all integer solutions can be derived. Namely, if (x_1, y_1) is the fundamental solution of $x^2 - Dy^2 = 1$, then the other solutions are (x_n, y_n), where $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$ for $n \geq 1$ and if (x_1, y_1) is the fundamental solution of $x^2 - Dy^2 = -1$, then the other solutions are (x_{2n+1}, y_{2n+1}), where $x_{2n+1} + y_{2n+1}\sqrt{D} = (x_1 + y_1\sqrt{D})^{2n+1}$ for $n \geq 0$ (see [1,5,7,8,9,11]).

Let $\alpha = [q_0; q_1, \cdots, q_l]$ for $l \in \mathbb{N}$ be a finite continued fraction expansion. Define two sequences $A_{-2} = 0, A_{-1} = 1, A_k = q_k A_{k-1} + A_{k-2}$ and $B_{-2} = 1, B_{-1} = 0, B_k = q_k\ B_{k-1} + B_{k-2}$ for a nonnegative integer k. Then $C_k = \frac{A_k}{B_k}$ is the k^{th} convergent of α for any nonnegative integer $k \leq l$. Then the fundamental solution is given below.

Lemma 3.1. [9, Corollary 5.7] *If $D > 0$ is not a perfect square and $\sqrt{D}$ has continued fraction expansion of period length l, then the fundamental solution of $x^2 - Dy^2 = 1$ is given by $(x_1, y_1) = (A_{l-1}, B_{l-1})$ if l is even or (A_{2l-1}, B_{2l-1}) if l is odd. If l is odd, then the fundamental solution of $x^2 - Dy^2 = -1$ is given by $(x_1, y_1) = (A_{l-1}, B_{l-1})$.*

In this section, we try to determine the set of all integer solutions of the Diophantine equation $F_\gamma^{\pm k}(x,y) = \pm Q$, that is,

$$F_\gamma^{\pm k}(x,y) = kx^2 + (2k+4)xy + 4y^2 = \pm k. \tag{3.2}$$

Before consider this problem, we need some notations. Let Δ be a non–square discriminant. The $\Delta-$order O_Δ is defined for non square discriminants Δ to be the ring $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 (\text{mod } 4)$ or $\rho_\Delta = \frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 (\text{mod } 4)$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta})$. The **unit group** O_Δ^* is defined for nonsquare discriminants Δ to be the group of units of the ring O_Δ. Let $O_{\Delta,1}^* = \{\alpha \in O_\Delta^* : N(\alpha) = 1\}$ to be the group of units with norm 1. The **module** M_F of a quadratic form F is the $O_\Delta-$module $M_F = \{xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta})$. So we get $(u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2}$, where

$$[x' \ \ y'] = \begin{cases} [x \ \ y]\begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 (\text{mod } 4) \\ [x \ \ y]\begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 (\text{mod } 4). \end{cases} \tag{3.3}$$

Therefore, there is a bijection

$$\Psi : \Omega = \{(x, y) : F(x, y) = m\} \to \{\gamma \in M_F : N(\gamma) = am\}$$

for solving the equation $F(x, y) = m$. The action of $O^*_{\Delta,1} = \{\alpha \in O^*_\Delta : N(\alpha) = 1\}$ on the set Ω is the most interesting when Δ is a positive non–square since $O^*_{\Delta,1}$ is infinite. So the orbit of each solution will then be infinite and hence the set Ω is either empty or infinite. Since $O^*_{\Delta,1}$ can be explicitly determined, Ω is satisfactorily described by the representation of such a list, called a **set of representatives** of the orbits. Let ε_Δ be the **smallest unit** of O_Δ that is grater than 1 and let $\tau_\Delta = \varepsilon_\Delta$ if $N(\varepsilon_\Delta) = 1$; or ε^2_Δ if $N(\varepsilon_\Delta) = -1$. Then every $O^*_{\Delta,1}$ orbit of integral solutions of $F(x, y) = m$ contains a solution $(x, y) \in \mathbb{Z}^2$ such that $0 \le y \le U$, where $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}} (1-\frac{1}{\tau_\Delta})$ if $am > 0$ or $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}} (1+\frac{1}{\tau_\Delta})$ if $am < 0$. So for finding a set of representatives of the $O^*_{\Delta,1}$ orbits of $F(x, y) = m$, we must determine for which values of y, $\Delta y^2 + 4am$ is a perfect square in the range $0 \le y \le U$ since $\Delta y^2 + 4am = (2ax + by)^2$.

3.1. Case 1) Let $k \ge 1$ be odd.

In order to determine τ_Δ, we have to know the simple continued fraction expansion of $\sqrt{D}$ which is given below.

Theorem 3.2. *Simple continued fraction expansion of $\sqrt{D}$ is*

$$\sqrt{D} = \begin{cases} [2; \overline{4}] & \text{for } k = 1 \\ [k; \overline{\tfrac{k-1}{2}, 1, 1, \tfrac{k-1}{2}, 2k}] & \text{for } k \ge 3. \end{cases}$$

Proof. Let $k = 1$. Then it is easily seen that $\sqrt{5} = [2; \overline{4}]$. Let $k \ge 3$. Then we easily get

$$\sqrt{k^2+4} = k + (\sqrt{k^2+4} - k) = k + \cfrac{1}{\frac{k-1}{2} + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{k-1}{2}+\cfrac{1}{2k+(\sqrt{k^2+4}-k)}}}}}.$$

So $\sqrt{D} = [k; \overline{\frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2k}]$.

By virtue of Lemma 3.1, we get form Theorem 3.2 that $A_9 = (k^6 + 6k^4 + 9k^2 + 2)/2$ and $B_9 = (k^5 + 4k^3 + 3k)/2$. So $\tau_\Delta = \frac{k^6+6k^4+9k^2+2}{2} + (\frac{k^5+4k^3+3k}{2})\sqrt{k^2+4}$ since $N(\tau_\Delta) = 1$. For the positive Diophantine equation

$$F^k_\gamma(x, y) = kx^2 + (2k + 4)xy + 4y^2 = k,$$

we have two cases:

Case 1) Let $k = 1$. Then the set of representatives is $\{[\pm 1 \;\; 0]\}$, and in this case $[1 \;\; 0]M^n$ generates the solutions (x_{2n}, y_{2n}) for $n \ge 1$ and $[-1 \;\; 0]M^{-n}$ generates the solutions (x_{2n+1}, y_{2n+1}) for $n \ge 0$, where $M = \begin{bmatrix} -3 & 4 \\ -16 & 21 \end{bmatrix}$ by (3.3). Thus we can give the following theorem.

Theorem 3.3. *If $k=1$, then the set of all integer solutions of F_γ^1 is*

$$\Psi(F_\gamma^1)=\pm\{(x_{2n+1},y_{2n+1})_{n\geq 0},(x_{2n},y_{2n})_{n\geq 1}\},$$

where

$$[x_{2n+1}\quad y_{2n+1}]=[-1\quad 0]M^{-n} \text{ for } n\geq 0$$
$$[x_{2n}\quad y_{2n}]=[1\quad 0]M^{n} \text{ for } n\geq 1,$$

and $M=\begin{bmatrix}-3 & 4\\ -16 & 21\end{bmatrix}$.

Case 2) Let $k\geq 3$ be an integer. Then we have two cases.

(i) If k is not a perfect square, then the set of representatives is

$$\{[\pm 1\quad 0],[-k^3+k^2-2k+1\quad \frac{k^3-2k^2+3k-2}{2}]\}.$$

Also

$$M=\begin{bmatrix}-k^5+k^4-4k^3+3k^2-3k+1 & \frac{k^6+4k^4+3k^2}{2}\\ -2k^5-8k^3-6k & k^6+k^5+5k^4+4k^3+6k^2+3k+1\end{bmatrix} \tag{3.4}$$

by (3.3). Here we see that

1. $[1\quad 0]M^n$ generates the solutions (x_{4n},y_{4n}) for $n\geq 1$,
2. $[-1\quad 0]M^{-n}$ generates the solutions (x_{4n+1},y_{4n+1}) for $n\geq 0$,
3. $[k^3-k^2+2k-1\quad \frac{-k^3+2k^2-3k+2}{2}]M^n$ generates the solutions (x_{4n-2},y_{4n-2}) for $n\geq 1$,
4. $[-k^3+k^2-2k+1\quad \frac{k^3-2k^2+3k-2}{2}]M^{-n}$ generates the solutions (x_{4n+3},y_{4n+3}) for $n\geq 0$.

Thus we can give the following theorem.

Theorem 3.4. *The set of all integer solutions of F_γ^k is*

$$\Psi(F_\gamma^k)=\pm\{(x_{4n+1},y_{4n+1})_{n\geq 0},(x_{4n+3},y_{4n+3})_{n\geq 0},$$
$$(x_{4n-2},y_{4n-2})_{n\geq 1},(x_{4n},y_{4n})_{n\geq 1}\},$$

where

$$[x_{4n+1}\quad y_{4n+1}]=[-1\quad 0]M^{-n},\ n\geq 0$$
$$[x_{4n-2}\quad y_{4n-2}]=[k^3-k^2+2k-1\quad \frac{-k^3+2k^2-3k+2}{2}]M^{n},\ n\geq 1$$
$$[x_{4n+3}\quad y_{4n+3}]=[-k^3+k^2-2k+1\quad \frac{k^3-2k^2+3k-2}{2}]M^{-n},\ n\geq 0$$
$$[x_{4n}\quad y_{4n}]=[1\quad 0]M^{n},\ n\geq 1$$

and M is defined in (3.4).

(ii) If k is a perfect square, then the set of representatives is

$$\{[\pm 1 \quad 0], [-k^{\frac{3}{2}} \quad \frac{k^{\frac{3}{2}}-k^{\frac{1}{2}}}{2}], [-k^3+k^2-2k+1 \quad \frac{k^3-2k^2+3k-2}{2}]\}.$$

Here we see that

1. $[1 \quad 0]M^n$ generates the solutions (x_{6n}, y_{6n}) for $n \geq 1$,
2. $[-1 \quad 0]M^{-n}$ generates the solutions (x_{6n+1}, y_{6n+1}) for $n \geq 0$,
3. $[k^{\frac{3}{2}} \quad \frac{-k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}]M^n$ generates the solutions (x_{6n-1}, y_{6n-1}) for $n \geq 1$,
4. $[-k^{\frac{3}{2}} \quad \frac{k^{\frac{3}{2}}-k^{\frac{1}{2}}}{2}]M^{-n}$ generates the solutions (x_{6n+2}, y_{6n+2}) for $n \geq 0$,
5. $[k^3-k^2+2k-1 \quad \frac{-k^3+2k^2-3k+2}{2}]M^n$ generates the solutions (x_{6n-3}, y_{6n-3}) for $n \geq 1$,
6. $[-k^3+k^2-2k+1 \quad \frac{k^3-2k^2+3k-2}{2}]M^{-n}$ generates the solutions (x_{6n+4}, y_{6n+4}) for $n \geq 0$.

Thus we can give the following theorem.

Theorem 3.5. *The set of all integer solutions of F_γ^k is*

$$\Psi(F_\gamma^k) = \pm \left\{ \begin{array}{l} (x_{6n+1}, y_{6n+1})_{n\geq 0}, (x_{6n+2}, y_{6n+2})_{n\geq 0}, (x_{6n+4}, y_{6n+4})_{n\geq 0}, \\ (x_{6n-3}, y_{6n-3})_{n\geq 1}, (x_{6n-1}, y_{6n-1})_{n\geq 1}, (x_{6n}, y_{6n})_{n\geq 1} \end{array} \right\},$$

where

$$\begin{aligned}
[x_{6n+1} \quad y_{6n+1}] &= [-1 \quad 0]M^{-n}, \ n \geq 0 \\
[x_{6n+2} \quad y_{6n+2}] &= [-k^{\frac{3}{2}} \quad \frac{k^{\frac{3}{2}}-k^{\frac{1}{2}}}{2}]M^{-n}, \ n \geq 0 \\
[x_{6n-3} \quad y_{6n-3}] &= [k^3-k^2+2k-1 \quad \frac{-k^3+2k^2-3k+2}{2}]M^n, \ n \geq 1 \\
[x_{6n+4} \quad y_{6n+4}] &= [-k^3+k^2-2k+1 \quad \frac{k^3-2k^2+3k-2}{2}]M^{-n}, \ n \geq 0, \\
[x_{6n-1} \quad y_{6n-1}] &= [k^{\frac{3}{2}} \quad \frac{-k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}]M^n, \ n \geq 1 \\
[x_{6n} \quad y_{6n}] &= [1 \quad 0]M^n, \ n \geq 1
\end{aligned}$$

and M is defined in (3.4).

For the negative Diophantine equation

$$F_\gamma^{-k}(x,y) = kx^2 + (2k+4)xy + 4y^2 = -k,$$

we have two cases:

Case 1) Let $k = 1$. Then the set of representatives is $\{[-5\ \ 1], [-1\ \ 1]\}$, and in this case, $[-1\ \ 1]M^n$ generates the solutions (x_{2n+1}, y_{2n+1}) for $n \geq 0$ ($[-5\ \ 1]M^n$ generates the solutions (x_{2n-1}, y_{2n-1}) for $n \geq 1$, these solutions are coincide) and $[-1\ \ 1]M^{-n}$ generates the solutions (x_{2n}, y_{2n}) for $n \geq 1$ ($[-5\ \ 1]M^{-n}$ generates the solutions (x_{2n+2}, y_{2n+2}) for $n \geq 0$, these solutions are coincide). Thus we can give the following theorem.

Theorem 3.6. *If $k = 1$, then the set of all integer solutions of F_γ^{-1} is*

$$\Psi(F_\gamma^{-1}) = \pm\{(x_{2n+1}, y_{2n+1})_{n\geq 0}, (x_{2n}, y_{2n})_{n\geq 1}\},$$

where

$$[x_{2n+1}\ \ y_{2n+1}] = [-1\ \ 1]M^n \text{ for } n \geq 0$$
$$[x_{2n}\ \ y_{2n}] = [-1\ \ 1]M^{-n} \text{ for } n \geq 1$$

and $M = \begin{bmatrix} -3 & 4 \\ -16 & 21 \end{bmatrix}$.

Case 2) Now let $k \geq 3$ be an integer. Then we have two cases.

(i) If k is not a perfect square, then the set of representatives is

$$\{[-1\ \ 1], [-k^2 + k - 1\quad \frac{k^3+k}{2}], [-k^3 - k^2 - 2k - 1\quad \frac{k^3+k}{2}]\}.$$

Here we see that

1. $[-1\ \ 1]M^n$ generates the solutions (x_{4n}, y_{4n}) for $n \geq 1$,
2. $[-1\ \ 1]M^{-n}$ generates the solutions (x_{4n+1}, y_{4n+1}) for $n \geq 0$,
3. $[-k^2 + k - 1\quad \frac{k^3+k}{2}]M^n$ generates the solutions (x_{4n+2}, y_{4n+2}) for $n \geq 0$ ($[-k^3 - k^2 - 2k - 1\quad \frac{k^3+k}{2}]M^n$ generates the solutions (x_{4n-2}, y_{4n-2}) for $n \geq 1$, these solutions are coincide),
4. $[-k^2 + k - 1\quad \frac{k^3+k}{2}]M^{-n}$ generates the solutions (x_{4n-1}, y_{4n-1}) for $n \geq 1$ ($[-k^3 - k^2 - 2k - 1\quad \frac{k^3+k}{2}]M^{-n}$ generates the solutions (x_{4n+3}, y_{4n+3}) for $n \geq 0$, these solutions are coincide).

Thus we can give the following theorem.

Theorem 3.7. *If $k \geq 3$ is not a perfect square, then the set of all integer solutions of F_γ^{-k} is*

$$\Psi(F_\gamma^{-k}) = \pm\{(x_{4n+1}, y_{4n+1})_{n\geq 0}, (x_{4n+2}, y_{4n+2})_{n\geq 0},$$
$$(x_{4n-1}, y_{4n-1})_{n\geq 1}, (x_{4n}, y_{4n})_{n\geq 1}\},$$

where

$$\begin{aligned}
&[x_{4n+1} \quad y_{4n+1}] = [-1\ 1]M^{-n},\ n \geq 0 \\
&[x_{4n+2} \quad y_{4n+2}] = [-k^2+k-1 \quad \frac{k^3+k}{2}]M^{n},\ n \geq 0 \\
&[x_{4n-1} \quad y_{4n-1}] = [-k^2+k-1 \quad \frac{k^3+k}{2}]M^{-n},\ n \geq 1 \\
&[x_{4n} \quad y_{4n}] = [-1 \quad 1]M^{n},\ n \geq 1
\end{aligned}$$

and M is defined in (3.4).

(ii) If k is a perfect square, then the set of representatives is

$$\{[-1 \quad 1], [-k^{\frac{1}{2}} \quad \frac{k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}], [-k^2+k-1 \quad \frac{k^3+k}{2}], [-k^3-k^2-2k-1 \quad \frac{k^3+k}{2}]\}.$$

Here we see that

1. $[-1 \quad 1]M^n$ generates the solutions (x_{6n}, y_{6n}) for $n \geq 1$,
2. $[-1 \quad 1]M^{-n}$ generates the solutions (x_{6n+1}, y_{6n+1}) for $n \geq 0$,
3. $[-k^{\frac{1}{2}} \quad \frac{k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}]M^n$ generates the solutions (x_{6n+2}, y_{6n+2}) for $n \geq 0$,
4. $[-k^{\frac{1}{2}} \quad \frac{k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}]M^{-n}$ generates the solutions (x_{6n-1}, y_{6n-1}) for $n \geq 1$,
5. $[-k^2+k-1 \quad \frac{k^3+k}{2}]M^n$ generates the solutions (x_{6n+3}, y_{6n+3}) for $n \geq 0$ ($[-k^3-k^2-2k-1 \quad \frac{k^3+k}{2}]M^n$ generates the solutions (x_{6n-3}, y_{6n-3}) for $n \geq 1$, these solutions are coincide),
6. $[-k^2+k-1 \quad \frac{k^3+k}{2}]M^{-n}$ generates the solutions (x_{6n-2}, y_{6n-2}) for $n \geq 1$ ($[-k^3-k^2-2k-1 \quad \frac{k^3+k}{2}]M^{-n}$ generates the solutions (x_{6n+4}, y_{6n+4}) for $n \geq 0$, these solutions are coincide).

Thus we can give the following theorem.

Theorem 3.8. *If $k \geq 3$ is a perfect square, then the set of all integer solutions of F_γ^{-k} is*

$$\Psi(F_\gamma^{-k}) = \pm \left\{ \begin{array}{l} (x_{6n+1}, y_{6n+1})_{n\geq 0}, (x_{6n+2}, y_{6n+2})_{n\geq 0}, (x_{6n+3}, y_{6n+3})_{n\geq 0}, \\ (x_{6n-2}, y_{6n-2})_{n\geq 1}, (x_{6n-1}, y_{6n-1})_{n\geq 1}, (x_{6n}, y_{6n})_{n\geq 1} \end{array} \right\},$$

where

$$
\begin{aligned}
[x_{6n+1}\quad y_{6n+1}] &= [-1\quad 1]M^{-n},\ n\geq 0\\
[x_{6n+2}\quad y_{6n+2}] &= [-k^{\frac{1}{2}}\quad \frac{k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}]M^{n},\ n\geq 0\\
[x_{6n+3}\quad y_{6n+3}] &= [-k^2+k-1\quad \frac{k^3+k}{2}]M^{n},\ n\geq 0\\
[x_{6n-2}\quad y_{6n-2}] &= [-k^2+k-1\quad \frac{k^3+k}{2}]M^{-n},\ n\geq 1\\
[x_{6n-1}\quad y_{6n-1}] &= [-k^{\frac{1}{2}}\quad \frac{k^{\frac{3}{2}}+k^{\frac{1}{2}}}{2}]M^{-n},\ n\geq 1\\
[x_{6n}\quad y_{6n}] &= [-1\quad 1]M^{n},\ n\geq 1,
\end{aligned}
$$

and M is defined in (3.4).

3.2. Case 2) Let $k \geq 2$ be even.

As in Case 1), we can give the following results without giving their proofs.

Theorem 3.9. *Let $k \geq 2$ be an even integer. Then $\sqrt{D} = [k; \overline{\frac{k}{2}, 2k}]$.*

From Theorem 3.9, we get $A_1 = (k^2+2)/2$ and $B_1 = k/2$. So $\tau_\Delta = \frac{k^2+2}{2} + \frac{k}{2}\sqrt{k^2+4}$. For the positive Diophantine equation

$$F_\gamma^k(x,y) = kx^2 + (2k+4)xy + 4y^2 = k,$$

we have two cases:
Case 1) Let $k = 2$. Then the set of representatives is $\{[\pm 1\ \ 0]\}$, and in this case $[1\ \ 0]M^n$ generates the solutions (x_{2n-1}, y_{2n-1}) for $n \geq 1$ and $[-1\ \ 0]M^{-n}$ generates the solutions (x_{2n+2}, y_{2n+2}) for $n \geq 0$, where $M = \begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$. Thus we can give the following theorem.

Theorem 3.10. *If $k = 2$, then the set of all integer solutions of F_γ^2 is*

$$\Psi(F_\gamma^2) = \pm\{(x_{2n+2}, y_{2n+2})_{n\geq 0}, (x_{2n-1}, y_{2n-1})_{n\geq 1}\},$$

where

$$
\begin{aligned}
[x_{2n-1}\quad y_{2n-1}] &= [1\quad 0]M^{n} \text{ for } n\geq 1,\\
[x_{2n+2}\quad y_{2n+2}] &= [-1\quad 0]M^{-n} \text{ for } n\geq 0,
\end{aligned}
$$

and $M = \begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$.

Remark 3.11. *For $k = 2$, we can also deduce the set of all integer solutions of F_γ^2 in terms of balancing numbers* (see [2] and [12]) *as follows: It can be proved by* *induction on n that the n^{th} power of* $M = \begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$ *is*

$$M^n = \begin{bmatrix} -c_n & 2B_n \\ -4B_n & c_{n+1} \end{bmatrix}$$

for $n \geq 1$, where B_n is the n^{th} balancing number and c_n is the n^{th} Lucas–balancing number. Therefore, $[x_{2n-1} \;\; y_{2n-1}] = [-c_n \;\; 2B_n]$ *for $n \geq 1$ and* $[x_{2n+2} \;\; y_{2n+2}] = [-c_{n+1} \;\; 2B_n]$ *for $n \geq 0$. Consequently,*

$$\Psi(F_\gamma^2) = \pm\{(-c_{n+1}, 2B_n)_{n\geq 0}, (-c_n, 2B_n)_{n\geq 1}\}.$$

Case 2) Let $k > 2$ be an integer. Then we have two cases.
(i) If k is not a perfect square, then the set of representatives is

$$\{[\pm 1 \;\; 0], [1-k \;\; \frac{k-2}{2}]\}.$$

Also

$$M = \begin{bmatrix} 1-k & \frac{k^2}{2} \\ -2k & k^2+k+1 \end{bmatrix}. \tag{3.5}$$

Here we see that

1. $[1 \;\; 0]M^n$ generates the solutions (x_{4n-1}, y_{4n-1}) for $n \geq 1$,
2. $[-1 \;\; 0]M^{-n}$ generates the solutions (x_{4n+2}, y_{4n+2}) for $n \geq 0$,
3. $[k-1 \;\; \frac{2-k}{2}]M^n$ generates the solutions $(x_{4n-3},\, y_{4n-3})$ for $n \geq 1$,
4. $[1-k \;\; \frac{k-2}{2}]M^{-n}$ generates the solutions $(x_{4n+4},\, y_{4n+4})$ for $n \geq 0$.

Thus we can give the following theorem.

Theorem 3.12. *The set of all integer solutions of F_γ^k is*

$$\Psi(F_\gamma^k) = \pm\{(x_{4n+2}, y_{4n+2})_{n\geq 0}, (x_{4n+4}, y_{4n+4})_{n\geq 0}, (x_{4n-1}, y_{4n-1})_{n\geq 1}, (x_{4n-3}, y_{4n-3})_{n\geq 1}\},$$

where

$$\begin{aligned}
[x_{4n-1} \;\; y_{4n-1}] &= [1 \;\; 0]M^n, \;\; n \geq 1 \\
[x_{4n+2} \;\; y_{4n+2}] &= [-1 \;\; 0]M^{-n}, \;\; n \geq 0 \\
[x_{4n-3} \;\; y_{4n-3}] &= [k-1 \;\; \frac{2-k}{2}]M^n, \;\; n \geq 1 \\
[x_{4n+4} \;\; y_{4n+4}] &= [1-k \;\; \frac{k-2}{2}]M^{-n}, \;\; n \geq 0
\end{aligned}$$

and M is defined in (3.5).

(ii) If k is a perfect square, then the set of representatives is

$$\{[\pm 1 \quad 0], [0 \quad \frac{k^{\frac{1}{2}}}{2}], [1-k \quad \frac{k-2}{2}]\}.$$

Here we see that

1. $[1 \quad 0]M^n$ generates the solutions (x_{6n-3}, y_{6n-3}) for $n \geq 1$,
2. $[-1 \quad 0]M^{-n}$ generates the solutions (x_{6n+2}, y_{6n+2}) for $n \geq 0$,
3. $[0 \quad \frac{k^{\frac{1}{2}}}{2}]M^n$ generates the solutions (x_{6n-1}, y_{6n-1}) for $n \geq 1$,
4. $[0 \quad -\frac{k^{\frac{1}{2}}}{2}]M^{-n}$ generates the solutions (x_{6n}, y_{6n}) for $n \geq 1$,
5. $[k-1 \quad \frac{2-k}{2}]M^n$ generates the solutions (x_{6n-5}, y_{6n-5}) for $n \geq 1$,
6. $[1-k \quad \frac{k-2}{2}]M^{-n}$ generates the solutions (x_{6n+4}, y_{6n+4}) for $n \geq 0$.

Thus we can give the following theorem.

Theorem 3.13. *The set of all integer solutions of F_γ^k is*

$$\Psi(F_\gamma^k) = \pm \left\{ \begin{array}{c} (x_{6n-3}, y_{6n-3})_{n\geq 1}, (x_{6n+2}, y_{6n+2})_{n\geq 0}, (x_{6n-1}, y_{6n-1})_{n\geq 1}, \\ (x_{6n}, y_{6n})_{n\geq 1}, (x_{6n-5}, y_{6n-5})_{n\geq 1}, (x_{6n+4}, y_{6n+4})_{n\geq 0} \end{array} \right\}$$

$$\cup \{(0, \pm\frac{k^{\frac{1}{2}}}{2})\}$$

where

$$\begin{array}{l} [x_{6n-3} \quad y_{6n-3}] = [1 \quad 0]M^n, \ n \geq 1 \\ [x_{6n+2} \quad y_{6n+2}] = [-1 \quad 0]M^{-n}, n \geq 0 \\ [x_{6n-1} \quad y_{6n-1}] = [0 \quad \frac{k^{\frac{1}{2}}}{2}]M^n, \ n \geq 1 \\ [x_{6n} \quad y_{6n}] = [0 \quad -\frac{k^{\frac{1}{2}}}{2}]M^{-n}, \ n \geq 1, \\ [x_{6n-5} \quad y_{6n-5}] = [k-1 \quad \frac{2-k}{2}]M^n, \ n \geq 1 \\ [x_{6n+4} \quad y_{6n+4}] = [1-k \quad \frac{k-2}{2}]M^{-n}, \ n \geq 0 \end{array}$$

and M is defined in (3.5).

Finally, we can consider the negative Diophantine equation

$$F_\gamma^{-k}(x, y) = kx^2 + (2k+4)xy + 4y^2 = -k.$$

Again we have two cases:

Case 1) Let $k = 2$. Then the set of representatives is $\{[-1 \;\; 1], [-3 \;\; 1]\}$. Here $[-1 \;\; 1]M^n$ generates the solutions (x_{2n}, y_{2n}) for $n \geq 1$ ($[-3 \;\; 1]M^n$ generates the solutions (x_{2n-2}, y_{2n-2}) for $n \geq 2$, these solutions are coincide) and $[-1 \;\; 1]M^{-n}$ generates the solutions (x_{2n+1}, y_{2n+1}) for $n \geq 0$ ($[-3 \;\; 1]M^{-n}$ generates the solutions (x_{2n+3}, y_{2n+3}) for $n \geq -1$, these solutions are coincide). Thus we can give the following theorem.

Theorem 3.14. *If $k = 2$, then the set of all integer solutions of F_γ^{-2} is*

$$\Psi(F_\gamma^{-k}) = \pm\{(x_{2n}, y_{2n})_{n\geq 1}, (x_{2n+1}, y_{2n+1})_{n\geq 0}\},$$

where

$$\begin{aligned} [x_{2n} \;\; y_{2n}] &= [-1 \;\; 1]M^n \text{ for } n \geq 1 \\ [x_{2n+1} \;\; y_{2n+1}] &= [-1 \;\; 1]M^{-n} \text{ for } n \geq 0, \end{aligned}$$

and $M = \begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$.

Remark 3.15. *Again, for $k = 2$, we can give the set of all integer solutions of F_γ^{-2} in terms of balancing numbers as*

$$\Psi(F_\gamma^{-2}) = \pm\{(c_n - 4B_n, -2B_n + c_{n+1})_{n\geq 1}, (-c_{n+1} + 4B_n, 2B_n - c_n)_{n\geq 1}, (-1, 1)\}.$$

Case 2) Now let $k > 2$ be an integer. Then we have two cases.

(i) If k is not a perfect square, then the set of representatives is

$$\{[-1 \;\; 1], [-k-1 \;\; \tfrac{k}{2}], [-1 \;\; \tfrac{k}{2}]\}.$$

Here we see that

1. $[-1 \;\; 1]M^n$ generates the solutions (x_{4n-1}, y_{4n-1}) for $n \geq 1$,
2. $[-1 \;\; 1]M^{-n}$ generates the solutions (x_{4n+2}, y_{4n+2}) for $n \geq 0$,
3. $[-k-1 \;\; \frac{k}{2}]M^n$ generates the solutions (x_{4n-3}, y_{4n-3}) for $n \geq 1$ ($[-1 \;\; \frac{k}{2}]M^n$ generates the solutions (x_{4n+1}, y_{4n+1}) for $n \geq 0$, these solutions are coincide),
4. $[-k-1 \;\; \frac{k}{2}]M^{-n}$ generates the solutions (x_{4n+4}, y_{4n+4}) for $n \geq 0$ ($[-1 \;\; \frac{k}{2}]M^{-n}$ generates the solutions (x_{4n}, y_{4n}) for $n \geq 1$, these solutions are coincide),

Thus we can give the following theorem.

Theorem 3.16. *If $k > 2$ is not a perfect square, then the set of all integer solutions of F_γ^{-k} is*

$$\begin{aligned} \Psi(F_\gamma^{-k}) = \pm\{&(x_{4n-1}, y_{4n-1})_{n\geq 1}, (x_{4n+2}, y_{4n+2})_{n\geq 0}, \\ &(x_{4n-3}, y_{4n-3})_{n\geq 1}, (x_{4n+4}, y_{4n+4})_{n\geq 0}\}, \end{aligned}$$

where

$$[x_{4n-1} \quad y_{4n-1}] = [-1 \quad 1]M^n, \ n \geq 1$$
$$[x_{4n+2} \quad y_{4n+2}] = [-1 \quad 1]M^{-n}, \ n \geq 0$$
$$[x_{4n-3} \quad y_{4n-3}] = [-k-1 \quad \frac{k}{2}]M^n, \ n \geq 1$$
$$[x_{4n+4} \quad y_{4n+4}] = [-k-1 \quad \frac{k}{2}]M^{-n}, \ n \geq 0$$

and M is defined in (3.5).

(ii) If k is a perfect square, then the set of representatives is

$$\{[-1 \quad 1], [-k^{\frac{1}{2}} \quad \frac{k^{\frac{1}{2}}}{2}], [-k-1 \quad \frac{k}{2}], [-1 \quad \frac{k}{2}]\}.$$

Here we see that

1. $[-1 \quad 1]M^n$ generates the solutions (x_{6n-1}, y_{6n-1}) for $n \geq 1$,
2. $[-1 \quad 1]M^{-n}$ generates the solutions (x_{6n+2}, y_{6n+2}) for $n \geq 0$,
3. $[-k^{\frac{1}{2}} \quad \frac{k^{\frac{1}{2}}}{2}]M^n$ generates the solutions (x_{6n-3}, y_{6n-3}) for $n \geq 1$,
4. $[-k^{\frac{1}{2}} \quad \frac{k^{\frac{1}{2}}}{2}]M^{-n}$ generates the solutions (x_{6n+4}, y_{6n+4}) for $n \geq 0$,
5. $[-k-1 \quad \frac{k}{2}]M^n$ generates the solutions (x_{6n-5}, y_{6n-5}) for $n \geq 1$, ($[-1 \quad \frac{k}{2}]M^n$ generates the solutions (x_{6n+1}, y_{6n+1}) for $n \geq 0$, these solutions are coincide),
6. $[-k-1 \quad \frac{k}{2}]M^{-n}$ generates the solutions (x_{6n+6}, y_{6n+6}) for $n \geq 0$, ($[-1 \quad \frac{k}{2}]M^{-n}$ generates the solutions (x_{6n}, y_{6n}) for $n \geq 1$, these solutions are coincide),

Thus we can give the following theorem.

Theorem 3.17. *If $k > 2$ is a perfect square, then the set of all integer solutions of F_γ^{-k} is*

$$\Psi(F_\gamma^{-k}) = \pm \left\{ \begin{array}{l} (x_{6n-1}, y_{6n-1})_{n\geq 1}, (x_{6n+2}, y_{6n+2})_{n\geq 0}, (x_{6n-3}, y_{6n-3})_{n\geq 1}, \\ (x_{6n+4}, y_{6n+4})_{n\geq 0}, (x_{6n-5}, y_{6n-5})_{n\geq 1}, (x_{6n+6}, y_{6n+6})_{n\geq 0} \end{array} \right\},$$

where

$$[x_{6n-1} \quad y_{6n-1}] = [-1 \quad 1]M^n, \ n \geq 1$$
$$[x_{6n+2} \quad y_{6n+2}] = [-1 \quad 1]M^{-n}, \ n \geq 0$$
$$[x_{6n-3} \quad y_{6n-3}] = [-k^{\frac{1}{2}} \quad \frac{k^{\frac{1}{2}}}{2}]M^n, \ n \geq 1$$
$$[x_{6n+4} \quad y_{6n+4}] = [-k^{\frac{1}{2}} \quad \frac{k^{\frac{1}{2}}}{2}]M^{-n}, \ n \geq 0$$
$$[x_{6n-5} \quad y_{6n-5}] = [-k-1 \quad \frac{k}{2}]M^n, \ n \geq 1$$
$$[x_{6n+6} \quad y_{6n+6}] = [-k-1 \quad \frac{k}{2}]M^{-n}, \ n \geq 0$$

and M is defined in (3.5).

4. Conclusion.

In Section 2, we derived the set of proper automorphisms of $F_\gamma = (k, 2k+4, 4)$ by considering its right neighbors. But, there is another way to get the set $Aut^+(F_\gamma)$ and also the set $Aut^*(F_\gamma)$ given below.

Theorem 4.1. *For the form $F_\gamma = (k, 2k+4, 4)$, we have*

$$Aut^+(F_\gamma) = \{\pm M^t : t \in \mathbb{Z}\} \text{ and } Aut^*(F_\gamma) = \{\pm (M^*)^{2t+1} : t \in \mathbb{Z}\},$$

where M is defined in (3.4) *and*

$$M^* = \begin{bmatrix} -k^2+k-1 & \frac{k^3+k}{2} \\ -2k^2-2 & k^3+k^2+2k+1 \end{bmatrix}$$

for every integer $k \geq 1$.

Proof. First note that for the matrix M in (3.4), we have $\det(M) = 1$. Also from (1.1) we get

$$\begin{aligned} MF_\gamma &= F_\gamma \begin{pmatrix} (-k^5+k^4-4k^3+3k^2-3k+1)x + (-2k^5-8k^3-6k)y, \\ (\frac{k^6+4k^4+3k^2}{2})x + (k^6+k^5+5k^4+4k^3+6k^2+3k+1)y \end{pmatrix} \\ &= k((-k^5+k^4-4k^3+3k^2-3k+1)x + (-2k^5-8k^3-6k)y)^2 \\ &\quad + (2k+4)((-k^5+k^4-4k^3+3k^2-3k+1)x + (-2k^5-8k^3-6k)y) \\ &\quad \times ((\frac{k^6+4k^4+3k^2}{2})x + (k^6+k^5+5k^4+4k^3+6k^2+3k+1)y) \\ &\quad + 4((\frac{k^6+4k^4+3k^2}{2})x + (k^6+k^5+5k^4+4k^3+6k^2+3k+1)y)^2 \\ &= kx^2 + (2k+4)xy + 4y^2 \\ &= F_\gamma. \end{aligned}$$

So M is a proper automorphism of F_γ. It can be proved by induction on t that M^t is also a proper automorphism. So $Aut^+(F_\gamma) = \{\pm M^t : t \in \mathbb{Z}\}$.

For the second matrix M^*, we have $\det(M^*) = -1$ and also

$$\begin{aligned} M^*F_\gamma &= F_\gamma((-k^2+k-1)x + (-2k^2-2)y, (\frac{k^3+k}{2})x + (k^3+k^2+2k+1)y) \\ &= k((-k^2+k-1)x + (-2k^2-2)y)^2 \\ &\quad + (2k+4)((-k^2+k-1)x + (-2k^2-2)y) \\ &\quad \times ((\frac{k^3+k}{2})x + (k^3+k^2+2k+1)y) + 4((\frac{k^3+k}{2})x + (k^3+k^2+2k+1)y)^2 \\ &= -kx^2 - (2k+4)xy - 4y^2 \\ &= -F_\gamma. \end{aligned}$$

So $M^* \in Aut^*(F_\gamma)$. It can be proved by induction that $Aut^*(F_\gamma) = \{\pm (M^*)^{2t+1} : t \in \mathbb{Z}\}$.

Remark 4.2. *In above theorem we see that odd powers of M^* are in $Aut^*(F_\gamma)$. Even powers of M^* are in $Aut^+(F_\gamma)$. This is because of $(M^*)^2 = M$.*

References

1. E. Barbeau. *Pell's Equation.* Springer Verlag, 2003.
2. A. Behera and G.K. Panda. *On the Square Roots of Triangular Numbers.* The Fibonacci Quarterly, **37**(2)(1999), 98-105.
3. J. Buchmann and U. Vollmer. *Binary Quadratic Forms: An Algorithmic Approach.* Springer-Verlag, Berlin, Heidelberg, 2007.
4. D.A. Buell. *Binary Quadratic Forms, Clasical Theory and Modern Computations.* Springer-Verlag, New York, 1989.
5. H.M. Edwards. *Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory.* Corrected reprint of the 1977 original. Graduate Texts in Mathematics, 50. Springer-Verlag, New York, 1996.
6. D.E. Flath. *Introduction to Number Theory.* Wiley, 1989.
7. M. Jacobson and H. Williams. *Solving the Pell Equation, CMS Books in Mathematics.* Springer, 2010.
8. K. Matthews. *The Diophantine Equation* $x^2 - Dy^2 = N, D > 0$. Expositiones Math. **18**(2000), 323–331.
9. R.A. Mollin. *Quadratics.* CRS Press, Boca Raton, New York, London, Tokyo, 1996.
10. R.A. Mollin. *Fundamental Number Theory with Applications.* Second Edition (Discrete Mathematics and Its Applications) Chapman & Hall/ CRC, Boca Raton, London, New York, 2008.
11. I. Niven, H.S. Zuckerman and H.L. Montgomery. *An Introduction to the Theory of Numbers.* Fifth Edition, John Wiley&Sons, Inc., New York, 1991.
12. G.K. Panda. *Some Fascinating Properties of Balancing Numbers.* Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications, Cong. Numer. **194** (2009), 185–189.
13. A. Tekcan. *On Indefinite Binary Quadratic Forms and Quadratic Ideals.* Novi Sad Journal of Maths. **38**(1)(2008), 83–96.
14. A. Tekcan. *On the Cycles of Indefinite Quadratic Forms and Cycles of Ideals II.* Southeast Asian Bulletin of Mathematics **34**(2010), 185–192.

A.Tekcan, S.Kutlu,
Uludag University
Faculty of Science
Department of Mathematics
Bursa, 16059, Turkiye.
E-mail address: tekcan@uludag.edu.tr, seymakutlu@gmail.com