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A note on eventually periodic endomorphisms and their maximizing measures

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abstract: Given an eventually periodic endomorphism T defined on a compact metric space K we constructed another endomorphism $\tilde{T} : K \to K$ that is C^0-close of T, has a nonperiodic orbit and such that $\sup_{\mu \in M_{\tilde{T}}} \int f d\mu \leq \sup_{\mu \in M_T} \int f d\mu$.

Key Words: Eventually Periodic Endomorphism, Maximizing measures

Contents

1. Introduction

Ergodic optimization is a relatively new field that use tecnhiques from ergodic theory and optimization to study the following problem: Given a metric space K, a potential function $f : K \to \mathbb{R}$ and a function $T : K \to K$, do exist T-invariant measures that maximize the functional $P_f : \mathcal{M}_T(K) \to \mathbb{R}$, $P_f(\mu) = \int f d\mu$ and how to characterize these maximizing measures in terms of their support?

Several problems can be put under this context, like finding Lyapunov exponents, action minimizing solutions to Lagrangian systems and the zero temperature limits of Gibbs equilibrium states in thermodynamical formalism. Some examples of this are [9,7,6,5,3,10,11]. A good introduction to the subject is [8], where the fundamental results of the theory are displayed and recentelly, in [4] an important conjecture was proved.

A research line that arises from this theory, seeks to characterize, when K is a compact metric space, f is a fixed continuous function and T is a surjective continuous function, the typical support of the maximizing measures, note that in this case, there exist maximizing measures due to P_f be a continuous functional defined on a compact set of T invariant Borel probability measures $\mathcal{M}_T(K)$.

Working in this line it has been showed in [1] and [2] that given an endomorphism $T : K \to K$ one can find some C^0 close endomorphism $\tilde{T} : K \to K$ with maximizing measures supported in periodic orbits. In [2] the pertubation $\tilde{T}$ of T had the property that for all point $x \in K$ there exist $n(x) \in \mathbb{N}$ such that $\tilde{T}^n(x)$ is a periodic point.

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Now given an endomorphism $T : K \to K$ it is called eventually periodic endomorphism if for all point $x \in K$ there exist some $n(x) \in \mathbb{N}$ such that $\tilde{T}^n(x)$ is a periodic point. We proved the following theorem:

Theorem 1.1. *Given a eventually periodic endomorphism T defined on a compact metric space K then there exist an endomorphim $\tilde{T} : K \to K$ that is C^0-close of T, has a nonperiodic orbit and such that $\sup_{\mu \in \mathcal{M}_{\tilde{T}}} \int f d\mu \leq \sup_{\mu \in \mathcal{M}_T} \int f d\mu$.*

Here we will denote by $\mathrm{End}(K)$ the set of continuous surjections of K endowed with the $\mathcal{C}^0$ metric, $d(T, G) = \sup_{x \in K} d(T(x), G(x))$.

The strategy of the proof is to create a perturbation $\tilde{T}$ of T with a nonperiodic orbit in a small region of K in a way all Birkoff sums will be least or equal to the $\int f d\mu = \alpha$, where μ is a f-maximizing measure for T.

The paper is organized as follows: In the next section we present some preliminary lemmas and notations, and in section 3 prove the theorem.

2. Preliminaries

We start with some notations and preliminary results. Let K be a compact metric space and $\mathrm{End}(K)$ the set of endomorphisms of K. We endow $\mathrm{End}(M)$ with its usual topology of uniform convergence defined by the metric $d(T, G) = \sup_{x \in K}(d(T(x), G(x)),\ T, G \in \mathrm{End(M)}$.

Given $T \in \mathrm{End}(M)$ we denote by $\mathcal{M}_T(K)$ the set of T invariant Borel probability measures, which is non-empty, convex and also compact in the weak-* topology. The subset of ergodic measures of $\mathcal{M}_T(K)$ is denoted by $\mathcal{M}_{erg}(T)$.

Given $f : K \to \mathbb{R}$ continuous and $T \in \mathrm{End}(K)$, we define $P_f : \mathcal{M}_T(K) \to \mathbb{R}$, $P_f(\mu) = \int f d\mu$. As the functional P_f is affine and $\mathcal{M}_T(K)$ is a convex compact set, P_f must have a maximum point at an extremal point of $\mathcal{M}_T(K)$. Since the extremal points of $\mathcal{M}_T$ are precisely the ergodic measures, there exists some $\mu_{\max} \in \mathcal{M}_{erg}(T)$ that maximizes P_f. We denote $S_nT(x) := \displaystyle\sum_{i=0}^{n-1} f(T^i(x))$ to the n Birkhoff sum of T.

The next lemmas are related to the perturbation $\tilde{T}$ of $T \in \mathrm{End}(K)$ and the set of its invarant measures $\mathcal{M}_{\tilde{T}}(K)$.

Lemma 2.1. *Given $T : K \to K$ an eventually periodic endomorphism defined in compact metric space K and $\varepsilon > 0$, then there exist an endomorphism $\tilde{T} : K \to K$, with $d(T, \tilde{T}) = \sup_{x \in K} d(T(x), \tilde{T}(x)) < \varepsilon$ such that it has a non periodic orbit for some $x \in K$.*

In order to proof lemma 2.1 we will considerate the following cases:

case 1) For all points $x \in K$ we have $T(x) = x$

case 2) For all points $x \in K$ we have $T^{k(x)}(x) = x$, where $k(x) \in \mathbb{N}$.

case 3) There is a finitely periodic orbit for some x with $n(x) > 1$, that is, for $x \in K$ there is for some $n(x) \in \mathbb{N}$ with $n(x) > 1$, such that $T^n(x)$ is a periodic point with period equal to $m(x) \in \mathbb{N}$.

Proof: To prove case 1) set a point $x_0 \in K$ and $B_\varepsilon(x_0)$ for some $\varepsilon > 0$. Inside this open ball we select any two-sided sequence $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ and for this sequence we define

$$\tilde{T}(x) = \begin{cases} x_{i+1} & \text{if } x = x_i \\ T(x) & \text{otherwise} \end{cases}$$

This function $\tilde{T} : K \to K$ is an endomorphism with $d(T, \tilde{T}) \leq 2\varepsilon$ and with a nonperiodic orbit.

For case 2): Given a point $x_0 \in K$ and $\varepsilon > 0$, we set a open ball $B_\varepsilon(x_0)$ and a sequence $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ such that $x_i \in B_\varepsilon(x_0)$ for all $i \in \mathbb{Z}$. For each x_i we have its periodic orbit $O_T(x_i) = \{x_i, T(x_i), \ldots T^{n(x_i)-1}(x_i)\}$ with period $n(x_i) \in \mathbb{N}$. The new endomorphism $\tilde{T} : K \to K$ will be defined by:

$$\tilde{T}(x) = \begin{cases} x_{i+1} & \text{if } x = T^{n(x_i)-1}(x_i) \\ T(x) & \text{otherwise} \end{cases}$$

This way $\tilde{T}$ will be an endomorphism such that has a nonperiodic orbit and $d(\tilde{T}, T) \leq 2\varepsilon$

For case 3) :

We set a eventually periodic point $x \in K$, that is, its positive orbit is

$$O_T^+(x) = \{x, T(x), \ldots, T^k(x), T^{k+1}(x), \ldots, T^{k+(n-1)}(x)\},$$

where $T^{k+n}(x) = T^k(x)$. Lets denote by $\Omega(x)$ the end point on the periodic part of the positive orbit $O_T^+(x)$, for x $\Omega(x) = T^{k+(n-1)}(x)$. Given $\varepsilon > 0$, we chose $z_1 \in B_\varepsilon(T(\Omega(x)))$ such that $z_1 \cap O_T(x) = \emptyset$, in the same way chose by induction a sequence $\{z_1, z_2, \ldots\}$ where $z_m \in B_\varepsilon(T(\Omega(z_{m-1})))$ such that $z_{m-1} \cap O_T(z_m) = \emptyset$

Now we define the endomorphism $\tilde{T} : K \to K$ by:

$$\tilde{T}(y) = \begin{cases} z_1 & , \text{ if } \quad y = \Omega(x) \\ z_i & , \text{ if } \quad y = \Omega(z_{i-1}) \\ T(y) & , \quad \text{otherwise} \end{cases}$$

Note that $\tilde{T}$ is an endomorphism such that $d(T, \tilde{T}) \leq \varepsilon$ and has a orbit that is not eventually periodic.

□

Proposition 2.2. *If $T : K \to K$ is a eventually periodic endomorphism then every measure supported on a periodic orbit is ergodic.*

Proof: Let A be a totally invariant set, that is, $T^{-1}A = A$, and $\mu \in M_T(K)$ a periodic measure suported on a periodic orbit $O(x_0)$ for some $x_0 \in K$. We have two possibilities: Or $A \cap O(x_0) = \emptyset$ in this case $A \subset (\mathrm{supp}(\mu))^c$ then $\mu(A) = 0$, or $A \cap O(x_0) \neq \emptyset$ in this case the intersection will have finitely many points, and as A is totally invariant we will have $\mu(A) = 1$. □

The next lemma is the other direction of proposition 2.2.

Lemma 2.3. *If $T : K \to K$ is a eventually periodic endomorphism defined on a compact metric space K, then all ergodic measures are periodics.*

Proof: Suppose that there is a ergodic measure μ that is not periodic, so there is $x \in \mathrm{supp}(\mu)$ a nonperiodic point and an open ball $A = B_\varepsilon(x)$ that contains x for some $\varepsilon > 0$. As T is eventually periodic we have that for all $x \in K$ there is $n(x) \in \mathbb{N}$ such that $T^n(x)$ is a periodic point, so by the Poincaré recurrence theorem $T^m(A) \cap A = \emptyset$ for some $\varepsilon > 0$ and for all $m > n(x)$, that is a contradiction. □

3. Proof of the main theorem

In this section we prove the main theorem:

Theorem 3.1. *Given a eventually periodic endomorphism $T : K \to K$ defined in a compact metric space K, $\varepsilon > 0$ and*

$$\alpha = \sup_{\mu \in M_T(K)} \int f d\mu,$$

where $f : K \to \mathbb{R}$ is a continuous function. Then there exist another endomorphism $\tilde{T} : K \to K$, with a nonperiodic orbit such that $d(T, \tilde{T}) < \varepsilon$ and

$$\beta = \sup_{\mu \in M_{\tilde{T}}(K)} \int f d\mu \leq \alpha$$

.

Proof: We will show that $\tilde{T} : K \to K$ constructed in lemma 2.1 is the endomorphism that we are looking for. By its construction we know that $d(\tilde{T}, T) < \varepsilon$ and it has a nonperiodic orbit. We need to proof that $\beta \leq \alpha$. In case 1 of lemma 2.1 since $T(x) = x$, we have that $S_n(T)(x) = nf(x)$ for all $x \in K$ and $n \in \mathbb{N}$. If x_∞ is some point in $\{x \in K : f(x) = sup_{y \in K} f(y)\}$ we will have that the Dirac measure δ_{x_∞} will be f-maximizing measure, and $\alpha = f(x_\infty)$. Note that if $x \notin \{\ldots, x_{-1}, x_0, x_1 \ldots\}$ then $S_n\tilde{T}(x) = S_nT(x) = nf(x)$ for all $n \in \mathbb{N}$, now if $x \in \{\ldots, x_{-1}, x_0, x_1 \ldots\}$ suppose $x = x_0$ we have

$$S_n\tilde{T}(x) = \sum_{i=0}^{n-1} f(\tilde{T}^i(x_0)) = \sum_{i=0}^{n-1} f(x_i) \leq nf(x_\infty).$$

As $\frac{1}{n}S_n\tilde{T}(x) \leq f(x_\infty)$ for all $x \in K$ and $n \in \mathbb{N}$, we have that $\beta \leq \alpha$.

Now we consider that we are in case 2 of lemma 2.1, we have the sequence $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ used to construct $\tilde{T}$. As any point $x \in K$ is a T-periodic point with period $m(x)$ from lemma 2.3 we have that there is a f-maximizing ergodic measure, suported in some periodic orbit, suppose $\mu = \frac{1}{n_p}\sum \delta_{T^i(x_p)}$. This way $\alpha = \frac{1}{n_p}S_{n_p}T(x_p)$ then

$$\lim \frac{1}{n}S_nT(x) \leq \frac{1}{n_p}S_{n_p}T(x),$$

for all $x \in K$. Now for $\tilde{T}$ we have that

$$\begin{aligned}
\frac{1}{n}S_n\tilde{T}(x) &= \frac{S_{n_1}T(x_1) + S_{n_2}T(x_2) + \ldots + S_{n_k}T(x_k)}{n_1 + n_2 + \ldots + n_k} \\
&\leq \frac{\frac{n_1}{n_p}S_{n_p}T(x_p) + \frac{n_2}{n_p}S_{n_p}T(x_p) + \ldots + \frac{n_k}{n_p}S_{n_p}T(x_p)}{n_1 + n_2 + \ldots + n_k} \\
&= \frac{1}{n_p}S_{n_p}T(x_p)\frac{n_1 + n_2 + \ldots + n_k}{n_1 + n_2 + \ldots + n_k} \leq \alpha
\end{aligned}$$

For case 3, just consider the sequence $\{z_1, z_2, \ldots\}$ use in this case on lemma 2.1. The rest of the proof will be similar with case 2. □

References

1. Tatiane C. Batista, Juliano S. Gonschorowksi, and Fabio A. Tal, *Density of the set of endomorphisms with a maximizing measure supported on a periodic orbit*, Discrete and Continuos Dynimical Systems **35** (2015), no. 8, 3315–3326.

2. Tatiane C. Batista, Juliano S. Gonschorowksi, and Fabio A. Tal, *Density of the set of symbolic dynamics with all ergodic measures supported on periodic orbits*, aceppted on Fundamenta Mathematicae **arXiv:1410.5857** (2015).

3. Thierry Bousch, *La condition de Walters*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 2, 287–311. MR 1841880 (2002h:37051)

4. G. Contreras, *Ground States are generically a Periodic Orbit*, ArXiv e-prints (2013).

5. G. Contreras, A. O. Lopes, and Ph. Thieullen, *Lyapunov minimizing measures for expanding maps of the circle*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1379–1409. MR 1855838 (2002i:37038)

6. Gonzalo Contreras, Jorge Delgado, and Renato Iturriaga, *Lagrangian flows: the dynamics of globally minimizing orbits. II*, Bol. Soc. Brasil. Mat. (N.S.) **28** (1997), no. 2, 155–196. MR 1479500 (98i:58093)

7. Eduardo Garibaldi and Philippe Thieullen, *Minimizing orbits in the discrete Aubry-Mather model*, Nonlinearity **24** (2011), no. 2, 563–611. MR 2765475 (2012b:37166)

8. Oliver Jenkinson, *Ergodic optimization*, Discrete Contin. Dyn. Syst. **15** (2006), no. 1, 197–224. MR 2191393 (2009i:37075)

9. Oliver Jenkinson and Ian D. Morris, *Lyapunov optimizing measures for C^1 expanding maps of the circle*, Ergodic Theory Dynam. Systems **28** (2008), no. 6, 1849–1860. MR 2465602 (2010b:37081)

10. Artur O. Lopes and Philippe Thieullen, *Sub-actions for Anosov diffeomorphisms*, Astérisque (2003), no. 287, xix, 135–146, Geometric methods in dynamics. II. MR 2040005 (2005c:37047)

11. Ian D. Morris, *Ergodic optimization for generic continuous functions*, Discrete Contin. Dyn. Syst. **27** (2010), no. 1, 383–388. MR 2600778 (2011e:37006)

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