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Some Identities Involving Multiplicative Generalized Derivations in Prime and Semiprime Rings*

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ABSTRACT: Let R be a ring with center $Z(R)$. A mapping $F : R \to R$ is called a multiplicative generalized derivation, if $F(xy) = F(x)y + xg(y)$ is fulfilled for all $x, y \in R$, where $g : R \to R$ is a derivation. In the present paper, our main object is to study the situations: (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$, (5) $F(xy) - g(y)F(x) \in Z(R)$; for all x, y in some suitable subset of R.

Key Words: Semiprime ring, derivation, generalized derivation, multiplicative generalized derivation.

Contents

1. Introduction

Let R be an associative ring with center $Z(R)$. For $x, y \in R$, $[x, y]$ stands for the commutator element $xy - yx$. Recall that a ring R is called prime, if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized derivation of R, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for any $x, y \in R$. If $d = 0$, then F is said to be a left centralizer map of R. The notion of generalized derivation was introduced by Brešar [5].

It is natural to investigate the above mappings without assumption of additivity condition. A mapping $D : R \to R$ (not necessarily additive) which satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$ is called a multiplicative derivation of R. Daif [9] introduced the concept of multiplicative derivations and it was motivated by the work of Martindale [16]. Then Daif and Tammam-El-Sayiad [8] introduced

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the notion of the multiplicative generalized derivation. The multiplicative generalized derivation of a ring R is a mapping $g : R \to R$ such that $g(xy) = g(x)y + xd(y)$, for all $x, y \in R$, where d is a derivation of R. Of course the mapping g is not necessarily additive. Thus multiplicative generalized derivations are the large number of maps containing derivations, generalized derivations and left multiplier maps etc. One can find an example of multiplicative generalized derivation, which is neither a derivation, nor a generalized derivation.

Example 1.1 Let $R = \begin{pmatrix} 0 & GF(2) & GF(2) \\ 0 & 0 & GF(2) \\ 0 & 0 & 0 \end{pmatrix}$. Define the mappings d and $F : R \longrightarrow R$ as follows: $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & ac^2 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is straightforward to verify that d is a derivation in R and F is not additive map in R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Hence, F is a multiplicative generalized derivation associated with a derivation d, but F is not a generalized derivation of R.

Let S be a nonempty subset of a ring R. The mapping $F : R \to R$ is said to be a homomorphism (anti-homomorphism) acting on S, if $F(xy) = F(x)F(y)$ holds for all $x, y \in S$ (respectively $F(xy) = F(y)F(x)$ holds for all $x, y \in S$).

In [3], Bell and Kappe showed that if a derivation d of a prime ring R can act as homomorphism or anti-homomorphism on a nonzero right ideal of R, then $d = 0$ on R. Then Ali, Rehman and Ali in [2] proved a similar result to Lie ideal case. They proved that if R is a 2-torsion free prime ring, L a nonzero Lie ideal of R such that $u^2 \in L$ for all $u \in L$ and d acts as a homomorphism or anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$.

On the other hand, the authors developed above results, replacing the derivation d with a generalized derivation F of R. In this view, Rehman [17] proved the following:

Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. Suppose $F : R \to R$ is a nonzero generalized derivation with d.

(i) If F acts as a homomorphism on I and if $d \neq 0$, then R is commutative.

(ii) If F acts as an anti-homomorphism on I and if $d \neq 0$, then R is commutative.

Recently, in [12] the first author of this article has studied the situations, when a generalized derivation F of a semiprime ring R acts as homomorphism or anti-homomorphism in a nonzero left ideal of R.

From above results, it is natural to consider the situations, when the generalized derivations F satisfy the identities: (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$; for all x, y in some suitable subset of R.

Albas [1] studied the above mentioned identities in prime rings. Albas proved the following theorems:

Theorem 1. *Let R be a prime ring with center $Z(R)$ and I be a nonzero ideal of R. If R admits a nonzero generalized derivation F of R, with associated derivation d such that $F(xy) - F(x)F(y) \in Z(R)$ or $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in I$, then either R is commutative or $F = I_{id}$ or $F = -I_{id}$, where I_{id} denotes the identity map of the ring R.*

Theorem 2. *Let R be a prime ring with center $Z(R)$ and I be a nonzero ideal of R. If R admits a nonzero generalized derivation F of R, with associated derivation d such that $F(xy) - F(y)F(x) \in Z(R)$ or $F(xy) + F(y)F(x) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Recently, in [11], Dhara et al. studied these situations of Albas [1] in semiprime rings.

In the present paper, our main object is to investigate the cases when a multiplicative generalized derivation F satisfies the identities: (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$, (5) $F(xy) - g(y)F(x) \in Z(R)$; for all x, y in some suitable subset of R.

2. Preliminaries

Following results are needed for the proof of our main results.

Lemma 2.1. *([14, Lemma 1.1.5] or [6, Lemma 2]) (a) If R is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of R; in particular, any commutative one-sided ideal is contained in the center of R.*

(b) If R is a prime ring with a nonzero central ideal, then R must be commutative.

Lemma 2.2. *([4, Theorem 3]) Let R be a semiprime ring and U a nonzero left ideal of R. If R admits a derivation d which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Lemma 2.3. *([7, Lemma 2]) If R is prime with a nonzero central ideal, then R is commutative.*

Lemma 2.4. *([4, Theorem 4]) Let R be a prime ring and I be a nonzero left ideal of R. If R admits a nonzero derivation d which is centralizing on I, then R is commutative.*

Lemma 2.5. *([15, Theorem 2 (ii)]) Let R be a noncommutative prime ring with extended centroid C, λ a nonzero left ideal of R and p, q, r, k are fixed positive integers. If d is a derivation of R such that $x^p[d(x^q), x^r]_k = 0$ for all $x \in \lambda$, then either $d = ad(b)$ and $\lambda b = (0)$ for some $b \in Q$ or $\lambda[\lambda, \lambda] = (0)$ and $d(\lambda) \subseteq \lambda C$.*

Lemma 2.6. *([13, Fact-4]) Let R be a semiprime ring, d a nonzero derivation of R such that $x[[d(x), x], x] = 0$ for all $x \in R$. Then d maps R into its center.*

Lemma 2.7. *([10, Lemma 2.4]) If R is a prime ring, $d : R \to R$ a derivation of R, I a nonzero left ideal of R and $0 \neq a \in R$ such that $[ad(x), x] = 0$ for all $x \in I$, then one of the following holds : (1) $a \in Z(R)$; (2) $Ia = (0)$; (3) $Id(I) = (0)$.*

Proof. We have

$$[ad(x), x] = 0 \tag{2.1}$$

for all $x \in I$. Linearizing above relation, we have

$$[ad(x), y] + [ad(y), x] = 0 \tag{2.2}$$

for all $x, y \in I$. Replacing y with yx in the relation (2), we get

$$[ad(x), yx] + [ad(y)x + ayd(x), x] = 0$$

which gives by (2.1) that

$$[ad(x), y]x + [ad(y), x]x + [ayd(x), x] = 0$$

for all $x, y \in I$. By (2.2), it reduces to $[ayd(x), x] = 0$ for all $x, y \in I$. Substituting ay for y, we get $0 = [aayd(x), x] = a[ayd(x), x] + [a, x]ayd(x) = [a, x]ayd(x)$ for all $x, y \in I$. This implies $[a, x]aRyd(x) = (0)$ for all $x, y \in I$. Since R is prime, for each $x \in I$, either $[a, x]a = 0$ or $Id(x) = (0)$. Since both the cases form additive subgroups of I whose union is I, it follows that either $[a, I]a = (0)$ or $Id(I) = (0)$. Since $0 \neq a$, $[a, I]a = (0)$ implies $(0) = [a, I]a = [a, RI]a = R[a, I]a + [a, R]Ia = [a, R]Ia = [a, R]RIa$. Since R is prime, either $a \in Z(R)$ or $Ia = (0)$. □

3. Main Results

Theorem 3.1. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Let $F : R \to R$ be a multiplicative generalized derivation associated with the derivation $g : R \to R$. If $F(xy) - F(x)F(y) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[g(x), x] = (0)$ for all $x \in \lambda$.*

In particular, when $\lambda = R$, then either $g = 0$ or R contains a nonzero central ideal.

Proof. By our assumption, we have

$$F(xy) - F(x)F(y) \in Z(R) \tag{3.1}$$

for all $x, y \in \lambda$. Replacing y with yz, where $z \in \lambda$, we get

$$F(xyz) - F(x)F(yz) \in Z(R) \tag{3.2}$$

which implies

$$F(xy)z + xyg(z) - F(x)\{F(y)z + yg(z)\} \in Z(R) \tag{3.3}$$

that is

$$(F(xy) - F(x)F(y))z + (x - F(x))yg(z) \in Z(R). \tag{3.4}$$

Commuting both sides with z, we get

$$[(F(xy) - F(x)F(y))z + (x - F(x))yg(z), z] = 0 \tag{3.5}$$

for all $x, y, z \in \lambda$. By the application of (3.1), (3.5) yields

$$[(x - F(x))yg(z), z] = 0 \tag{3.6}$$

for all $x, y, z \in \lambda$. Now we put $x = xz$, and then obtain that

$$[(xz - F(x)z - xg(z))yg(z), z] = 0 \tag{3.7}$$

which is

$$[(x - F(x))zyg(z), z] - [xg(z)yg(z), z] = 0 \tag{3.8}$$

for all $x, y, z \in \lambda$. In (3.6), replacing y with zy, we get $[(x - F(x))zyg(z), z] = 0$ for all $x, y, z \in \lambda$, and using this fact (3.8) gives

$$[xg(z)yg(z), z] = 0 \tag{3.9}$$

for all $x, y, z \in \lambda$. Now we put $x = g(z)x$ in (3.9), and then we see that

$$\begin{aligned} 0 &= [g(z)xg(z)yg(z), z] \\ &= g(z)[xg(z)yg(z), z] + [g(z), z]xg(z)yg(z) \end{aligned} \tag{3.10}$$

for all $x, y, z \in \lambda$. As an application of (3.9), (3.10) reduces to

$$[g(z), z]xg(z)yg(z) = 0 \tag{3.11}$$

for all $x, y, z \in \lambda$. Replacing x with xz and y with zy respectively in (3.11), we get

$$[g(z), z]xzg(z)yg(z) = 0 \tag{3.12}$$

and

$$[g(z), z]xg(z)zyg(z) = 0 \tag{3.13}$$

for all $x, y, z \in \lambda$. Subtracting one from another yields

$$[g(z), z]x[g(z), z]yg(z) = 0 \tag{3.14}$$

for all $x, y, z \in \lambda$. Replacing y with yz in (3.14) and right multiplying (3.14) by z respectively and then subtracting one from another yields

$$[g(z), z]x[g(z), z]y[g(z), z] = 0 \tag{3.15}$$

for all $x, y, z \in \lambda$, which implies $(\lambda[g(z), z])^3 = (0)$ for all $z \in \lambda$. Since R is semiprime, it contains no nonzero nilpotent left ideal, implying $\lambda[g(z), z] = (0)$ for all $z \in \lambda$.

In particular, when $\lambda = R$, then $[g(x), x] = 0$ for all $x \in R$. Then by Lemma 2.2, either $g = 0$ or R contains a nonzero central ideal. □

Theorem 3.1. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Let $F : R \to R$ be a multiplicative generalized derivation associated with the derivation $g : R \to R$. If $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[g(x), x] = 0$ for all $x \in \lambda$.*

In particular, when $\lambda = R$, then either $g = 0$ or R contains a nonzero central ideal.

Proof. If we replace F with $-F$ and g with $-g$ in Theorem 3.1, we conclude that $(-F)(xy)-(-F)(x)(-F)(y) \in Z(R)$ for all $x, y \in \lambda$, implies $\lambda[(-g)(x), x] = (0)$ for all $x \in \lambda$, that is, $F(xy)+F(x)F(y) \in Z(R)$ for all $x, y \in \lambda$, implies $\lambda[g(x), x] = (0)$ for all $x \in \lambda$, as desired.

In particular, when $\lambda = R$, then $[g(x), x] = 0$ for all $x \in R$. Then by Lemma 2.2, either $g = 0$ or R contains a nonzero central ideal. □

Corollary 3.2. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Let $F : R \to R$ be a multiplicative generalized derivation associated with the derivation $g : R \to R$. If $F(xy) \pm F(x)F(y) \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:*

(1) $g(x) = [b, x]$ for all $x \in R$ and for some $b \in Q$ with $\lambda b = 0$. Moreover in this case either $F(\lambda) = 0$ or $\lambda(\pm F(y) + y) = 0$ for all $y \in \lambda$;

(2) $\lambda[\lambda, \lambda] = 0$.

Proof. By Theorem 3.1 and Theorem 3.1, $\lambda[g(x), x] = (0)$ for all $x \in \lambda$. Then by Lemma 2.5, either $g(x) = [b, x]$ for all $x \in R$ and for some $b \in Q$ with $\lambda b = 0$ or $\lambda[\lambda, \lambda] = 0$. In the first case $\lambda g(\lambda) = 0$ and hence $F(xy) = F(x)y + xg(y) = F(x)y$ for all $x, y \in \lambda$. Thus by hypothesis $F(x)(y \pm F(y)) \in Z(R)$ for all $x, y \in \lambda$. Now replacing y with yz, where $z \in \lambda$ yields $F(x)(y \pm F(y))z \in Z(R)$. Since $F(x)(y \pm F(y)) \in Z(R)$, we conclude that either $F(x)(y \pm F(y)) = 0$ for all $x, y \in \lambda$ or $\lambda \subseteq Z(R)$. Now $\lambda \subseteq Z(R)$ implies $\lambda[\lambda, \lambda] = 0$. In case $F(x)(y \pm F(y)) = 0$ for all $x, y \in \lambda$, then replacing x with xt, where $t \in \lambda$ yields $F(x)\lambda(y \pm F(y)) = 0$ for all $x, y \in \lambda$. This implies either $F(\lambda) = 0$ or $\lambda(\pm F(y) + y) = 0$ for all $y \in \lambda$. □

Theorem 3.3. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Let $F : R \to R$ be a multiplicative generalized derivation associated with the derivation $g : R \to R$. If $F(xy) - F(y)F(x) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda x[g(x), x]_2 = (0)$ for all $x \in \lambda$.*

In particular, when $\lambda = R$, then either $g = 0$ or R contains a nonzero central ideal.

Proof. We have
$$F(xy) - F(y)F(x) \in Z(R) \tag{3.16}$$
for all $x, y \in \lambda$. Putting $x = xz$, we have
$$F(xzy) - F(y)(F(x)z + xg(z)) \in Z(R) \tag{3.17}$$

that is

$$F(xzy) - F(y)F(x)z - F(y)xg(z) \in Z(R). \tag{3.18}$$

This gives

$$F(xzy) - F(xy)z + (F(xy) - F(y)F(x))z - F(y)xg(z) \in Z(R) \tag{3.19}$$

for all $x, y, z \in \lambda$. Commuting both sides with z and using the fact of (3.16), we have from above,

$$[F(xzy), z] - [F(xy), z]z - [F(y)xg(z), z] = 0 \tag{3.20}$$

for all $x, y, z \in \lambda$. Putting $y = yz$ in (3.20), we have

$$[F(xzyz), z] - [F(xyz), z]z - [(F(y)z + yg(z))xg(z), z] = 0 \tag{3.21}$$

for all $x, y, z \in \lambda$. Putting $x = zx$ in (3.20), we have

$$[F(zxzy), z] - [F(zxy), z]z - [F(y)zxg(z), z] = 0 \tag{3.22}$$

for all $x, y, z \in \lambda$. Subtracting (3.22) from (3.21), we get

$$[F(xzyz), z] - [F(xyz), z]z - [F(zxzy), z] + [F(zxy), z]z - [yg(z)xg(z), z] = 0 \tag{3.23}$$

for all $x, y, z \in \lambda$. Now putting $y = z$ in above, we have

$$\begin{aligned} &[F(xz^3), z] - [F(xz^2), z]z - [F(zxz^2), z] + [F(zxz), z]z \\ &-[zg(z)xg(z), z] = 0. \end{aligned} \tag{3.24}$$

Now putting $x = zx$ in (3.24), we get

$$\begin{aligned} &[F(zxz^3), z] - [F(zxz^2), z]z - [F(z^2xz^2), z] + [F(z^2xz), z]z \\ &-[zg(z)zxg(z), z] = 0 \end{aligned} \tag{3.25}$$

for all $x, y, z \in \lambda$. Left multiplying (3.24) by z and then subtracting from (3.25), we get

$$A(x, y, z) - [z[g(z), z]xg(z), z] = 0, \tag{3.26}$$

where

$$\begin{aligned} A(x, y, z) = &[F(zxz^3), z] - [F(zxz^2), z]z - [F(z^2xz^2), z] + [F(z^2xz), z]z \\ &-z[F(xz^3), z] + z[F(xz^2), z]z + z[F(zxz^2), z] - z[F(zxz), z]z. \end{aligned}$$

We compute

$$\begin{aligned} A(x, y, z) &= [F(zxz^3) - F(zxz^2)z, z] - [F(z^2xz^2) - F(z^2xz)z, z] \\ &\quad -z[F(xz^3) - F(xz^2)z, z] + z[F(zxz^2) - F(zxz)z, z] \\ &= [zxz^2g(z), z] - [z^2xzg(z), z] - z[xz^2g(z), z] + z[zxzg(z), z] \\ &= 0. \end{aligned}$$

Hence (3.26) reduces to

$$[z[g(z), z]xg(z), z] = 0, \tag{3.27}$$

for all $x, z \in \lambda$. Putting $x = xz$ in above we have

$$[z[g(z), z]xzg(z), z] = 0 \tag{3.28}$$

for all $x, z \in \lambda$. Right multiplying (3.27) by z and then subtracting from (3.28), we get

$$[z[g(z), z]x[g(z), z], z] = 0 \tag{3.29}$$

for all $x, z \in \lambda$. This implies

$$[z[g(z), z]xz[g(z), z], z] = 0. \tag{3.30}$$

Let $f(z) = z[g(z), z]$. Then we have

$$f(z)xf(z)z - zf(z)xf(z) = 0 \tag{3.31}$$

for all $x, z \in \lambda$. In (3.31), replacing x with $xf(z)u$, where $u \in \lambda$, we obtain

$$f(z)xf(z)uf(z)z - zf(z)xf(z)uf(z) = 0 \tag{3.32}$$

for all $x, u, z \in \lambda$. Using (3.31), (3.32) gives

$$f(z)xzf(z)uf(z) - f(z)xf(z)zuf(z) = 0 \tag{3.33}$$

that is

$$f(z)x[f(z), z]uf(z) = 0 \tag{3.34}$$

for all $x, u, z \in \lambda$. This implies $[f(z), z]x[f(z), z]u[f(z), z] = 0$ for all $x, u, z \in \lambda$, which is $(\lambda[f(z), z])^3 = (0)$ for all $z \in \lambda$. Since R is semiprime, we conclude that $\lambda[f(z), z] = (0)$ for all $z \in \lambda$. Hence, $\lambda z[[g(z), z], z] = (0)$ for all $z \in \lambda$.

In particular, when $\lambda = R$, then $x[[g(x), x], x] = 0$ for all $x \in R$. Then by Lemma 2.6 and by Lemma 2.2, either $g = 0$ or R contains a nonzero central ideal. $\square$

Theorem 3.4. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Let $F : R \to R$ be a multiplicative generalized derivation associated with the derivation $g : R \to R$. If $F(xy) + F(y)F(x) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda x[g(x), x]_2 = (0)$ for all $x \in \lambda$.*

In particular, when $\lambda = R$, then either $g = 0$ or R contains a nonzero central ideal.

Proof. If we replace F with $-F$ and g with $-g$ in Theorem 3.3, we conclude that $(-F)(xy) - (-F)(y)(-F)(x) \in Z(R)$ for all $x, y \in \lambda$, implies $\lambda x[(-g)(x), x]_2 = (0)$ for all $x \in \lambda$, that is, $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in \lambda$, implies $\lambda x[g(x), x]_2 = (0)$ for all $x \in \lambda$, as desired.

In particular, when $\lambda = R$, then $x[[g(x), x], x] = 0$ for all $x \in R$. Then by Lemma 2.6 and by Lemma 2.2, either $g = 0$ or R contains a nonzero central ideal. $\square$

Corollary 3.5. *Let R be a prime ring with center $Z(R)$. Let $F : R \to R$ be a multiplicative generalized derivation of R associated with the derivation $g : R \to R$. If $F(xy) \pm F(y)F(x) \in Z(R)$ for all $x, y \in R$, then one of the following holds:*
(1) $\lambda[F(\lambda), \lambda] = 0$*;*
(2) $\lambda[\lambda, \lambda] = 0$.

Proof. By Theorem 3.3 and Theorem 3.4, we have $\lambda[g(x), x]_2 = (0)$ for all $x \in \lambda$. Then by Lemma 2.5, either $g(x) = [b, x]$ for all $x \in R$ and for some $b \in Q$ with $\lambda b = 0$ or $\lambda[\lambda, \lambda] = 0$. In the first case $\lambda g(\lambda) = 0$ and hence $F(xy) = F(x)y + xg(y) = F(x)y$ for all $x, y \in \lambda$.

Thus by hypothesis $F(x)y \pm F(y)F(x) \in Z(R)$ for all $x, y \in \lambda$. Now replacing y with yz, where $z \in \lambda$ yields $(F(x)y \pm F(y)F(x))z \mp F(y)F(x)z \pm F(y)zF(x) \in Z(R)$. Commuting both sides with z yields

$$[(F(x)y \pm F(y)F(x))z, z] + [\mp F(y)F(x)z \pm F(y)zF(x), z] = 0$$

Since $F(x)y \pm F(y)F(x) \in Z(R)$, above relation yields $[F(y)[F(x), z], z] = 0$ for all $x, y, z \in \lambda$. Then by Lemma 2.7, $\lambda[F(\lambda), \lambda] = 0$. □

Theorem 3.6. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Let $F : R \to R$ be a multiplicative generalized derivation associated with the derivation $g : R \to R$. If $F(xy) - g(y)F(x) \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[g(x), x]_2 = (0)$ for all $x \in \lambda$.*

In particular, when $\lambda = R$, then either $g = 0$ or R contains a nonzero central ideal.

Proof. We have

$$F(xy) - g(y)F(x) \in Z(R) \tag{3.35}$$

for all $x, y \in \lambda$. Putting $x = xz$, we have

$$F(xzy) - g(y)(F(x)z + xg(z)) \in Z(R) \tag{3.36}$$

that is

$$F(xzy) - F(xy)z + (F(xy) - g(y)F(x))z - g(y)xg(z) \in Z(R) \tag{3.37}$$

for all $x, y, z \in \lambda$. Since $F(xy) - g(y)F(x) \in Z(R)$, commuting both sides with z, we have

$$[F(xzy) - F(xy)z - g(y)xg(z), z] = 0 \tag{3.38}$$

for all $x, y, z \in \lambda$. In particular, for $y = z$,

$$[xzg(z) - g(z)xg(z), z] = 0 \tag{3.39}$$

for all $x, y, z \in \lambda$. Replacing x with zx, we obtain

$$[zxzg(z) - g(z)zxg(z), z] = 0 \tag{3.40}$$

for all $x, y, z \in \lambda$. Left multiplying by z in (3.39) and then subtracting from (3.40), we obtain

$$[[g(z), z]xg(z), z] = 0 \tag{3.41}$$

for all $x, y, z \in \lambda$. Replacing x with xz in above relation, we have

$$[[g(z), z]xzg(z), z] = 0 \tag{3.42}$$

for all $x, y, z \in \lambda$. Right multiplying by z in (3.41) and then subtracting from (3.42), we obtain that

$$[[g(z), z]x[g(z), z], z] = 0 \tag{3.43}$$

for all $x, y, z \in \lambda$. This can be re-written as

$$[g(z), z]x[g(z), z]z - z[g(z), z]x[g(z), z] = 0 \tag{3.44}$$

for all $x, z \in \lambda$. In (3.44), replacing x with $x[g(z), z]u$, where $u \in \lambda$, we obtain

$$[g(z), z]x[g(z), z]u[g(z), z]z - z[g(z), z]x[g(z), z]u[g(z), z] = 0 \tag{3.45}$$

for all $x, u, z \in \lambda$. Using (3.44), (3.45) gives

$$[g(z), z]xz[g(z), z]u[g(z), z] - [g(z), z]x[g(z), z]zu[g(z), z] = 0 \tag{3.46}$$

that is

$$[g(z), z]x[[g(z), z], z]u[g(z), z] = 0 \tag{3.47}$$

for all $x, u, z \in \lambda$. This implies $[g(z), z]_2x[g(z), z]_2u[g(z), z]_2 = 0$ for all $x, u, z \in \lambda$, which is $(\lambda[g(z), z]_2)^3 = (0)$ for all $z \in \lambda$. Since R is semiprime, we conclude that $\lambda[g(z), z]_2 = (0)$ for all $z \in \lambda$.

In particular, when $\lambda = R$, then $[g(x), x]_2 = 0$ for all $x \in R$. Then by Lemma 2.6 and by Lemma 2.2, either $g = 0$ or R contains a nonzero central ideal. □

Corollary 3.7. *Let R be a prime ring with center $Z(R)$. Let $F : R \to R$ be a multiplicative generalized derivation of R associated with the nonzero derivation $g : R \to R$. If any one of the following holds:*

(i) $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in R$,
(ii) $F(xy) - F(x)F(y) \in Z(R)$ for all $x, y \in R$,
(iii) $F(xy) + F(y)F(x) \in Z(R)$ for all $x, y \in R$,
(iv) $F(xy) - F(y)F(x) \in Z(R)$ for all $x, y \in R$,
(v) $F(xy) - g(y)F(x) \in Z(R)$ for all $x, y \in R$,
then R must be commutative.

Example 1.2 Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers. Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$, so R is not prime ring. Define the mappings d and $F : R \longrightarrow R$ as follows: $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & 0 \\ 0 & 0 & c^2 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is straightforward to verify that d is a derivation in R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Since F is not additive map, F is a multiplicative generalized derivation on R. Note that $F(xy) \pm F(x)F(y) \in Z(R)$, $F(xy) \pm F(y)F(x) \in Z(R)$ and $F(xy) - d(y)F(x) \in Z(R)$ for all $x, y \in R$. Since R is noncommutative, the primeness hypothesis in Corollary 3.7 is essential.

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