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On Statistical Convergent Sequence Spaces Of Intuitionistic Fuzzy Numbers

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abstract: In the present paper we introduce the classes of sequence ${}_{st}c^{IFN}$, ${}_{st}c_0^{IFN}$ and ${}_{st}\ell_\infty^{IFN}$ of statistically convergent, statistically null and statistically bounded sequences of intuitionistic fuzzy number based on the newly defined metric on the space of all intuitionistic fuzzy numbers (IFNs). We study some algebraic and topological properties of these spaces and prove some inclusion relations too.

Key Words: Intuitionistic fuzzy set, triangular intuitionistic fuzzy number, solidity, normal.

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Introduction

Intuitionistic fuzzy set (IFS) is one of the generalizations of fuzzy sets theory [1]. Out of several higher-order fuzzy sets, IFS was first introduced by Atanassov, have been found to be compatible to deal with vagueness. The conception of IFS can be viewed as an appropriate and alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set theory. In fuzzy sets the degree of acceptance is considered only but IFS is characterized by a membership function and a non-membership function such that the sum of both values is less than one. Presently IFSs are being studied and used in different fields of science. Let X be universe of discourse defined by $X = \{x_1, x_2, \dots x_n\}$. The grade of membership of an element $x_i \in X$ in a fuzzy set is represented by real value in $[0, 1]$. It does indicate the evidence for $x_i \in X$, but does not indicate the evidence against $x_i \in X$. Atanassov presented the concept of IFS, an IFS A in X is characterized by a membership function ${}_A(x)$ and a nonmembership function $\nu_A(x)$. Here ${}_A(x)$ and $\nu_A(x)$ are associated with each point in X, a real number in $[0, 1]$ with the values of ${}_A(x)$ and $\nu_A(x)$ at X representing the grade of membership and non-membership of x in A. When A is an ordinary (crisp) set, its membership function can take only two values zero and one. An IFS becomes a fuzzy set A when $\nu_A(x) = 0$ but ${}_A(x) \in [0, 1]$ for all

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$x \in A$. Burillo et. al. [4] proposed definition of intuitionistic fuzzy number (IFN). The notion of statistical convergence was introduced by Fast [11] and Schoenberg [20] independently. This concept has been generalized and developed by Fridy [12], ˇSalat [18], Connor [6], Connor et. al. [7], Et and Nuray [10] and many others. Nuray and Savas [17], extended the idea to apply to sequences of fuzzy numbers. Later on, Aytar and Pehlivan [2], Bilgin [3], Colak et. al. [5], Kwon [14], Tripathy and Baruah [21], Savas [18], Debnath et. al. [9] and many others extended the idea of statistical convergence to the sequences of fuzzy numbers. The existing literature on statistical convergence appears to have been restricted to sequences of real numbers, complex numbers or fuzzy numbers. As intuitionistic fuzzy numbers are generalization of fuzzy numbers, it is reasonable to think about the existing sequence spaces of real numbers, interval numbers and fuzzy numbers in terms of intuitionistic fuzzy numbers. Recently the authors have introduced convergent, null and bounded sequence spaces of intuitionistic fuzzy numbers. In the current paper we have introduced and studied the properties of the sequence spaces ${}_{st}c^{IFN}$, ${}_{st}c_0^{IFN}$ and ${}_{st}\ell_\infty^{IFN}$ of statistically convergent, null and bounded sequences of intuitionistic fuzzy numbers with the help of a newly defined metric.

1. Preliminaries

Definition 2.1. *A fuzzy number X is a fuzzy subset of the real line R, i.e, a mapping $X : R \to I = [0,1]$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is normal if there exists $t_0 \in R$ such that $X(t_0) = 1$. A fuzzy number X is upper semi continuous if for each $\varepsilon > 0$, $X^{-1}([0, \alpha + \varepsilon))$ is open in the usual topology for all $\alpha \in [0,1)$.*

Definition 2.2. *α–cuts of a fuzzy number X is defined by, $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}, \alpha \in [0,1]$.*

Definition 2.3. [15] *A sequence space E is said to be solid (or normal) if $(Y_k) \in E$ whenever $(X_k) \in E$ and $|Y_k| \leq |X_k|$ for all $k \in N$.*

Definition 2.4. [15] *A sequence space E is said to be monotone if E contains the canonical pre image of all its step spaces.*

Definition 2.5. [15] *A sequence space E is said to be sequence algebra if $(X_k \bigotimes Y_k) \in E$ whenever $(X_k), (Y_k) \in E$.*

Lemma 2.1. [15] A sequence space E is normal implies that it is monotone.

Definition 2.6. Let X be a given non-empty set. An intuitionistic fuzzy set(IFS) [1] in X is an object A given by

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ are functions such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Definition 2.7. [4] An Intuitionistic fuzzy number (IFN) A is

$i)$ an intuitionistic fuzzy subset of the real line.

$ii)$ normal, i.e., there is any $x_0 \in R$ such that $\mu_A(x_0) = 1$ and, $\nu_A(x_0) = 0$.

$iii)$ convex for the membership function $\mu_A(x)$

i.e $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\ (\mu_A(x_1), \mu_A(x_2))\ \forall\ x_1, x_2 \in R$ and $\lambda \in [0,1]$.
concave for the non-membership function $\nu_A(x)$
i.e $\nu_A(\lambda x_1 + (1-\lambda x_2)) \geq \min(\nu_A(x_1), \nu_A(x_2))\ \forall\ x_1, x_2 \in R$ and $\lambda \in [0,1]$.
iii) μ_A is upper semi continuous and ν_A is lower semi continuous
iv) supp $A=$ cl $(\{x \in X : \nu_A(x) < 1\})$ is bounded.

An IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ such that $\mu_A(x)$ and $1-\nu_A(x)$ are fuzzy numbers, where $(1-\nu_A)(x) = 1-\nu_A(x)$, and $\mu_A(x)+\nu_A(x) \leq 1$ is called an intuitionistic fuzzy number. We denote by $A = (\mu_A, \nu_A)$, an intuitionistic fuzzy number and by $R^2(I)$, the set of all IFN. It is obvious that any fuzzy number $B = \{< x, \mu_A(x) >: x \in X\}$ can be represented as an intuitionistic fuzzy number by

$$B = \{< x, \mu_A(x), 1-\mu_A(x) >: x \in X\}.$$

Let D^2 be the set of all closed and bounded intervals of the form $([x_{1l}, x_{1r}], [x_{2l}, x_{2r}])$ on $R^2 = R \times R$. For $X, Y \in D^2$ we have $X \leq Y$ iff $x_{1l} \leq y_{1l}$, $x_{1r} \leq y_{1r}$, $x_{2l} \leq y_{2l}$, $x_{2r} \leq y_{2r}$ where $X = ([x_{1l}, x_{1r}], [x_{2l}, x_{2r}])$, $Y = ([y_{1l}, y_{1r}], [y_{2l}, y_{2r}])$. The distance between X and Y is given by $d(X,Y) = max\{|x_{1l}-y_{1l}|, |x_{2l}-y_{2l}|, |x_{1r}-y_{1r}|, |x_{2r}-y_{2r}|\}$. Then d defines a metric on D^2 and (D^2, d) forms a complete metric space [8].

Let $R^2(I)$ denote the set of all intuitionistic fuzzy numbers. Then for $X, Y \in R^2(I)$ the $\alpha-$level set of X, Y are $[\mu_X(x)]^\alpha = \{x \in X : {}_\alpha(x) \geq \alpha\}$
$[1-\nu_X(x)]^\alpha = \{x \in X : 1-\nu_\alpha(x) \geq \alpha\}$
$= \{x \in X : \nu_\alpha(x) \leq 1-\alpha, \alpha \in [0,1]\} = [\nu_X(x)]^{1-\alpha}$
$[\mu_Y(y)]^\alpha = \{y \in Y : {}_\alpha(y) \geq \alpha\}$
$[1-\nu_Y(y)]^\alpha = \{y \in Y : 1-\nu_\alpha(y) \geq \alpha\}$
$= \{y \in Y : \nu_\alpha(y) \leq 1-\alpha, \alpha \in [0,1]\} = [\nu_Y(y)]^{1-\alpha}$
Then $[\mu_X(x)]^\alpha$, $[\mu_Y(y)]^\alpha$, $[1-\nu_X(x)]^\alpha$, $[1-\nu_Y(y)]^\alpha$ are closed and bounded intervals of the following form:
$[\mu_X(x)]^\alpha = [x_1^\alpha, x_2^\alpha]$
$[\mu_Y(y)]^\alpha = [y_1^\alpha, y_2^\alpha]$
$[1-\nu_X(x)]^\alpha = [x_3^{1-\alpha}, x_4^{1-\alpha}]$
$[1-\nu_Y(y)]^\alpha = [y_3^{1-\alpha}, y_4^{1-\alpha}]$
with respect to the $\alpha-$cuts of the fuzzy number $1-\nu_X$, the following equality is immediate: $(1-\nu_X)(\alpha) = \nu_X(1-\alpha)$ [13].

The additive identity and multiplicative identity of $R^2(I)$ are $\bar{0}$ and $\bar{1}$ respectively.

Let $X, Y \in R^2(I)$ and the $\alpha-$ level sets are
$[X]^\alpha = ([x_1^\alpha, x_2^\alpha], [x_3^{1-\alpha}, x_4^{1-\alpha}])$
$[Y]^\alpha = ([y_1^\alpha, y_2^\alpha], [y_3^{1-\alpha}, y_4^{1-\alpha}])$

Theorem 2.1. [16] If $A_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $B_{TIFN} = (b_1, b_2, b_3; b_1', b_2, b_3')$ are two $TIFN$, then
a) $A \oplus B = (a_1+b_1, a_2+b_2, a_3+b_3; a_1'+b_1', a_2+b_2, a_3'+b_3')$.
b) $A \otimes B = (a_1b_1, a_2b_2, a_3b_3; a_1'b_1', a_2b_2, a_3'b_3')$.

A sequence $X = (X_k)$ of intuitionistic fuzzy numbers is a function X from the set N of all positive integers into $R^2(I)$. Thus, a sequence of intuitionistic fuzzy numbers X is a correspondence from the set of positive integers to a set of intuitionistic fuzzy numbers, i.e., to each positive integer k there corresponds an intuitionistic fuzzy number $X(k)$. It is more common to write X_k rather than $X(k)$ and to denote the sequence by (X_k) rather than X. The intuitionistic fuzzy number X_k is called the $k-$th term of the sequence.

Definition 2.8. A sequence $X = (X_k)$ of IFN is said to be convergent to an intuitionistic fuzzy number X_0, if there exists a positive integer n_0 such that $\overline{d}(X_k, X_0) < \varepsilon$ for all $k > n_0$. We write $lim\ X_k = X_0$.

Definition 2.9. A sequence $X = (X_k)$ of IFN is said to be Cauchy sequence if there exists a positive integer n_0 such that $\overline{d}(X_k, X_m) < \varepsilon$ for all $n, m \geq n_0$.

Definition 2.10. A sequence $X = (X_k)$ of IFN is said to be bounded if the sets $\{\mu_{X_k} : k \in N\}$ and $\{1 - \nu_{X_k} : k \in N\}$ are bounded.

2. Main Result

Let us define a mapping $\bar{d}^2 : R^2(I) \times R^2(I) \to R^+ \cup \{0\}$ by

$\bar{d}^2(X,Y) = \sup_{\alpha\in[0,1]} d(X^\alpha, Y^\alpha)$

$= sup_{\alpha\in[0,1]} max\{|x_{1l}^\alpha - y_{1l}^\alpha|, |x_{1r}^\alpha - y_{1r}^\alpha|, |x_{2l}^{1-\alpha} - y_{2l}^{1-\alpha}|, |x_{2r}^{1-\alpha} - y_{2r}^{1-\alpha}|\}$

Where, $[X]^\alpha = ([x_{1l}^\alpha, x_{1r}^\alpha], [x_{2l}^{1-\alpha}, x_{2r}^{1-\alpha}])$ and $[Y]^\alpha = ([y_{1l}^\alpha, y_{1r}^\alpha], [y_{2l}^{1-\alpha}, y_{2r}^{1-\alpha}])$

Theorem 3.1. $(R^2(I), \bar{d}^2)$ is a complete metric space.

Proof: Let $\{x^k\}$ be any Cauchy sequence in $R^2(I)$. Then there is a $n_0 \in N$ such that $\bar{d}^2(x^k, x^m) < \varepsilon$ for all $k, m > n_0$.

$\Rightarrow \sup_{\alpha\in[0,1]} d(x^{k(\alpha)}, x^{m(\alpha)}) < \varepsilon.$

$\Rightarrow \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{m(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{m(\alpha)}|, |x_{2l}^{k(1-\alpha)} - x_{2l}^{m(1-\alpha)}|,$
$|x_{2r}^{k(1-\alpha)} - x_{2r}^{m(1-\alpha)}|\} < \varepsilon$

$\Rightarrow \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{m(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{m(\alpha)}|\} < \varepsilon$ and
$\sup_{\alpha\in[0,1]} \max\{|x_{2l}^{k(1-\alpha)} - x_{2l}^{m(1-\alpha)}|, |x_{2r}^{k(1-\alpha)} - x_{2r}^{m(1-\alpha)}|\} < \varepsilon$

$\Rightarrow \sup_{\alpha\in[0,1]} d(x_1^{k(\alpha)}, x_1^{m(\alpha)}) < \varepsilon$ and $\sup_{\alpha\in[0,1]} d(x_2^{k(1-\alpha)}, x_2^{m(1-\alpha)}) < \varepsilon$

$\Rightarrow \bar{d}(x_1^k, x_1^m) < \varepsilon$ and $\bar{d}(x_2^k, x_2^m) < \varepsilon$

$\Rightarrow \{x_1^k\}$ and $\{x_2^k\}$ are Cauchy sequence in $R(I)$.

But $R(I)$ is complete. Therefore we can write,

$\lim_k x_1^k = x_1^0$ and $\lim_k x_2^k = x_2^0 \Rightarrow \bar{d}(x_1^k, x_1^0) < \varepsilon$ and $\bar{d}(x_2^k, x_2^0) < \varepsilon$

Now, $\bar{d}(x_1^k, x_1^0) < \varepsilon \Rightarrow \sup_{\alpha\in[0,1]} d(x_1^{k(\alpha)}, x_1^{0(\alpha)}) < \varepsilon.$

$\Rightarrow \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{0(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{0(\alpha)}|\} < \varepsilon$

Similarly, $\bar{d}(x_2^k, x_2^0) < \varepsilon \Rightarrow \sup_{\alpha\in[0,1]} d(x_2^{k(\alpha)}, x_2^{0(\alpha)}) < \varepsilon.$

$\Rightarrow \sup_{\alpha\in[0,1]} \max\{|x_{2l}^{k(\alpha)} - x_{2l}^{0(\alpha)}|, |x_{2r}^{k(\alpha)} - x_{2r}^{0(\alpha)}|\} < \varepsilon$

From (1) and (2) we can write,

$\sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{0(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{0(\alpha)}|, |x_{2l}^{k(\alpha)} - x_{2l}^{0(\alpha)}|, |x_{2r}^{k(\alpha)} - x_{2r}^{0(\alpha)}|\} < \varepsilon.$

i.e, $\sup_{\alpha\in[0,1]} d(x^{k(\alpha)}, x^{0(\alpha)}) < \varepsilon$. i.e, $\bar{d}^2(x^k, x^0) < \varepsilon$. i.e, $\lim_k x^k = x^0$ and it is obvious that $x^0 \in R^2(I)$. This completes the proof.

Definition 3.1. A sequence $X = (X_k)$ of IFN is said to be statistically convergent to an IFN X_0 if for each $\varepsilon > 0$,

$$\delta(A(\varepsilon)) = \delta(\{k \in N : \bar{d}^2(X_k, X_0) \geq \varepsilon\}) = 0.$$

Definition 3.2. A sequence $X = (X_k)$ of IFN is said to be statistically null , if for each $\varepsilon > 0$, $\delta(A(\varepsilon)) = \delta$ $(\{k \in N : \bar{d}^2(X_k, \bar{0}) \geq \varepsilon\}) = 0$.
Definition 3.3. A sequence $X = (X_k)$ of IFN is said to be statistically bounded, if for each $\varepsilon > 0$, $\delta(A(\varepsilon)) = \delta(\{k \in N : \exists M > 0,\ \bar{d}^2(X_k, \bar{0}) \geq M\}) = 0$.
Definition 3.4. A sequence $X = (X_k)$ of IFN is said to be statistically Cauchy, if for each $\varepsilon > 0$, $\delta(A(\varepsilon)) = \delta(\{k \in N : \bar{d}^2(X_k, X_m) \geq \varepsilon\}) = 0$ for $k, m > n_0 \in N$.

Let w^{IFN} denotes the spaces of all sequences of IFN.

We introduce the following new sequence spaces

${}_{st}c^{IFN} = \{(X_k) \in w^{IFN} : st - lim X_k = X_0\}$

${}_{st}c_0^{IFN} = \{(X_k) \in w^{IFN} : st - lim X_k = \bar{0}\}$

${}_{st}\ell_\infty^{IFN} = \{(X_k) \in w^{IFN} : \exists K > 0, \delta(\{k \in N : \bar{d}^2(X_k, \bar{0}) \geq K\}) = 0\}$

Remark 3.1. ${}_{st}c_0^{IFN} \subset {}_{st}c^{IFN} \subset {}_{st}\ell_\infty^{IFN}$.
Example 3.1. Let $X = (X_k)$ be a sequence of IFN, where

$$X_k(t) = \begin{cases} tk + 1 & \text{for } \frac{-1}{k} \leq t \leq 0 \\ 1 - tk & \text{for } 0 \leq t \leq \frac{1}{k} \\ 0 \text{ , otherwise} \end{cases}$$

$$\nu_{X_k}(t) = \begin{cases} \frac{tk}{1-2k} & \text{for } \frac{1-2k}{k} \leq t \leq 0 \\ \frac{tk}{k} & \text{for } 0 \leq t \leq \frac{1}{k} \\ 1 \text{ , otherwise} \end{cases}$$

Then

$[X_k(t)]^\alpha = [\frac{\alpha-1}{k}, \frac{-\alpha+1}{k}]$

$[\nu_{X_k}(t)]^\alpha = [\frac{1-\alpha-2k+2k\alpha}{k}, k(1-\alpha)]$

$[X]^\alpha = ([\frac{\alpha-1}{k}, \frac{-\alpha+1}{k}], [\frac{1-\alpha-2k+2k\alpha}{k}, k(1-\alpha)])$

$\bar{d}^2(X_k, \bar{0}) = \sup_{\alpha \in [0,1]} d([X_k]^\alpha, [0]^\alpha) =$

$\sup_{\alpha \in [0,1]} \max \{|\frac{\alpha-1}{k}|, |\frac{-\alpha+1}{k}|, |\frac{1-\alpha-2k+2k\alpha}{k}|, |k(1-\alpha)|\} = \max\{\frac{1}{k}, \frac{1}{k}, 0, k\} = k.$

Now taking $k = n^2$ we have, $\delta\left(\{k \in N : \bar{d}^2(X_k, \bar{0}) \geq \varepsilon\}\right) = 0$.

The above example shows that (X_k) is statistically convergent but not ordinary convergent.

Theorem 3.2. The spaces ${}_{st}c^{IFN}$, ${}_{st}c_0^{IFN}$, ${}_{st}\ell_\infty^{IFN}$ are closed with respect to addition and scalar multiplication.

Proof: Let $X = (X_k)$ and $Y = (Y_k)$ be two element of ${}_{st}c_0^{IFN}$ and α, βany scalar. Then

$\delta(\{k \in N : \bar{d}^2(X_k, \bar{0}) \geq \varepsilon\}) = 0$ and $\delta(\{k \in N : \bar{d}^2(Y_k, \bar{0}) \geq \varepsilon\}) = 0$

Now, $\bar{d}^2(\alpha X_k + \beta Y_k, \bar{0}) \leq |\alpha| \bar{d}^2(X_k, \bar{0}) + |\beta| \bar{d}^2(Y_k, \bar{0})$

$\Rightarrow \delta(\{k \in N : \bar{d}^2(\alpha X_k + \beta Y_k, \bar{0}) \geq \varepsilon\}) \subseteq \delta(\{k \in N : (|\alpha| \bar{d}^2(X_k, \bar{0}) + |\beta| \bar{d}^2(Y_k, \bar{0})) \geq \varepsilon\})$

$= |\alpha| \delta(\{k \in N : \bar{d}^2(X_k, \bar{0}) \geq \varepsilon\}) + |\beta| \delta(\{k \in N : \bar{d}^2(Y_k, \bar{0}) \geq \varepsilon\}) = 0.$

This completes the proof.

Theorem 3.3. The spaces ${}_{st}c^{IFN}$, ${}_{st}c_0^{IFN}$, ${}_{st}\ell_\infty^{IFN}$ are complete metric spaces w.r.t the metric $\bar{d}^2(X, Y) = \sup_k \bar{d}^2(X_k, Y_k)$.

Proof: Let $\{x^k\}$ be any Cauchy sequence in ${}_{st}c_0^{IFN}$. Then there is a $n_0 \in N$ such that
$\delta(\{k \in N : \bar{d}^2(x^k, x^m) \geq \varepsilon\}) = 0$ for all $k, m > n_0$.
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} d(x^{k(\alpha)}, x^{m(\alpha)}) \geq \varepsilon\}) = 0$.
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{m(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{m(\alpha)}|, |x_{2l}^{k(1-\alpha)} - x_{2l}^{m(1-\alpha)}|,$
$|x_{2r}^{k(1-\alpha)} - x_{2r}^{m(1-\alpha)}|\} \geq \varepsilon\}) = 0$.
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{m(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{m(\alpha)}|\} \geq \varepsilon\}) = 0$ and
$\delta(\{k \in N : \sup_{\alpha\in[0,1]} \max\{|x_{2l}^{k(1-\alpha)} - x_{2l}^{m(1-\alpha)}|, |x_{2r}^{k(1-\alpha)} - x_{2r}^{m(1-\alpha)}|\} \geq \varepsilon\}) = 0$
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} d(x_1^{k(\alpha)}, x_1^{m(\alpha)}) \geq \varepsilon\}) = 0$ and
$\delta(\{k \in N : \sup_{\alpha\in[0,1]} d(x_2^{k(1-\alpha)}, x_2^{m(1-\alpha)}) \geq \varepsilon\}) = 0$
$\Rightarrow \delta(\{k \in N : \bar{d}(x_1^k, x_1^m) \geq \varepsilon\}) = 0$ and $\delta(\{k \in N : \bar{d}(x_2^k, x_2^m) \geq \varepsilon\}) = 0$
$\{x_1^k\}$ and $\{x_2^k\}$ are Cauchy sequence in $(R(I), \bar{d})$.
But $(R(I), \bar{d})$ is complete. Therefore we can write,
st-$\lim_k x_1^k = x_1^0$ and st-$\lim_k x_2^k = x_2^0$
$\Rightarrow \delta(\{k \in N : \bar{d}(x_1^k, x_1^0) \geq \varepsilon\}) = 0$ and
$\delta(\{k \in N : \bar{d}(x_2^k, x_2^0) \geq \varepsilon\}) = 0$
Now, $\delta(\{k \in N : \bar{d}(x_1^k, x_1^0) \geq \varepsilon\}) = 0$
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} d(x_1^{k(\alpha)}, x_1^{0(\alpha)}) \geq \varepsilon\}) = 0$.
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{0(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{0(\alpha)}|\} \geq \varepsilon\}) = 0$ (1)
Similarly,
$\delta(\{k \in N : \bar{d}(x_2^k, x_2^0) \geq \varepsilon\}) = 0 \Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} d(x_2^{k(1-\alpha)}, x_2^{0(1-\alpha)}) \geq \varepsilon\}) = 0$.
$\Rightarrow \delta(\{k \in N : \sup_{\alpha\in[0,1]} \max\{|x_{2l}^{k(1-\alpha)} - x_{2l}^{0(1-\alpha)}|, |x_{2r}^{k(1-\alpha)} - x_{2r}^{0(1-\alpha)}|\} \geq \varepsilon\}) = 0$ (2)
From (1) and (2) we can write,
$\delta(\{k \in N : \sup_{\alpha\in[0,1]} \max\{|x_{1l}^{k(\alpha)} - x_{1l}^{0(\alpha)}|, |x_{1r}^{k(\alpha)} - x_{1r}^{0(\alpha)}|, |x_{2l}^{k(1-\alpha)} - x_{2l}^{0(1-\alpha)}|,$
$|x_{2r}^{k(1-\alpha)} - x_{2r}^{0(1-\alpha)}|\} \geq \varepsilon\}) = 0$.
i.e, $\delta(\{k \in N : \sup_{\alpha\in[0,1]} d(x^{k(\alpha)}, x^{0(\alpha)}) \geq \varepsilon\}) = 0$. i.e,
$\delta(\{k \in N : \bar{d}^2(x^k, x^0) \geq \varepsilon\}) = 0$.
i.e, st-$\lim_k x^k = x^0$ and it is obvious that $x^0 \in R^2(I)$.
This completes the proof.

Theorem 3.4. The spaces ${}_{st}c^{IFN}$, ${}_{st}c_0^{IFN}$, ${}_{st}\ell_\infty^{IFN}$ are normal and monotone.

Proof: Let $X = (X_k) \in {}_{st}c_0^{IFN}$, and $Y = (Y_k)$ be such that $\bar{d}^2(X_k, \bar{0}) \geq \bar{d}^2(Y_k, \bar{0})$.
$\Rightarrow \delta(\{k \in N : \bar{d}^2(Y_k, \bar{0}) \geq \varepsilon\}) \subseteq \delta(\{k \in N : \bar{d}^2(X_k, \bar{0}) \geq \varepsilon\}) = 0$
$\Rightarrow \delta(\{k \in N : \bar{d}^2(Y_k, \bar{0}) \geq \varepsilon\}) = 0$
Hence ${}_{st}c_0^{IFN}$ is normal and hence monotone.

Theorem 3.5. The spaces ${}_{st}c^{IFN}$, ${}_{st}c_0^{IFN}$, ${}_{st}\ell_\infty^{IFN}$ are sequence algebra.

Proof: Proof is obvious.

3. Conclusion

It is the first time we have introduced this idea based on IFNs. This paper may be useful for those working in various fields using IFNs.

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