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Exponential Differential Operators For Singular Integral Equations and Space Fractional Fokker-Planck Equation

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ABSTRACT: In this article, it has been shown that the combined use of exponential operators and integral transform provides a powerful tool to evaluate integrals, solution to certain type of fractional differential equations and families of singular integral equations. It is shown that exponential operators are powerful and effective method for solving certain space fractional Fokker-Planck equation with non-constant coefficients. Constructive examples are provided.

Key Words: Fractional partial differential equations; Riemann – Liouville fractional derivative; Nonlinear differential equations; Laplace transforms; Fokker-Planck equation.

Contents

Introduction

We present a general method of operational nature to obtain solutions for several types of partial differential equations. The integral transform technique is one of the most useful tools of applied mathematics, employed in many branches of science, mathematical physics and engineering. The most interesting and useful applications of the Laplace transformation are solving linear differential equations with discontinuous or impulsive forcing functions which are common place in mechanical systems and circuit analysis problems.

1. One dimensional Laplace transform

Definition 1.1. *Laplace transform of function $f(t)$ is as follows*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt := F(s). \tag{1.1}$$

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If $\mathcal{L}\{f(t)\} = F(s)$*, then* $\mathcal{L}^{-1}\{F(s)\}$ *is given by [3]*

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{1.2}$$

where $F(s)$ *is analytic in the region* $\mathrm{Re}(s) > c$*.*

Definition 1.2. *If the function* $\Phi(t)$ *belongs to* $C[a,b]$ *and* $a < t < b$*,*

$$I_a^{RL,\alpha}\{\Phi(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\Phi(\xi)}{(t-\xi)^{1-\alpha}}d\xi. \tag{1.3}$$

The left Riemann-Liouville fractional integral of order $\alpha > 0$ *is defined as [7]*

$$D_a^{RL,\alpha}\phi(x) = \tfrac{1}{\Gamma(1-\alpha)}\tfrac{d}{dx}\int_a^t \tfrac{\Phi(\xi)}{(t-\xi)^{\alpha}}d\xi. \tag{1.4}$$

It follows that $D_a^{RL,\alpha}\phi(x)$ *exists for all* $\Phi(t)$ *belongs to* $C[a,b]$ *,and* $a < t < b$ *.*
Note: *A very useful fact about the R-L operators is that they satisfy semi group properties of fractional integrals.*
The special case of fractional derivative when $\alpha = 0.5$ *is called semi –derivative.*

Definition 1.3. *The left Caputo fractional derivative of order* α *(*$0 < \alpha < 1$*) of* $\phi(t)$ *is as follows [8]*

$$D_a^{c,\alpha}\phi(x) = \tfrac{1}{\Gamma(1-\alpha)}\int_a^t \tfrac{1}{(t-\xi)^{\alpha}}\phi'(\xi)d\xi. \tag{1.5}$$

Lemma 1.1: The following identities hold true.

1. $\mathcal{L}^{-1}F(s^{\beta}) = \frac{1}{2\pi}\int_0^{\infty} f(\tau)(\int_0^{\infty} e^{-tr-r^{\beta}(\tau cos\beta\pi)}sin(r^{\beta}\tau sin\beta\pi))dr)d\tau,$
2. $\mathcal{L}^{-1}F(\sqrt{s}) = \frac{1}{(2t\sqrt{t\pi})}\int_0^{\infty} e^{-\frac{\xi^2}{4t}}f(\xi)d\xi,$
3. $\mathcal{L}^{-1}(e^{-k\sqrt{s}}) = \frac{k}{(2\sqrt{\pi})}\int_0^{\infty} e^{-t\xi-\frac{k^2}{4\xi}}d\xi,$
4. $e^{-ks^{\beta}} = \frac{1}{\pi}\int_0^{\infty} e^{-r^{\beta}(kcos\beta\pi)}sin(kr^{\beta}sin\beta\pi)(\int_0^{\infty} e^{-s\tau-r\tau}d\tau)dr.$

Proof. See [1].
Lemma 1.2: The following exponential identities hold true.

1. $\exp(\pm\alpha\frac{d}{dt})\Phi(t) = \Phi(t\pm\alpha),$
2. $\exp(\pm\alpha t\frac{d}{dt})\Phi(t) = \Phi(te^{\pm\alpha}),$
3. $\exp(\alpha q(t)\frac{d}{dt})\Phi(t) = \Phi(Q(F(t)+\alpha)),$
4. $\exp(-\alpha\sinh^2 t\frac{d}{dt})\Phi(t) = \Phi(\ ln(\sqrt{\frac{1+\alpha+\coth t}{1-\alpha-\coth t}})),$

where $F(t)$ is primitive of $(q(t))^{-1}$ and $Q(t)$ is inverse of $F(t)$.
Proof. See [4].
Like Fourier transform, the Laplace transforms is used in a variety of applications. The most common usage of the Laplace transforms is in the solution of initial value problems. The Laplace transform is useful tool in applied mathematics, for instance for solving singular integral equations, partial differential equations, and in automatic control, where it defines a transfer function.

Problem 1. *Let us consider the following non- linear impulsive differential equation*

$$(\sqrt{D_t^2 - a^2})y(t) = t^k\delta(t-\lambda).$$

Solution. *The above differential equation can be written as below*

$$y(t) = \frac{1}{(\sqrt{D_t^2 - a^2})} t^k\delta(t-\lambda),$$

from which we deduce that

$$y(t) = (\int_0^\infty d\xi e^{-\xi D_t} I_0(a\xi)) t^k\delta(t-\lambda),$$

finally,

$$y(t) = \int_0^\infty d\xi I_0(a\xi) e^{-\xi D_t} t^k\delta(t-\lambda).$$

Using Lemma 1.2 leads to the following

$$y(t) = \int_0^\infty I_0(a\xi)(t-\xi)^k\delta(t-\xi-\lambda)d\xi,$$

thus, by using elementary properties of Dirac-delta function, we obtain

$$y(t) = \lambda^k I_0(a(t-\lambda)).$$

Note:*In the above relation I_0 stands for the modified Bessel's function of the first kind of order zero.*

Problem 2. *Let us solve the following fractional Volterra integral equation of convolution type. The Laplace transform provides a useful technique for the solution of such integral equations.*

$$\lambda \int_\beta^{t+\beta} \sin(a(t-\xi+\beta)) D^{c,\alpha}\phi(\xi-\beta)d\xi = (\tfrac{t}{a})^{\frac{\nu}{2}} J_\nu(2\sqrt{at}), \tag{1.6}$$

$$\phi(\beta) = 0, \alpha+\nu-2 > 0, 0 < \alpha < 1.$$

Solution. *Let us make a change of variable $\xi - \beta = \eta$ and taking the Laplace transform of the given integral equation, we obtain*

$$s^\alpha \Phi(s)\frac{a\lambda}{(s^2+a^2)} = \frac{e^{\frac{-a}{s}}}{s^{1+\nu}},$$

solving the above equation, leads to

$$\Phi(s)=\frac{(s^2+a^2)e^{\frac{-a}{s}}}{(a\lambda)s^{1+\alpha+\nu}},$$

or equivalently

$$\Phi(s)=\frac{(s^2e^{\frac{-a}{s}}+a^2e^{\frac{-a}{s}})}{(a\lambda)s^{1+\alpha+\nu}},$$

at this point, taking inverse Laplace transform term wise, after simplifing we have

$$\phi(t)=\frac{a}{\lambda}(\frac{(t-\beta)}{a})^{\frac{\alpha+\nu}{2}}J_{\alpha+\nu}(2\sqrt{a(t-\beta)})+\frac{1}{a\lambda}(\frac{(t-\beta)}{a})^{\frac{(\alpha+\nu-2)}{2}}J_{\alpha+\nu-2}(2\sqrt{a(t-\beta)}).$$

In this section, we will also develop a more general procedure to treat singular integral equation whose solution requires exponential differential operators. Singular integral equations arise in many problems of mathematical physics. The mathematical formulation of physical phenomena often involves singular integral equations. Applications in many important fields like elastic contacts problems, the theory of porous filtering, fracture mechanics contain integral and integro-differential equation with singular kernel.

Corollary 1.1 Let us consider the following singular integral equation

$$exp(-\omega x^2)=\int_0^\infty \xi^\nu e^{-\xi}g(x\xi^\mu)d\xi, \tag{1.7}$$

the above integral equation has the following formal solution

$$g(x)=J_\nu(\sqrt{w}x:2\mu).$$

Where $J_\nu(.:.)$ stands for the Bessel – Wright function of order ν.

Note:the special function of the form defined by the series representation
$J_\nu(x:\mu)=\sum_{n=0}^\infty\frac{(-1)^n}{n!\Gamma(1+\nu+n\mu)}(x)^n$

is known as Bessel- Maitland function or the Bessel- Wright function. It has a wide application in the problem of physics, chemistry, applied sciences.

Proof. Let us rewrite the left hand side of the above equation(1.7) as below

$$exp(-\omega x^2)=(\int_0^\infty d\xi e^{-\xi}\xi^\nu\xi^{\mu xD_x}g(x)), \tag{1.8}$$

in relation (1.8), we used the following exponential identity

$$\Phi(\lambda^k x)=\Phi(e^{kln(\lambda)}x)=\Phi(e^{ln(\lambda)^k}x)=e^{ln(\lambda)^k xD_x}\Phi(x)=\lambda^{kxD_x}\Phi(x),$$

thus,

$$exp(-\omega x^2)=(\int_0^\infty e^{-\xi}\xi^{\nu+\mu xD_x}d\xi)g(x). \tag{1.9}$$

At this point, we may rewrite relation (1.9) in terms of Gamma function as follows

$$exp(-\omega x^2)=\Gamma(1+\nu+\mu xD_x)g(x). \tag{1.10}$$

From the above operational relationship and Taylor expansion of the exponential function results in

$$g(x) = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{\Gamma(1+\nu+\mu x D_x)} (x)^{2n}, \tag{1.11}$$

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(1+\nu+2n\mu)} (\sqrt{\omega} x)^{2n} = J_\nu(\sqrt{w} x : 2\mu).$$

Corollary 1.2. Let us consider the following singular integral equation

$$x^\nu exp(x) = (\int_0^\infty \xi^\nu e^{-\xi} g(x\xi) d\xi), \tag{1.12}$$

the above integral equation has the following formal solution

$$g(x) = I_\nu(2\sqrt{x\mu}),$$

where $I_\nu(2\sqrt{x\mu})$, stands for the modified Bessel function of the first kind of order ν . **Proof.** Let us rewrite the left hand side of the above equation as below

$$x^\nu exp(x) = (\int_0^\infty d\xi e^{-\xi} \xi^{xD_x+\nu}) g(x), \tag{1.13}$$

and treating the derivative operator as a constant, the evaluation of the integral yields

$$g(x) = \Gamma^{-1}(1+\nu+x\mu D_x)(x^\nu exp(x)), \tag{1.14}$$

after writing Taylor expansion of exponential function, we arrive at

$$\begin{aligned} g(x) =& \sum_{n=0}^{\infty} \frac{(1)}{n!\Gamma(1+\nu+\mu x D_x)} (x)^{n+\nu} \\ =& \sum_{n=0}^{\infty} \frac{(1)}{n!\Gamma(1+\nu+\mu(n+\nu))} (x)^{n+\nu} \\ =& I_\nu(2\sqrt{x\mu}). \end{aligned} \tag{1.15}$$

Note: From operational relation $(xD_x)x^n = nx^n$ we get the following identity

$$g(cxD_x)x^n = g(cn)x^n, \tag{1.16}$$

and $g(x)$ has Taylor series expansion.

Lemma 1.3. The following exponential operator identity holds true

$$(\tfrac{d}{dt})^{(1-\frac{1}{n})} \exp(k\tfrac{d}{dt})\Phi(t) = \frac{1}{\Gamma(\frac{1}{n})} \int_0^\infty \frac{\Phi(t+k-\xi)}{\sqrt[n]{\xi}} d\xi. \tag{1.17}$$

Proof. Let us introduce the following integral

$$y(t) = \frac{1}{\Gamma(\frac{1}{n})} \int_{-k}^\infty \frac{\exp(-p\xi)}{\sqrt[n]{(k+\xi)^{n-1}}} d\xi. \tag{1.18}$$

By making the change of variable $k + \xi = \zeta$ in the above integral we get

$$y(t) = \frac{1}{\Gamma(\frac{1}{n})}\int_0^\infty \frac{\exp(pk-p\zeta)}{\sqrt[n]{(\zeta)^{n-1}}}d\zeta, \tag{1.19}$$

after simplifying, we get

$$y(t) = \frac{exp(kp)}{\Gamma(\frac{1}{n})}\int_0^\infty \frac{\exp(-p\zeta)}{\sqrt[n]{(\zeta)^{n-1}}}d\zeta, \tag{1.20}$$

or,

$$y(t) = \frac{exp(kp)}{\sqrt[n]{p}}.$$

Let us choose $p = \frac{d}{dt}$, we obtain

$$(\frac{d}{dt})^{(1-\frac{1}{n})}\exp(k\frac{d}{dt})\Phi(t) = \frac{1}{\Gamma(\frac{1}{n})}\int_0^\infty \frac{\Phi(t+k-\xi)}{\sqrt[n]{\xi}}d\xi. \tag{1.21}$$

Special case . For $n = 2$, we have

$$(\frac{d}{dt})^{\frac{1}{2}}\exp(k\frac{d}{dt})\Phi(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{\Phi(t+k-\xi)}{\sqrt{\xi}}d\xi,$$

from which we deduce

$$\sqrt{(\frac{d}{dt})}exp(k\frac{d}{dt})\Phi(t) = \frac{1}{\Gamma(\frac{1}{n})}\int_{-k}^\infty \frac{d\xi}{\sqrt[n]{(k+\xi)^{n-1}}}exp(-\xi\frac{d}{dt})\Phi(t)$$
$$= \int_0^\infty \frac{\Phi(t+k-\xi)}{\sqrt{\pi\xi}}d\xi. \tag{1.22}$$

Lemma1.4. The following second order exponential operator relations hold true.

$$1. \exp(r(\frac{\partial}{\partial x})^2)\Phi(x) = \frac{1}{(2r\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4r}}(\Phi(x+iu)+\Phi(x-iu))du, \tag{1.23}$$

$$2. \exp(kx(\frac{\partial}{\partial x})^2)\Phi(x) = \frac{1}{(2kx\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4kx}}(\Phi(x+iu)+\Phi(x-iu))du. \tag{1.24}$$

Proof. Let us consider the following elementary integral

$$r\sqrt{\pi}\exp -r(b^2-a^2) = \int_0^\infty e^{-\frac{u^2}{4r}}\cos(au)\cosh(bu)du. \tag{1.25}$$

By integration by parts, one can easily find the value of the integral and after some algebra we obtain

$$\exp -r(b^2-a^2) = \frac{1}{(4r\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4r}}(\exp(iau)+exp(-iau))(\exp(bu)+exp(-bu))du. \tag{1.26}$$

1. In the above integral relation, we set $a = (\frac{\partial}{\partial x})$, $b = 0$ to obtain

$$\exp(r(\frac{\partial}{\partial x})^2)\Phi(x) = \frac{1}{(2r\sqrt{\pi})}\int_0^\infty du(e^{-\frac{u^2}{4r}}(exp(iu)(\frac{\partial}{\partial x}) + \exp(-iu)(\frac{\partial}{\partial x}))\Phi(x), \tag{1.27}$$

by using lemma 1.1, we get finally

$$\exp(r(\frac{\partial}{\partial x})^2)\Phi(x) = \frac{1}{(2r\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4r}}(\Phi(x+iu) + \Phi(x-iu))du.$$

2. In the above integral relation, we set $r = kx$ $a = \frac{\partial}{\partial x}$, $b = 0$ to obtain

$$\begin{aligned}\exp(kx(\frac{\partial}{\partial x})^2)\Phi(x) =& \frac{1}{(2kx\sqrt{\pi})}\int_0^\infty du(e^{-\frac{u^2}{4kx}}(\exp(iu)(\frac{\partial}{\partial x})\\ &+ \exp(-iu)(\frac{\partial}{\partial x}))\Phi(x),\end{aligned} \tag{1.28}$$

by using Lemma 1.1, we get finally

$$\exp(kx(\frac{\partial}{\partial x})^2)\Phi(x) = \frac{1}{(2kx\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4kx}}(\Phi(x+iu) + \Phi(x-iu))du. \tag{1.29}$$

Corollary 1.3. Let us consider the following Fredholm singular integral equation

$$\exp(\beta x^2) = \int_{-\infty}^\infty e^{-\xi^2}\phi(x - 2\xi\sqrt{k})d\xi, \tag{1.30}$$

the above integral equation has the following formal solution

$$\Phi(x) = \frac{exp((-\frac{1+8\lambda\beta}{1+4\lambda\beta})\beta x^2)}{\sqrt{\lambda\pi(1+4\lambda\beta)}}.$$

Proof. Let us rewrite the right hand side of the above equation as below

$$exp(\beta x^2) = \int_{-\infty}^\infty d\xi e^{-\xi^2}e^{-2\sqrt{k}\xi D_x}\Phi(x), \qquad (\tag{1.31}$$

and treating the derivative operator as a constant, the evaluation of the integral yields

$$\Phi(x) = \frac{1}{\sqrt{\pi}}e^{-\lambda D_x^2}\exp(\beta x^2), \tag{1.32}$$

at this point, using relation (1.23) leads to

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\frac{1}{(2\lambda\sqrt{\pi})}\int_0^\infty e^{-\frac{u^2}{4\lambda}}(exp(\beta(x+iu)^2) + \exp(\beta(x-iu)^2))du, \tag{1.33}$$

from which and after some easy calculations, we get

$$\Phi(x) = \frac{\exp((-\frac{1+8\lambda\beta}{1+4\lambda\beta})\beta x^2)}{\sqrt{\lambda\pi(1+4\lambda\beta)}}.$$

2. Evaluation of certain integrals

The main purpose of this section is to introduce the use of the exponential differential operator technique for evaluation of certain types of integrals.

Lemma 2.1. Considering the integral

$$I_r = I(x,\nu) = \int_0^\infty J_\nu(\tfrac{x}{(k^2+t^2)^\mu})dt, \tag{2.1}$$

as a function of parameters ν ,μ, show that$I(x,\nu)$ satisfies the following relationship

$$I_r = \int_0^\infty J_\nu(\tfrac{x}{(k^2+t^2)^\mu})dt = \tfrac{k\sqrt{\pi}}{2}\sum_{n=0}^\infty \tfrac{(-1)^n}{n!\Gamma(1+\nu+n)}\tfrac{\Gamma(\mu(n+\nu)-0.5)}{\Gamma(\mu(n+\nu))}(0.5k^{-2\mu}x)^{n+\nu}. \tag{2.2}$$

Proof. By making a change of variable $t = ky$,and letting $x = k^{2\mu}r$, we get

$$I_r = k\int_0^\infty J_\nu(\tfrac{r}{(1+y^2)^\mu})dy. \tag{2.3}$$

The above integral can be written in the following operational form

$$I_r = k\int_0^\infty J_\nu(\tfrac{r}{(1+y^2)^\mu})dy = k(\int_0^\infty (\tfrac{1}{1+y^2})^{\mu r D_r}dy)J_\nu(r), \tag{2.4}$$

after evaluation and simplifying the right hand side integral, this last result leads to

$$I_r = k\int_0^\infty J_\nu(\tfrac{r}{(1+y^2)^\mu})dy = \tfrac{k\sqrt{\pi}}{2}\tfrac{\Gamma(\mu r D_r-0.5)}{\Gamma(\mu r D_r)}J_\nu(r). \tag{2.5}$$

By using Taylor expansion of the Bessel's function of order ν, we obtain

$$I_r = \tfrac{k\sqrt{\pi}}{2}\sum_{n=0}^\infty \tfrac{(-1)^n}{n!\Gamma(1+\nu+n)}\tfrac{\Gamma(\mu r D_r-0.5)}{\Gamma(\mu r D_r)}(0.5r)^{n+\nu}, \tag{2.6}$$

finally,

$$I_r = \tfrac{k\sqrt{\pi}}{2}\sum_{n=0}^\infty \tfrac{(-1)^n}{n!\Gamma(1+\nu+n)}\tfrac{\Gamma(\mu(n+\nu)-0.5)}{\Gamma(\mu(n+\nu))}(0.5r)^{n+\nu}. \tag{2.7}$$

Lemma 2.2. Let us Consider the following integral

$$I_\alpha = \int_0^\infty Erf(\tfrac{x}{(k^2+t^2)^\mu}dt, \tag{2.8}$$

as a function of parameters k ,μ , show that I_0 satisfies the following relationship

$$I_\alpha = k\int_0^\infty Erf(\tfrac{r}{(1+y^2)^\mu})dy = \sum_{n=0}^\infty \tfrac{k(-1)^n}{n!(1+2n)}\tfrac{\Gamma(\mu(2n+1)-0.5)}{\Gamma(\mu(2n+1))}(k^{-2\mu}x)^{2n+1}. \tag{2.9}$$

Proof. By making a change of variable $t = ky$,and letting $x = k^{2\mu}r$, we get

$$I_\alpha = k\int_0^\infty Erf(\tfrac{r}{(1+y^2)^\mu})dy. \tag{2.10}$$

The above integral can be written in the following operational form

$$I_\alpha = k\int_0^\infty Erf(\tfrac{r}{(1+y^2)^\mu})dy = k(\int_0^\infty (\tfrac{1}{1+y^2})^{\mu r D_r}dy)Erf(r), \tag{2.11}$$

after evaluation and simplifying the right hand side integral, this last result leads to

$$I_\alpha = k\int_0^\infty Erf(\tfrac{r}{(1+y^2)^\mu})dy = \tfrac{k\sqrt{\pi}}{2}\tfrac{\Gamma(\mu r D_r - 0.5)}{\Gamma(\mu r D_r)} Erf(r). \quad (2.12)$$

By using Taylor expansion of the Error - function, one has

$$I_\alpha = k\int_0^\infty Erf(\tfrac{r}{(1+y^2)^\mu})dy = \sum_{n=0}^{\infty} \tfrac{k(-1)^n}{n!(1+2n)}(\tfrac{\Gamma(\mu(2n+1)-0.5)}{\Gamma(\mu(2n+1))} r^{2n+1}, \qquad) \quad (2.13)$$

finally we get

$$I_\alpha = k\int_0^\infty Erf(\tfrac{r}{(1+y^2)^\mu})dy = \sum_{n=0}^{\infty} \tfrac{k(-1)^n}{n!(1+2n)}\tfrac{\Gamma(\mu(2n+1)-0.5)}{\Gamma(\mu(2n+1))}(k^{-2\mu}x)^{2n+1}. \quad (2.14)$$

3. Main Results

Solution to generalized space fractional Fokker-Planck equation with non-constant coefficients, which is used to study the beam life time due to quantum fluctuation in the storage ring. Fokker-Planck equation arises frequently in the theory of stochastic processes. The physical interpretation of the variables in this equation is that, the probability that a random variable has the value x at time t. For example, u(x,t) might be the probability distribution of the position of a harmonically bound particle in Brownian motion, or probability distribution of the deflection x of an electrical noise traces at time t.

Problem 3. *Let us consider the following generalized Fokker-Planck equation*

$$(\tfrac{t^{-\nu}}{\nu+1})u_t + \alpha(\tfrac{\partial}{\partial x}x\tfrac{\partial}{\partial x})^\lambda u + \beta x^\mu u(x,t) = 0,$$
$$u(x,0) = f(x).$$

Solution: *In order to obtain a solution for equation (3.1) in view of [2],[3] first by solving the first order PDE with respect to t, and applying the initial condition (3.2), we get the following relationship*

$$u(x,t) = \exp(-\beta t^{\nu+1}x^\mu)\exp(-ax^{\nu+1}(\tfrac{\partial}{\partial x}x\tfrac{\partial}{\partial x})^\lambda)f(x).$$

In order to find the result of the action of exponential operator, we make use of part three of Lemma 1.1, by choosing $\lambda = \beta = 0.5$, $k = at^{\nu+1}$, $s = (\frac{\partial}{\partial x}x\frac{\partial}{\partial x},)$ *to obtain*

$$u(x,t) = exp(-\beta t^{\nu+1}x^\mu)\exp(-ax^{\nu+1}(\tfrac{\partial}{\partial x}x\tfrac{\partial}{\partial x})^\lambda)f(x), \qquad) \quad (3.1)$$

or

$u(x,t) = \exp(-\lambda t^{\nu+1}x^\mu)\frac{1}{\pi}\int_0^\infty g(r)dr,$

where, $g(r) = e^{-r^\lambda(at^{\nu+1}\cos\lambda\pi)}\sin(at^{\nu+1}r^\lambda\sin\lambda\pi)(\int_0^\infty d\tau e^{-r\tau-\tau\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}f(x)).$

Let us take $f(x) = \exp(-qx)$, $\lambda = 0.5$,

$u(x,t) = \exp(-0.5t^{\nu+1}x^\mu)\frac{1}{\pi}\int_0^\infty e^{-ar^{0.5}t^{\nu+1}cos0.5\pi}......$

$$.........\sin(at^{\nu+1}r^{0.5}\sin 0.5\pi)(\int_0^\infty d\tau e^{-r\tau-\tau\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}\exp(-qx))dr,$$

then after some manipulation, and using the following relationship [5]

$$\exp(-\tau\frac{\partial}{\partial x}x\frac{\partial}{\partial x})\exp(-qx)=\sum_{n=1}^{\infty}L_n(qx,\tau), \tag{3.2}$$

we get the formal solution as below

$u(x,t)=\exp(-0.5t^{\nu+1}x^{\mu})\frac{1}{\pi}\int_0^\infty sin(\pi a t^{\nu+1}\sqrt{r})(\int_0^\infty d\tau e^{-r\tau-\tau\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}\exp(-qx))dr,$
or,

$u(x,t)=\exp(-0.5t^{\nu+1}x^{\mu})\frac{1}{\pi}\int_0^\infty \sin(\pi a t^{\nu+1}\sqrt{r})(\int_0^\infty e^{-r\tau}\sum_{n=1}^{\infty}L_n(qx,\tau)d\tau)dr.$
At this point, in order to simplify the above relationship, we consider the following - well – known relationship for Laguerre polynomials of two variable as below.

$$\sum_{n=1}^{\infty}L_n(x,\tau)=\frac{1}{1-\tau}\exp(\frac{x}{1-\tau}), \tag{3.3}$$

$u(x,t)=\exp(-0.5t^{\nu+1}x^{\mu})\frac{1}{\pi}\int_0^\infty \sin(\pi a t^{\nu+1}\sqrt{r})(\int_0^\infty e^{-r\tau}\frac{1}{1-\tau}\exp(\frac{qx}{1-\tau})d\tau)dr,$
the above double integral may be simplified as following

$u(x,t)=\exp(-0.5t^{\nu+1}x^{\mu})\frac{1}{\pi}\int_0^\infty \frac{1}{1-\tau}\exp(\frac{qx}{1-\tau})d\tau(\int_0^\infty e^{-r\tau}\sin(\pi a t^{\nu+1}\sqrt{r})dr,$
thus the result will become

$u(x,t)=\frac{(a\pi t^{1+\nu})^2}{2}\exp(-0.5t^{\nu+1}x^{\mu})\int_0^\infty\frac{1}{\tau(1-\tau)\sqrt{\tau}}\exp(\frac{qx}{1-\tau}-\frac{(a\pi t^{1+\nu})^2}{4\tau})d\tau.$

Remark. For the general case,$0<\lambda<1$,$\lambda\neq 0.5$, and $f(x)$ is any differentiable function of all orders (we assume that,$f(x)$ has Taylor series expansion), we may solve the above fractional partial differential equation by making use of the above procedure. The procedure as described above should be generally applicable to the most fractional partial differential equations with non - constant coefficients.

4. Conclusion

Operational methods provide fast and universal mathematical tool for obtaining the solution of PDEs or even FPDEs. Combination of integral transforms, operational methods and special functions give more powerful analytical instrument for solving a wide range of engineering and physical problems. The paper is devoted to study exponential operators and their applications in solving certain boundary value problems such as the space fractional Fokker-Planck equation. We show that the present technique could be used to solve different kind of fractional partial differential equations. The present method can be readily applied to certain singular integral equations such as generalized Lamb - Bateman equation. The results of these developments will be published in future papers.

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