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Some properties of Generalized Fibonacci difference bounded and p-absolutely convergent sequences*

Anupam Das and Bipan Hazarika*

ABSTRACT: The main objective of this paper is to introduced a new sequence space $l_p(\hat{F}(r,s))$, $1 \le p \le \infty$ by using the band matrix $\hat{F}(r,s)$. We also establish a few inclusion relations concerning this space and determine its $\alpha-, \beta-, \gamma-$duals. We also characterize some matrix classes on the space $l_p(\hat{F}(r,s))$ and examine some geometric properties of this space.

Key Words: Fibonacci numbers; Difference matrix; α-,β-,γ-duals; Matrix Transformations; fixed point property; Banach-Saks type p.

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1. Introduction

Let ω be the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. By l_∞, c, c_0 and l_p $(1 \le p < \infty)$, we denote the sets of all bounded, convergent, null sequences and $p-$absolutely convergent series, respectively. Also we use the convensions that $e = (1,1,...)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,...\}$.

Let $X$and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,k \in \mathbb{N}$. We write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Then we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \ \ (n \in \mathbb{N}). \tag{1.1}$$

* *The corresponding author
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For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Also if $x \in \omega$, then we write $x = (x_k)_{k=0}^{\infty}$.

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X, Y)$ iff the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$.

The approach constructing a new sequence space by means of matrix domain has recently employed by several authors.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}.$$

Let Δ denote the matrix $\Delta = (\Delta_{nk})$ defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n \\ 0, & 0 \le k < n-1 \quad or \quad k > n \end{cases}$$

The concept of matrix domain we refer to [2,3,4,9,12,13,14,15,16,17,18].
Define the sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \ge 2$.
Fibonacci numbers have many interesting properties and applications. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also some basic properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha \quad (golden\ ratio),$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}),$$

$$\sum_k \frac{1}{f_k} \text{ converges},$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \ge 1)(Cassini formula)$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.
For the properties of Fibonnaci numbers and matrix domain related to Fibonnaci numbers we refer to [1,8,11].

A sequence space X is called a $FK-$space if it is complete linear metric space with continuous coordinates $p_n : X \to \mathbb{R}(n \in \mathbb{N})$, where $\mathbb{R}$ denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A $BK-$ space is a normed $FK-$ space, that is a $BK-$space is a Banach space with continuous coordinates. The space $l_p(1 \le p < \infty)$ is a BK-space with the norm

$$\| x \|_p = \left(\sum_{k=0}^{\infty} | x_k |^p \right)^{1/p}$$

and c_0, c and l_∞ are BK-spaces with the norm

$$\| x \|_\infty = \sup_k |x_k|.$$

The sequence space λ is said to be solid if and only if

$$\tilde{\lambda} = \{(u_k) \in \omega : \exists (x_k) \in \lambda \text{ such that } \mid u_k \mid \leq \mid x_k \mid \forall k \in \mathbb{N}\} \subset \lambda.$$

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if every $x \in X$, there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_{m\to\infty} \| x - \sum_{n=0}^{m} \alpha_n b_n \| = 0$.

The $\alpha-, \beta-, \gamma-$duals of the sequence space X are respectively defined by
$X^\alpha = \{a = (a_k) \in \omega : ax = (a_k x_k) \in l_1 \,\forall\, x = (x_k) \in X\}$,
$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \,\forall\, x = (x_k) \in X\}$,
and
$X^\gamma = \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \,\forall\, x = (x_k) \in X\}$,
where cs and bs are the sequence spaces of all convergent and bounded series, respectively (see for instance [2,7,15]).
Now let $A = (a_{nk})$ be an infinite matrix and consider the following conditions:

$$\sup_n \sum_k |a_{nk}|^q < \infty, q = \frac{p}{p-1} \tag{1.2}$$

$$\lim_n a_{nk} \text{ exists } \forall\, k \tag{1.3}$$

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} a_{nk} \right|^q < \infty, q = \frac{p}{p-1} \tag{1.4}$$

$$\lim_n \sum_k |a_{nk}| = \sum_k \left| \lim_n a_{nk} \right| \tag{1.5}$$

Now we may give the following lemma due to Stieglitz and Tietz [12] on the characterization of the matrix transformations between some sequence spaces.

Lemma 1.1. *The following statements hold:*

(a) $A = (a_{nk}) \in (l_p, c)$ *iff (1.2),(1.3) holds,* $1 < p < \infty$.

(b) $A = (a_{nk}) \in (l_p, l_1)$ *iff (1.4) holds,* $1 < p < \infty$.

(c) $A = (a_{nk}) \in (l_\infty, c)$ *iff (1.3),(1.5) holds.*

(d) $A = (a_{nk}) \in (l_p, l_\infty)$ *iff (1.2) holds,* $1 < p < \infty$.

2. Fibonacci difference sequence space $l_p(\hat{F}(r,s))$

In this section, we have used the Fibonacci band matrix $\hat{F}(r,s) = (f_{nk}(r,s))$,, which was introduced by Candan [5], and introduce the sequence space $l_p(\hat{F}(r,s))$. Also we present some inclusion theorems and construct the Schauder basis of the space $l_p(\hat{F}(r,s))$.

Let f_n be the nth Fibonacci number for every $n \in \mathbb{N}$. Then we define the infinite matrix $\hat{F}(r,s) = (f_{nk}(r,s))$ by

$$f_{nk}(r,s) = \begin{cases} s\frac{f_{n+1}}{f_n}, & k = n-1 \\ r\frac{f_n}{f_{n+1}}, & k = n \\ 0, & 0 \leq k < n-1 \quad or \quad k > n \end{cases} \tag{2.1}$$

where $n, k \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$.

Define the sequence $y = (y_n)$, which will be frequently used, by the $\hat{F}(r,s)$ - transform of a sequence $x = (x_n)$, i.e., $y_n = \hat{F}(r,s)_n(x)$, where

$$y_n = \begin{cases} r\frac{f_0}{f_1}x_0 = rx_0, & n = 0 \\ r\dfrac{f_n}{f_{n+1}}x_n + s\dfrac{f_{n+1}}{f_n}x_{n-1}, & n \geq 1 \end{cases} \tag{2.2}$$

where $n \in \mathbb{N}$.

Moreover it is obvious that $\hat{F}(r,s)$ is a triangle. Thus it has a unique inverse $\hat{F}(r,s)^{-1} = (\hat{f}_{nk}(r,s)^{-1})$ and it is given by

$$\hat{f}_{nk}(r,s)^{-1} = \begin{cases} \frac{1}{r}\left(-\frac{s}{r}\right)^{n-k}\frac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \tag{2.3}$$

for all $n, k \in \mathbb{N}$. There we have by (2.3) that

$$x_k = \sum_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j; (k \in \mathbb{N}). \tag{2.4}$$

Now we introduce new Fibonacci sequence spaces as follows

$$l_p(\hat{F}(r,s)) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} \right|^p < \infty \right\}, 1 \leq p < \infty$$

and

$$l_\infty(\hat{F}(r,s)) = \left\{ x = (x_n) \in \omega : \sup_n \left| r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} \right| < \infty \right\}.$$

The sequence spaces $l_p(\hat{F}(r,s))$ and $l_\infty(\hat{F}(r,s))$ may be redefined as

$$l_p(\hat{F}(r,s)) = (l_p)_{\hat{F}(r,s)}, l_\infty(\hat{F}(r,s)) = (l_\infty)_{\hat{F}(r,s)}. \tag{2.5}$$

In this section, we give some results related to the space $l_p(\hat{F}(r,s)), 1 \leq p \leq \infty$.

Theorem 2.1. *Let $1 \leq p < \infty$. Then $l_p(\hat{F}(r,s))$ is a BK-space with norm*

$$\| x \|_{l_p(\hat{F}(r,s))} = \left(\sum_k \left| \hat{F}(r,s)_k(x) \right|^p \right)^{1/p}$$

and $l_\infty(\hat{F}(r,s))$ is a BK-space with norm

$$\| x \|_{l_\infty(\hat{F}(r,s))} = \sup_k |\hat{F}(r,s)_k(x)|.$$

Proof. Since (2.5) holds, l_p and l_∞ are BK-spaces with respect to their natural norm and the matrix $\hat{F}(r,s)$ is triangular matrix. By Theorem 4.3.3 of Wilansky [18] gives the fact that the spaces $l_p(\hat{F}(r,s)), 1 \leq p < \infty$ and $l_\infty(\hat{F}(r,s))$ are BK space with the given norms. □

Remark 2.2. *The spaces $l_p(\hat{F}(r,s))$ for $1 \leq p < \infty$ and $l_\infty(\hat{F}(r,s))$ are non-absolute type because $\| x \|_{l_p(\hat{F}(r,s))} \neq \| |x| \|_{l_p(\hat{F})(r,s)}$ and $\| x \|_{l_\infty(\hat{F}(r,s))} \neq \| |x| \|_{l_\infty(\hat{F})(r,s)}$, where $|x| = (|x_k|)$.*

Theorem 2.3. *The sequence spaces $l_p(\hat{F}(r,s)), 1 \leq p < \infty$ and $l_\infty(\hat{F}(r,s))$ of non-absolute type are linearly isomorphic to the spaes l_p and l_∞, respectively, i.e. $l_p(\hat{F}(r,s)) \cong l_p$ and $l_\infty(\hat{F}(r,s)) \cong l_\infty$.*

Proof. To prove this, we have to show that there exists a linear bijective mapping between $l_p(\hat{F}(r,s))$ and l_p for $1 \leq p \leq \infty$.
Let us consider a mapping T defined from $l_p(\hat{F}(r,s))$ to l_p by $Tx = \hat{F}(r,s)(x) = y \in l_p$ for every $x \in l_p(\hat{F}(r,s))$, where $x = (x_k)$ and $y = (y_k)$.

It is obvious that T is linear. Further, it is trivial that $x = 0$ whenever $Tx = 0$. Hence T is injective.

Let $y = (y_k) \in l_p$, $1 \leq p \leq \infty$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \frac{1}{r} \left(-\frac{s}{r} \right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \text{ for all } k \in \mathbb{N}.$$

Then, in the cases $1 \leq p < \infty$ and $p = \infty$ we get

$$\| x \|_{l_p(\hat{F}(r,s))} = \left(\sum_k \left| r \tfrac{f_k}{f_{k+1}} x_k + s \tfrac{f_{k+1}}{f_k} x_{k-1} \right|^p \right)^{1/p}$$

$$= \left(\sum_k \left| r \tfrac{f_k}{f_{k+1}} \sum_{j=1}^{k} \tfrac{1}{r} \left(-\tfrac{s}{r} \right)^{k-j} \tfrac{f_{k+1}^2}{f_j f_{j+1}} y_j + s \tfrac{f_{k+1}}{f_k} \sum_{j=1}^{k-1} \tfrac{1}{r} \left(-\tfrac{s}{r} \right)^{k-j-1} \tfrac{f_{k+1}^2}{f_j f_{j+1}} y_j \right|^p \right)^{1/p}$$

$$= \left(\sum_k |y_k|^p \right)^{1/p}$$

$$= \| y \|_{l_p} < \infty.$$

Similarly we can show that $\| x \|_{l_\infty(\hat{F}(r,s))} = \| y \|_\infty$.
Thus we have $x \in l_p(\hat{F}(r,s))$ for $1 \leq p \leq \infty$. Hence T is surjective and norm

preserving. Consequently T is a linear bijection which proves that the spaces $l_p(\hat{F})(r,s)$ and l_p are linearly isomorphic for $1 \leq p \leq \infty$. □

Theorem 2.4. *$l_p \subset l_p(\hat{F}(r,s))$ holds for $1 \leq p \leq \infty$ and for finite r, s such that $\left|-\frac{s}{r}\right| \geq 1$, $\mid r \mid \leq 1$ and $\mid s \mid \leq 1/2$.*

Proof. Let $x = (x_k) \in l_p$ and $1 \leq p \leq \infty$. Since the inequalities $\frac{f_k}{f_{k+1}} \leq 1$ and $\frac{f_{k+1}}{f_k} \leq 2$ for every $k \in \mathbb{N}$ therefore we have

$\sum_k \mid \hat{F}(r,s)_k(x) \mid^p$

$= \sum_k \left| r\frac{f_k}{f_{k+1}}x_k + s\frac{f_{k+1}}{f_k}x_{k-1}\right|^p$

$\leq \mid r \mid^p \sum_k \mid x_k \mid^p + \mid 2s \mid^p \sum_k \mid x_{k-1} \mid^p$

and

$\sup_{k\in\mathbb{N}} \mid \hat{F}(r,s)_k(x) \mid \leq (\mid r \mid + \mid 2s \mid) \sup_{k\in\mathbb{N}} \mid x_k \mid$

which together gives

$\parallel x \parallel_{l_p(\hat{F}(r,s))} \leq (\mid r \mid + \mid 2s \mid) \parallel x \parallel_{l_p}$ for $1 \leq p \leq \infty$, where r, s are finite.

Therefore $\parallel x \parallel_{l_p(\hat{F})(r,s)} < \infty$, since $x \in l_p$.

Hence $l_p \subseteq l_p(\hat{F}(r,s))$. Further since $x = (x_k) = \left(\frac{1}{r}\left(-\frac{s}{r}\right)^k f_{k+1}^2\right)$ is in $l_p(\hat{F}(r,s)) - l_p$ for $\left|-\frac{s}{r}\right| \geq 1$. Therefore $l_p \subset l_p(\hat{F}(r,s))$ for $1 \leq p \leq \infty$. □

Theorem 2.5. *For $1 \leq p < q$, $l_p(\hat{F}(r,s)) \subset l_q(\hat{F}(r,s))$ holds.*

Proof. Let $1 \leq p < q$ and $x \in l_p(\hat{F}(r,s))$. Then we obtain from Theorem 2.3 that $y \in l_p$, where $y = \hat{F}(r,s)(x)$. We have $l_p \subset l_q$ which gives $y \in l_q$. This means that $x \in l_q(\hat{F}(r,s))$. Hence we have $l_p(\hat{F}(r,s)) \subset l_q(\hat{F}(r,s))$. □

Theorem 2.6. *If $\left|-\frac{s}{r}\right| \geq 1$ then the space l_∞ does not include the space $l_p(\hat{F}(r,s))$.*

Proof. Let $\left|-\frac{s}{r}\right| \geq 1$ and $x = (x_k) = \left(\frac{1}{r}\left(-\frac{s}{r}\right)^k f_{k+1}^2\right)$. We know that $f_{k+1}^2 \to \infty$ as $k \to \infty$ and $\hat{F}(r,s)(x) = (1,0,0,0,...)$. Therefore the sequence lies in $l_p(\hat{F}(r,s))$ but not in l_∞. This completes the proof. □

Theorem 2.7. *If $\left|-\frac{s}{r}\right| \geq 1$ then the space bv_p does not include the space $l_p(\hat{F}(r,s))$.*

Proof. Let $\left|-\frac{s}{r}\right| \geq 1$ and $x = (x_k) = \left(\frac{1}{r}\left(-\frac{s}{r}\right)^k f_{k+1}^2\right)$. We know that $f_{k+1}^2 \to \infty$ as $k \to \infty$ and $\hat{F}(r,s)(x) = (1,0,0,0,...)$ and $\Delta x = (\Delta x_k) = \left(-\frac{1}{r}\left(-\frac{s}{r}\right)^{k-1}\left(\frac{s}{r}f_{k+1}^2 + f_k^2\right)\right)$.
Clearly for $\left|-\frac{s}{r}\right| \geq 1$, $\Delta x \notin l_p$. Therefore the sequence lies in $l_p(\hat{F}(r,s))$ but not in bv_p. This completes the proof. □

Lemma 2.8. *[2] Let λ be a BK-space including the space ϕ. Then λ is solid if and only if $l_\infty\lambda \subset \lambda$.*

Now we give a sequence of points of the space $l_p(\hat{F}(r,s))$ which will form the basis for the space $l_p(\hat{F}(r,s))$ for $1 \le p < \infty$.

Theorem 2.9. *Let $1 \le p < \infty$ and define the sequence $c^{(n)} \in l_p(\hat{F}(r,s))$ for every fixed $n \in \mathbb{N}$ by*

$$(c^{(n)})_k = \begin{cases} 0, & 0 \le k \le n-1 \\ \frac{1}{r}.\left(-\frac{s}{r}\right)^{k-n}.\frac{f_{k+1}^2}{f_n f_{n+1}}, & k \ge n \end{cases} \tag{2.6}$$

where $n \in \mathbb{N}$. Then the sequence $(c^{(n)})_{n=0}^{\infty}$ is a basis for the space $l_p(\hat{F}(r,s))$, and every $x \in l_p(\hat{F}(r,s))$ has a unique representation of the form

$$x = \sum_n \hat{F}(r,s)_n(x)c^{(n)}. \tag{2.7}$$

Proof. Let $1 \le p < \infty$. It is obvious by that $\hat{F}(r,s)(c^{(n)}) = e^{(n)} \in l_p$ $(k \in \mathbb{N})$ and hence $c^{(n)} \in l_p(\hat{F}(r,s))$ for all $k \in \mathbb{N}$.

Further, let $x \in l_p(\hat{F}(r,s))$. For any non-negative integer m, we put $x^{(m)} = \sum\limits_{n=0}^{m} \hat{F}(r,s)_n(x)c^{(n)}$.

Then we have that

$$\hat{F}(r,s)(x^{(m)}) = \sum_{n=0}^{m} \hat{F}(r,s)_n(x)\hat{F}(r,s)(c^{(n)}) = \sum_{n=0}^{m} \hat{F}(r,s)_n(x)e^{(n)}$$

and hence

$$\hat{F}(r,s)_k(x - x^{(m)}) = \begin{cases} 0, & 0 \le k \le m \\ \hat{F}(r,s)_k(x), & k > m \end{cases}$$

where $k, m \in \mathbb{N}$.

For any given $\epsilon > 0$, there is a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} \left|\hat{F}(r,s)_n(x)\right|^p \le \left(\frac{\epsilon}{2}\right)^p.$$

Therefore we have for every $m \ge m_0$ that

$$\begin{aligned} &\| x - x^{(m)} \|_{l_p(\hat{F}(r,s))} \\ &= \left(\sum_{n=m+1}^{\infty} \left|\hat{F}(r,s)_n(x)\right|^p\right)^{1/p} \\ &\le \left(\sum_{n=m_0+1}^{\infty} \left|\hat{F}(r,s)_n(x)\right|^p\right)^{1/p} \le \frac{\epsilon}{2} < \epsilon \end{aligned}$$

which shows that $\lim_{m\to\infty} \| x - x^{(m)} \|_{l_p(\hat{F}(r,s))} = 0$ and hence x is represented as in (2.7).

Now we are going to show the uniqueness of the representation (2.7) of $x \in l_p(\hat{F}(r,s))$. Let $x = \sum_k \mu_k(x) c^{(k)}$. We have $\hat{F}(r,s)$ is a linear mapping from $l_p(\hat{F}(r,s))$ to l_p. Since any matrix mapping between FK spaces is continuous, so $\hat{F}(r,s)$ is continuous.

Now

$$\hat{F}(r,s)_n(x) = \sum_k \mu_k(x) \hat{F}(r,s)_n(c^{(k)}) = \mu_n(x) \quad (n \in \mathbb{N}).$$

Hence the representation (2.7) is unique. □

3. The α-,β- and γ-duals of the space $l_p(\hat{F}(r,s))$

In this section, we determine the $\alpha-$, $\beta-$ and $\gamma-$duals of the sequence space $l_p(\hat{F}(r,s))$. Since the case $p = 1$ can be proved by analogy, we omit the proof of that case and consider only the case $1 < p \leq \infty$.

Theorem 3.1. *The α-dual of the sequence space $l_p(\hat{F}(r,s))$ is the set $d_1 = \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} b_{nk} \right|^q < \infty, q = \frac{p}{p-1} \right\}$ where $1 < p \leq \infty$ and the matrix $B = (b_{nk})$ is defined as follows*

$$b_{nk} = \begin{cases} \frac{1}{r}\left(-\frac{s}{r}\right)^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$ and $a = (a_n) \in \omega$.

Proof. Let $a = (a_n) \in \omega$. Also for every $x = (x_n) \in \omega$, we put $y = (y_n) = \hat{F}(r,s)(x)$. Then it follows by (2.4) that $x_k = \sum_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j$ and

$$B_n(y) = \sum_{k=0}^{n} b_{nk} y_k = \sum_{k=0}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{n-k} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = a_n x_n. \tag{3.1}$$

where $n \in \mathbb{N}$.

Thus we observe by (3.1) that $ax = (a_n x_n) \in l_1$ whenever $x \in l_p(\hat{F}(r,s))$ if and only if $By \in l_1$ whenever $y \in l_p$. Therefore we derive by using the Lemma 1.1 that $\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} b_{nk} \right|^q < \infty$ which implies that $\left\{ l_p(\hat{F}(r,s)) \right\}^\alpha = d_1$. □

Theorem 3.2. *Define the sets d_2, d_3 and d_4 by*
$d_2 = \left\{ a = (a_k) \in \omega : \sup_n \sum_k | d_{nk} |^q < \infty, q = \frac{p}{p-1} \right\}$,
$d_3 = \{ a = (a_k) \in \omega : \lim_n d_{nk} \text{ exists } \forall\, k \}$,

and $d_4 = \left\{ a = (a_k) \in \omega : \lim_n \sum_{k=0}^{n} \mid d_{nk} \mid = \sum_k \mid \lim_n d_{nk} \mid \right\}$.

Then $\left\{ l_p(\hat{F}(r,s)) \right\}^{\beta} = d_2 \cap d_3$ *and* $\left\{ l_\infty(\hat{F}(r,s)) \right\}^{\beta} = d_2 \cap d_4$ *where* $1 < p < \infty$ *and* $D = (d_{nk})$ *is defined by*

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_n, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

Proof. Let $a = (a_k) \in \omega$ and consider the equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \right) = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right) y_k = D_n(y) \tag{3.2}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_n, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

where $n, k \in \mathbb{N}$. Then we deduce from Lemma 1.1 that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in l_p(\hat{F}(r,s))$ if and only if $Dy \in c$ whenever $y \in l_p$. Thus $a \in \left\{ l_p(\hat{F}(r,s)) \right\}^{\beta}$ if and only if $a \in d_2$, $a \in d_3$. Hence $\left\{ l_p(\hat{F}(r,s)) \right\}^{\beta} = d_2 \cap d_3$. Similarly, we can show that $\left\{ l_\infty(\hat{F}(r,s)) \right\}^{\beta} = d_3 \cap d_4$. □

Theorem 3.3. $\left\{ l_p(\hat{F}(r,s)) \right\}^{\gamma} = d_2, 1 < p \le \infty$.

Proof. This result can be obtained from Lemma 1.1. □

4. Some matrix transformations related to the sequence space $l_p(\hat{F}(r,s))$

In this section, we characterize the classes $\left(l_p(\hat{F}(r,s)), X \right)$, where $1 \le p \le \infty$ and X is any of the spaces l_∞, l_1, c and c_0.

We use the following lemma to prove our results.

Lemma 4.1. *[2] Let* $C = (c_{nk})$ *be defined via a sequence* $a = (a_k) \in \omega$ *and the inverse matrix* $V = (v_{nk})$ *of the triangle matrix* $U = (u_{nk})$ *by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. *Then for any sequence space* λ,
$\lambda_U^{\gamma} = \{a = (a_k) \in \omega : C \in (\lambda, l_\infty)\}$ *and* $\lambda_U^{\beta} = \{a = (a_k) \in \omega : C \in (\lambda, c)\}$.

Theorem 4.2. *Let* $\lambda = l_p$, $1 \le p \le \infty$ *and* μ *be an arbitrary subset of* ω. *Then* $A = (a_{nk}) \in (\lambda_{\hat{F}(r,s)}, \mu)$ *if and only if*

$$D^{(m)} = \left(d_{nk}^{(m)}\right) \in (\lambda, c)\, for\, all\, n \in \mathbb{N}, \tag{4.1}$$

$$D = (d_{nk}) \in (\lambda, \mu), \tag{4.2}$$

where

$$d_{nk}^{(m)} = \begin{cases} \sum\limits_{j=k}^{m} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}, & 0 \le k \le m \\ 0, & k > m \end{cases}$$

and $d_{nk} = \sum\limits_{j=k}^{\infty} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}$ *for all* $k, m, n \in \mathbb{N}$.

Proof. To prove this theorem, we follow the similar way due to Kirişçi and Başar [9]. Let $A = (a_{nk}) \in (\lambda_{\hat{F}(r,s)}, \mu)$ and $x = (x_k) \in \lambda_{\hat{F}(r,s)}$. We have from (2.4),
$x_k = \sum\limits_{j=0}^{k} \frac{1}{r}\left(-\frac{s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j$ for all $k \in \mathbb{N}$.
From (3.2) we get

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\left(\sum_{j=k}^{m} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}\right) y_k = \sum_{k=0}^{m} d_{nk}^{(m)} y_k = D_n^{(m)}(y), \tag{4.3}$$

for all $m, n \in \mathbb{N}$.
Since Ax exists, $D^{(m)} \in (\lambda, c)$. As $m \to \infty$ in the equality (4.3), we obtain $Ax = Dy$ which implies $D \in (\lambda, \mu)$.

Conversely, suppose (4.1) and (4.2) holds and take any $x = (x_k) \in \lambda_{\hat{F}(r,s)}$. Then we have $(d_{nk}) \in \lambda^{\beta}$ which gives together with (4.1) that $A_n = (a_{nk})_{k\in\mathbb{N}} \in \lambda_{\hat{F}(r,s)}^{\beta}$ for all $n \in \mathbb{N}$. Thus Ax exists. Therefore we derive by equality (4.3) as $m \to \infty$ that $Ax = Dy$ and this shows that $A \in (\lambda_{\hat{F}(r,s)}, \mu)$. □

Now we consider the following conditions

$$\sup_n \sum_k \left|d_{nk}^{(m)}\right|^q < \infty, q = \frac{p}{p-1} \tag{4.4}$$

$$\lim_n d_{nk}^{(m)} \text{ exists } \forall\, k \tag{4.5}$$

$$\lim_n \sum_k \left|d_{nk}^{(m)}\right| = \sum_k \left|\lim_n d_{nk}^{(m)}\right| \tag{4.6}$$

$$\sup_n \sum_k |d_{nk}|^q < \infty, q = \frac{p}{p-1} \tag{4.7}$$

$$\lim_{n} d_{nk} \text{ exists } \forall\, k \tag{4.8}$$

$$\lim_{n} \sum_{k} |d_{nk}| = \sum_{k} \left| \lim_{n} d_{nk} \right| \tag{4.9}$$

$$\sup_{K \in \mathcal{F}} \sum_{k} \left| \sum_{n \in K} d_{nk} \right|^{q} < \infty, q = \frac{p}{p-1} \tag{4.10}$$

Combining Theorems 4.2 and Lemma 1.1, we derive the following results:

Corollary 4.3. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

(a) $A \in (l_p(\hat{F}(r,s)), c), 1 < p < \infty$ if and only if (4.4),(4.5),(4.7),(4.8).

(b) $A \in (l_p(\hat{F}(r,s)), l_1), 1 < p < \infty$ if and only if (4.4),(4.5),(4.10).

(c) $A \in (l_\infty(\hat{F}(r,s)), c)$ if and only if (4.5),(4.6),(4.8),(4.9).

(d) $A \in (l_p(\hat{F}(r,s)), l_\infty), 1 < p < \infty$ if and only if (4.4),(4.5),(4.7).

(e) $A \in (l_\infty(\hat{F}(r,s)), l_1)$ if and only if (4.5),(4.6),(4.10).

(f) $A \in (l_\infty(\hat{F}(r,s)), l_\infty)$ if and only if (4.5),(4.6),(4.7).

5. Some geometric properties of the space $l_p(\hat{F}(r,s))\,(1 < p < \infty)$

In this section, we study some geometric properties of the space $l_p(\hat{F}(r,s))\,(1 < p < \infty)$.

For geometric properties we refer to [8,6,10].

A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X (see [17]), where

$$t_k(z) = \frac{1}{k+1}\,(z_0 + z_1 + ... + z_k)\ (k \in \mathbb{N}) \tag{5.1}$$

A Banach space X is said to have the weak Banach-Saks property whenever, given any weakly null sequence $(x_n) \subset X$, there exists a subsequence (z_n) of (x_n) such that the sequence $\{t_k(z)\}$ is strongly convergent to zero.

In [6], García-Falset introduced the following coefficient:

$$R(X) = \sup\left\{\liminf_{n\to\infty} \parallel x_n - x \parallel : (x_n) \subset B(X), x_n \to 0(weakly), x \in B(X)\right\} \tag{5.2}$$

where $B(X)$ denotes the unit ball of X.

Remark 5.1. *[6] A Banach space X with $R(X) < 2$ has the weak fixed point property.*

Let $1 < p < \infty$. A Banach space is said to have the Banach-Saks type p or the property $(BS)_p$ if every weakly null sequence (x_k) has a subsequence (x_{k_l}) such that for some $C > 0$,

$$\| \sum_{l=0}^{n} x_{k_l} \| < C(n+1)^{1/p} \tag{5.3}$$

for all $n \in \mathbb{N}$ (see [10]).

Now we are going to prove some geometric properties of the space $l_p(\hat{F}(r,s))$ for $1 < p < \infty$.

Theorem 5.2. *Let $1 < p < \infty$. Then the space $l_p(\hat{F}(r,s))$ has the Bnach-Saks type p.*

Proof. Let (ϵ_n) be a sequence of positive numbers for which $\sum \epsilon_n \leq 1/2$, and also let (x_n) be a weakly null sequence in $B(l_p(\hat{F}(r,s)))$. Set $z_0 = x_0 = 0$ and $z_1 = x_{n_1} = x_1$. Then there exists $m_1 \in \mathbb{N}$ such that

$$\| \sum_{i=m_1+1}^{\infty} z_1(i) e^{(i)} \|_{l_p(\hat{F}(r,s))} < \epsilon_1 \tag{5.4}$$

Since (x_n) being a weakly null sequence implies $x_n \to 0$ coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\| \sum_{i=0}^{m_1} x_n(i) e^{(i)} \|_{l_p(\hat{F}(r,s))} < \epsilon_1, \text{ when } n \geq n_2.$$

Set $z_2 = x_{n_2}$. Then there exists an $m_2 > m_1$ such that

$$\| \sum_{i=m_2+1}^{\infty} z_2(i) e^{(i)} \|_{l_p(\hat{F}(r,s))} < \epsilon_2.$$

Again using the fact that $x_n \to 0$ coordinatewise, there exists an $n_3 \geq n_2$ such that

$$\| \sum_{i=0}^{m_2} x_n(i) e^{(i)} \|_{l_p(\hat{F}(r,s))} < \epsilon_2, \text{ when } n \geq n_3.$$

If we continue this process, we can find two increasing subsequences (m_i) and (n_i) such that

$$\| \sum_{i=0}^{m_j} x_n(i) e^{(i)} \|_{l_p(\hat{F}(r,s))} < \epsilon_j \text{ for each } n \geq n_{j+1}$$

and

$$\| \sum_{i=m_j+1}^{\infty} z_j(i) e^{(i)} \|_{l_p(\hat{F}(r,s))} < \epsilon_j, \text{ where } z_j = x_{n_j}.$$

Hence

$\| \sum_{j=0}^{n} z_j \|_{l_p(\hat{F}(r,s))}$

$$= \| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} z_j(i)e^{(i)} + \sum_{i=m_{j-1}+1}^{m_j} z_j(i)e^{(i)} + \sum_{i=m_j+1}^{\infty} z_j(i)e^{(i)} \right) \|_{l_p(\hat{F}(r,s))}$$
$$\leq \| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} z_j(i)e^{(i)} \right) \|_{l_p(\hat{F}(r,s))} + 2\sum_{j=0}^{n} \epsilon_j.$$

Since $z \in l_p(\hat{F}(r,s))$ therefore there exists $C > 0$ such that $\|z\|_{l_p(\hat{F}(r,s))} \leq C$. Therefore we have that

$$\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} z_j(i)e^{(i)} \right) \|_{l_p(\hat{F}(r,s))}$$
$$\leq \sum_{j=0}^{n} \sum_{i=m_{j-1}+1}^{m_j} \left| r\frac{f_i}{f_{i+1}} z_j(i) + s\frac{f_{i+1}}{f_i} z_j(i-1) \right|^p$$
$$\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} \left| r\frac{f_i}{f_{i+1}} z_j(i) + s\frac{f_{i+1}}{f_i} z_j(i-1) \right|^p$$
$$\leq \sum_{j=0}^{n} \|z\|_{l_p(\hat{F}(r,s))}$$
$$\leq C^p(n+1).$$

Hence we obtain

$$\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} z_j(i)e^{(i)} \right) \|_{l_p(\hat{F}(r,s))} \leq C(n+1)^p.$$

By using the fact that $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}$ and $1 < p < \infty$, we have

$$\| \sum_{j=0}^{n} z_j \|_{l_p(\hat{F}(r,s))} \leq C(n+1)^p + 1 \leq (C+1)(n+1)^p.$$

Hence $l_p(\hat{F}(r,s))$ has the Banach-Saks type p. □

Remark 5.3. *Note that* $R\left(l_p(\hat{F}(r,s))\right) = R\left(l_p\right) = 2^{1/p}$ *since* $l_p(\hat{F}(r,s))$ *is linearly isomorphic to* l_p.

By Remarks 5.1 and 5.3, we have the following theorem.

Theorem 5.4. *The space* $l_p(\hat{F}(r,s))$ *has the weak fixed point property, where* $1 < p < \infty$.

References

1. A. Alotaibi, M. Mursaleen, B. A. Alamri and S. A. Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, J. Inequa. Appl. (2015) 2015:203, 7 pages.

2. B. Altay, F. Başar, Certain topological properties and duals of the domain of a triangle matrix in a sequence space, J. Math. Anal. Appl., 336(2007), 632-645

3. B. Altay, F. Başar, M. Mursaleen, Some generalizations of the space bv_p of p-bounded variation sequences. Nonlinear Anal. TMA, 68(2008), 273-287.

4. C. Aydin, F. Başar, Some new sequence spaces which include the spaces l_p and l_∞, Demonstr. Math., 38(3)(2005), 641-656.

5. M. Candan, A New Approach on the Spaces of Generalized Fibonacci Difference Null and Convergent Sequences, Mathematica Aeterna, 5(1)(2015) 191-210.

6. J. García-Falset, The fixed point property in Banach spaces with the NUS-property. J. Math. Anal. Appl., 215(2)(1997), 532-542.

7. P.K. Kamthan, M. Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York and Basel, 1981.

8. E.V. Kara, Some topological and geometrical properties of new Banach sequence spaces, J. Inequa. Appl., 2013, 2013:38

9. M. Kirişçi, F. Başar , Some new sequence spaces derived by the domain of generalized difference matrix. Comput. Math. Appl., 60(2010), 1229-1309.

10. H. Knaust, Orlicz sequence spaces of Banach-Saks type. Arch. Math., 59(6)(1992), 562-565.

11. T. Koshy, Fibonacci and Lucas Numbers with applications, Wiley, 2001.

12. Michael Stieglitz, Hubert Tietz, Matrixtransformationen von Folgenräumen Eine Ergebnisübersicht, Math. Z., 154(1977), 1-16.

13. S. K. Mishra, Matrix maps involving certain sequence spaces. Indian J. Pure Appl. Math., 24(2)(1993), 125-132.

14. M. Mursaleen, Generalized spaces of difference sequences. J. Math. Anal. Appl., 203(3)(1996), 738-745.

15. M. Mursaleen, A.K. Gaur, A.H. Saifi, Some new sequence spaces and their duals and matrix transformations, Bull. Calcutta Math. Soc., 88(3)(1996), 207-212.

16. M. Mursaleen, A.K. Noman, On some new sequence spaces of non-absolute type related to the spaces l_p and l_∞ I. Filomat 25(2)(2011), 33-51.

17. M. Mursaleen, F. Başar, B. Altay, On the Euler sequence spaces which include the spaces l_p and l_∞II. Nonlinear Anal. TMA, 65(3)(2006), 707-717.

18. A. Wilansky, Sumability Through Functional Analysis. North-Holland Mathematics Studies, vol. 85. Elsevier Amsterdam.(1984)

Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791 112, Arunachal Pradesh, India.
E-mail address: anupam.das@rgu.ac.in

and

Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791 112, Arunachal Pradesh, India.
E-mail address: bh_rgu@yahoo.co.in