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Existence of Positive Periodic Solutions for Nonlinear Neutral Dynamic Equations with Variable Coefficients on a Time Scale

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ABSTRACT: Let $\mathbb{T}$ be a periodic time scale. The purpose of this paper is to use Krasnosel'skiĭ's fixed point theorem to prove the existence of positive periodic solutions for nonlinear neutral dynamic equations with variable coefficients on a time scale. We invert these equations to construct a sum of a contraction and a compact map which is suitable for applying the Krasnosel'skiĭ's theorem. The results obtained here extend the work of Candan [11].

Key Words: Positive periodic solutions, nonlinear neutral dynamic equations, fixed point theorem, time scales

Contents

1. Introduction

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of dynamic equations. Motivated by the papers [1]–[7], [10]–[20] and the references therein, we consider the following two kinds of nonlinear neutral dynamic equations with variable coefficients

$$(x(t) - c(t)\,x(t-\tau))^{\triangle} = -a(t)\,x^{\sigma}(t) + f(t, x(t-\tau)), \tag{1.1}$$

where $x^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$ (see [8]). Throughout this paper we assume that $\tau = m\omega$ if $\mathbb{T}$ has period ω and τ is fixed if $\mathbb{T} = \mathbb{R}$. Our purpose here is to use the Krasnosel'skiĭ's fixed point theorem to show the existence of positive periodic solutions on time scales for equation (1.1). To reach our desired end we have to transform (1.1) into integral equation written as a sum of two mapping; one is a contraction and the other is compact. After that, we use Krasnosel'skiĭ's fixed point theorem, to show the existence of a positive periodic solution for equation

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(1.1). In the special case $\mathbb{T} = \mathbb{R}$, in [11] we show that (1.1) have a positive periodic solution by using Krasnosel'skiĭ's fixed point theorem.

The organization of this paper is as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the Krasnosel'skiĭ's fixed point theorem. For details on Krasnosel'skiĭ's theorem we refer the reader to [21]. Also, we present the inversion of (1.1), and we give the Green's functions of (1.1), which play an important role in this paper. In Section 3 and Section 4, we present our main results on existence of positive periodic solutions of (1.1). The results presented in this paper extend the main results in [11].

2. Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1], [3], [5]–[9], [16], [17] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [8] and [9] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et al. [6] and Kaufmann and Raffoul [16]. The following two definitions are borrowed from [6] and [16].

Definition 2.1. *We say that a time scale $\mathbb{T}$ is periodic if there exists a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.*

Below are examples of periodic time scales taken from [16].

Example 2.2. *The following time scales are periodic.*

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ *has period* $\omega = 2h$.
(2) $\mathbb{T} = h\mathbb{Z}$ *has period* $\omega = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ *where,* $0 < q < 1$ *has period* $\omega = 1$.

Remark 2.3 ([16])**.** *All periodic time scales are unbounded above and below.*

Definition 2.4. *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.*

Remark 2.5 ([16])**.** *If $\mathbb{T}$ is a periodic time scale with period ω, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.*

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales ([8], Theorem 1.93).

Theorem 2.6 (Chain Rule)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule ([8], Theorem 1.98).

Theorem 2.7 (Substitution)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\, \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\, \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\, f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log\left(1 + \mu(z)\, p(z)\right) \triangle z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t)\, y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.8 ([8])**.** *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\, p(t))\, e_p(t, s)$;

(iii) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t)\, p(t)}$;

(iv) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;

(v) $e_p(t, s)\, e_p(s, r) = e_p(t, r)$;

(vi) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\dfrac{1}{e_p(., s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(., s)}$.

Theorem 2.9 ([7], Theorem 2.1)**.** *Let $\mathbb{T}$ be a periodic time scale with period $\omega > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = n\omega$, then*

$$\int_{a+T}^{b+T} p(u) \triangle u = \int_a^b p(u) \triangle u, \;\; e_p(b + T, a + T) = e_p(b, a) \;\; \text{if } p \in \mathcal{R},$$

and $e_p(t+T,t)$ *is independent of* $t \in \mathbb{T}$ *whenever* $p \in \mathcal{R}$.

Lemma 2.10 ([1])**.** *If* $p \in \mathcal{R}^+$, *then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \ \forall t \in \mathbb{T}.$$

Corollary 2.11 ([1])**.** *If* $p \in \mathcal{R}^+$ *and* $p(t) < 0$ *for all* $t \in \mathbb{T}$, *then for all* $s \in \mathbb{T}$ *with* $s \leq t$ *we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right) < 1.$$

We state Krasnosel'skiĭ's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [21].

Theorem 2.12 (Krasnosel'skiĭ)**.** *Let* $\mathbb{D}$ *be a closed convex nonempty subset of a Banach space* $(\mathbb{B}, \|.\|)$. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *map* $\mathbb{D}$ *into* $\mathbb{B}$ *such that*
(*i*) $x, y \in \mathbb{D}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(*ii*) $\mathcal{A}$ *is compact and continuous,*
(*iii*) $\mathcal{B}$ *is a contraction mapping.*
Then there exists $z \in \mathbb{D}$ *with* $z = \mathcal{A}z + \mathcal{B}z$.

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

Define $P_T = \{\varphi : \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t \in \mathbb{T}} |\varphi(t)| = \sup_{t \in [0,T]} |\varphi(t)|.$$

We will need the following lemma whose proof can be found in [16].

Lemma 2.13. *Let* $x \in P_T$. *Then* $\|x^\sigma\| = \|x \circ \sigma\|$ *exists and* $\|x^\sigma\| = \|x\|$.

In this paper we assume that $a \in \mathcal{R}^+$, c are continuous and for all $t \in \mathbb{T}$,

$$a(t+T) = a(t), \ c(t+T) = c(t), \tag{2.2}$$

where $c^\triangle$ is continuous. Also, we assume

$$\int_0^T a(s) \triangle s > 0. \tag{2.3}$$

We also assume that the function $f(t,x)$ is continuous in their respective arguments and periodic in t with period T, that is,

$$f(t+T,x)=f(t,x). \tag{2.4}$$

The following lemma is essential for our results on existence of positive periodic solutions of (1.1).

Lemma 2.14. *Suppose (2.2)–(2.4) hold. If $x\in P_T$, then x is a solution of equation (1.1) if and only if*

$$x(t)=c(t)\,x(t-\tau)+\int_t^{t+T}G(t,s)\left[f(s,x(s-\tau))-a(s)\,c^{\sigma}(s)\,x^{\sigma}(s-\tau)\right]\triangle s, \tag{2.5}$$

where

$$G(t,s)=\frac{e_a(s,t)}{e_a(T,0)-1}. \tag{2.6}$$

Proof. Let $x\in P_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned}&(x(t)-c(t)\,x(t-\tau))^{\triangle}+a(t)\,(x^{\sigma}(t)-c^{\sigma}(t)\,x^{\sigma}(t-\tau))\\&=f(t,x(t-\tau))-a(t)\,c^{\sigma}(t)\,x^{\sigma}(t-\tau).\end{aligned}$$

Multiply both sides of the above equation by $e_a(t,0)$ we get

$$\begin{aligned}&\left\{(x(t)-c(t)\,x(t-\tau))^{\triangle}+a(t)\,(x^{\sigma}(t)-c^{\sigma}(t)\,x^{\sigma}(t-\tau))\right\}e_a(t,0)\\&=\{f(t,x(t-\tau))-a(t)\,c^{\sigma}(t)\,x^{\sigma}(t-\tau)\}\,e_a(t,0).\end{aligned}$$

Since $e_a(t,0)^{\triangle}=a(t)\,e_a(t,0)$ we find

$$\begin{aligned}&[(x(t)-c(t)\,x(t-\tau))\,e_a(t,0)]^{\triangle}\\&=\{f(t,x(t-\tau))-a(t)\,c^{\sigma}(t)\,x^{\sigma}(t-\tau)\}\,e_a(t,0).\end{aligned}$$

Taking the integral from t to $t+T$, we obtain

$$\begin{aligned}&\int_t^{t+T}[(x(s)-c(s)\,x(s-\tau))\,e_a(s,0)]^{\triangle}\triangle s\\&=\int_t^{t+T}\{f(s,x(s-\tau))-a(s)\,c^{\sigma}(s)\,x^{\sigma}(s-\tau)\}\,e_a(s,0)\triangle s.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&(x(t+T)-c(t+T)\,x(t+T-\tau))\,e_a(t+T,0)-(x(t)-c(t)\,x(t-\tau))\,e_a(t,0)\\&=\int_t^{t+T}\{f(s,x(s-\tau))-a(s)\,c^{\sigma}(s)\,x^{\sigma}(s-\tau)\}\,e_a(s,0)\triangle s.\end{aligned}$$

Dividing both sides of the above equation by $e_a(t,0)$ and using the fact that $x(t+T) = x(t)$, (2.2) and (2.4), we obtain

$$\begin{aligned} &x(t) - c(t)x(t-\tau) \\ &= \int_t^{t+T} \frac{e_a(s,t)}{e_a(T,0)-1} \{f(s,x(s-\tau)) - a(s)c^\sigma(s)x^\sigma(s-\tau)\} \triangle s. \end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

Corollary 2.15. *Suppose $c(t) \neq 0$ for all $t \in \mathbb{T}$ and (2.2)–(2.4) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) =& \frac{x(t+\tau)}{c(t+\tau)} \\ &+ \frac{1}{c(t+\tau)} \int_{t+\tau}^{t+\tau+T} G(t+\tau,s)\left[a(s)c^\sigma(s)x^\sigma(s-\tau) - f(s,x(s-\tau))\right] \triangle s, \end{aligned} \tag{2.7}$$

where G is given by (2.6).

From Lemma 2.8 and Theorem 2.9, we have for all $t, s \in \mathbb{R}$,

$$G(t+T,s+T) = G(t,s), \ G(t+\tau+T,s+T) = G(t+\tau,s), \tag{2.8}$$

and

$$\int_t^{t+T} G(t,s)a(s)\triangle s = 1, \int_{t+\tau}^{t+\tau+T} G(t+\tau,s)a(s)\triangle s = 1. \tag{2.9}$$

3. Existence of positive periodic solutions in the case $|c(t)| > 1$

To apply Theorem 2.12, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B}, \|\cdot\|) = (P_T, \|\cdot\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \leq \varphi \leq K\}$, where L is a non-negative constant and K is a positive constant. We express equation (2.7) as

$$\varphi(t) = (\mathcal{B}_1\varphi)(t) + (\mathcal{A}_1\varphi)(t) := (H_1\varphi)(t),$$

where $\mathcal{A}_1, \mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_1\varphi)(t) = \frac{1}{c(t+\tau)} \int_{t+\tau}^{t+\tau+T} G(t+\tau,s)\left[a(s)c^\sigma(s)\varphi^\sigma(s-\tau) - f(s,\varphi(s-\tau))\right] \triangle s, \tag{3.1}$$

and

$$(\mathcal{B}_1\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}. \tag{3.2}$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $1 < c(t) < \infty$ and $-\infty < c(t) < -1$ for all $t \in \mathbb{T}$.

Denote

$$F(t, x, y) = c^{\sigma}(t)\, y - \frac{f(t, x)}{a(t)}.$$

In the case $1 < c(t) < \infty$, we assume that there exist positive constants c_1 and c_2 such that

$$c_1 \le c(t) \le c_2, \text{ for all } t \in [0, T], \tag{3.3}$$

$$c_1 > 1, \tag{3.4}$$

and for all $t \in [0, T]$, $x, y \in \mathbb{D}$

$$(c_2 - 1) L \le F(t, x, y) \le (c_1 - 1) K. \tag{3.5}$$

Lemma 3.1. *For $\mathcal{A}_1$ defined in (3.1), Suppose that the conditions (2.2)–(2.4) and (3.3)–(3.5) hold. Then $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is compact.*

Proof. We first show that $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$. Clearly, if φ is continuous, then $\mathcal{A}_1\varphi$ is. Evaluating (3.1) at $t + T$ gives

$$\begin{aligned}&(\mathcal{A}_1\varphi)(t+T)\\ &= \frac{1}{c(t+\tau+T)} \int_{t+\tau+T}^{t+\tau+2T} G(t+\tau+T, s)\left[a(s)\, c^{\sigma}(s)\, \varphi^{\sigma}(s-\tau) - f(s, \varphi(s-\tau))\right] \triangle s.\end{aligned}$$

Use Theorem 2.7 with $u = s - T$ to get

$$\begin{aligned}(\mathcal{A}_1\varphi)(t+T) &= \frac{1}{c(t+\tau+T)} \int_{t+\tau}^{t+\tau+T} G(t+\tau+T, u+T)\\ &\quad \times \left[a(u+T)\, c^{\sigma}(u+T)\, \varphi^{\sigma}(u+T-\tau) - f(u+T, \varphi(u+T-\tau))\right] \triangle u.\end{aligned}$$

From (2.2), (2.3) and (2.8), we obtain

$$\begin{aligned}(\mathcal{A}_1\varphi)(t+T) &= \frac{1}{c(t+\tau)} \int_{t+\tau}^{t+\tau+T} G(t+\tau, u)\left\{a(u)\, c^{\sigma}(u)\, \varphi^{\sigma}(u-\tau) - f(u, \varphi(u-\tau))\right\} \triangle u\\ &= (\mathcal{A}_1\varphi)(t).\end{aligned}$$

That is, $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$.

We show that $\mathcal{A}_1(\mathbb{D})$ is uniformly bounded. For $t \in [0, T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{A}_1\varphi)(t)| &= \left| \frac{1}{c(t+\tau)} \int_{t+\tau}^{t+\tau+T} G(t+\tau, s)\left[a(s)\, c^{\sigma}(s)\, \varphi^{\sigma}(s-\tau) - f(s, \varphi(s-\tau))\right] \triangle s \right|\\ &\le \frac{(c_1 - 1) K}{c_1}.\end{aligned}$$

by (2.9) and (3.5). Thus from the estimation of $|(\mathcal{A}_1\varphi)(t)|$ we arrive

$$\|\mathcal{A}_1\varphi\| \leq \frac{(c_1-1)K}{c_1}.$$

This shows that $\mathcal{A}_1(\mathbb{D})$ is uniformly bounded.

It remains to show that $\mathcal{A}_1(\mathbb{D})$ is equicontinuous. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $(\mathcal{A}_1\varphi_n)^{\triangle}(t)$ and show that it is uniformly bounded. By making use of (2.2) and (2.4) we obtain by taking the derivative in (3.1) that

$$\begin{aligned}(\mathcal{A}_1\varphi_n)^{\triangle}(t) &= -\frac{c^{\triangle}(t+\tau)}{c^{\sigma}(t+\tau)}(\mathcal{A}_1\varphi_n)(t) - a(t+\tau)(\mathcal{A}_1\varphi_n)^{\sigma}(t) \\ &\quad + \frac{a(t+\tau)}{c^{\sigma}(t+\tau)}\left\{c^{\sigma}(t+\tau)\varphi_n^{\sigma}(t) - \frac{f(t+\tau,\varphi_n(t))}{a(t+\tau)}\right\}.\end{aligned}$$

Consequently, by invoking (2.9), (3.5) and Lemma 2.13, we obtain

$$\left|(\mathcal{A}_1\varphi_n)^{\triangle}(t)\right| \leq \frac{(c_1-1)K}{c_1}\left(\frac{1}{c_1}\left\|c^{\triangle}\right\| + 2\|a\|\right) \leq D,$$

for some positive constant D. Hence the sequence $(\mathcal{A}_1\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}_1\varphi_{n_k})$ of $(\mathcal{A}_1\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}_1$ is continuous and $\mathcal{A}_1(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. □

Lemma 3.2. *Suppose that (2.2)–(2.4), (3.3) and (3.4) hold. Then $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathcal{B}_1$ be defined by (3.2). Obviously, $\mathcal{B}_1\varphi$ is continuous and it is easy to show that $(\mathcal{B}_1\varphi)(t+T) = (\mathcal{B}_1\varphi)(t)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{B}_1\varphi)(t) - (\mathcal{B}_1\psi)(t)| &\leq \left|\frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)}\right| \\ &\leq \frac{1}{c_1}|\varphi(t+\tau) - \psi(t+\tau)| \\ &\leq \frac{1}{c_1}\|\varphi - \psi\|.\end{aligned}$$

Then $\|\mathcal{B}_1\varphi - \mathcal{B}_1\psi\| \leq \frac{1}{c_1}\|\varphi - \psi\|$. Thus $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction by (3.4). □

Theorem 3.3. *Suppose (2.2)–(2.4), (3.3)–(3.5) hold and there exists a $t_0 \in [0,T]$ such that $F(t_0,x,y) > (c_2-1)L$ for any $x, y \in \mathbb{D}$. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}_1 = \{\varphi \in \mathbb{B} : L < \varphi \leq K\}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= \frac{\psi(t+\tau)}{c(t+\tau)} + \frac{1}{c(t+\tau)} \int_{t+\tau}^{t+\tau+T} G(t+\tau, s)\left[a(s)c^\sigma(s)\varphi^\sigma(s-\tau) - f(s,\varphi(s-\tau))\right] \triangle s \\
&\le \frac{K}{c_1} + \frac{(c_1-1)K}{c_1} \int_{t+\tau}^{t+\tau+T} G(t+\tau, s)\, a(s) \triangle s \\
&= \frac{K}{c_1} + \frac{(c_1-1)K}{c_1} = K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= \frac{\psi(t+\tau)}{c(t+\tau)} + \frac{1}{c(t+\tau)} \int_{t+\tau}^{t+\tau+T} G(t+\tau, s)\left[a(s)c^\sigma(s)\varphi^\sigma(s-\tau) - f(s,\varphi(s-\tau))\right] \triangle s \\
&\ge \frac{L}{c_2} + \frac{(c_2-1)L}{c_2} \int_{t+\tau}^{t+\tau+T} G(t+\tau, s)\, a(s) \triangle s \\
&= \frac{L}{c_2} + \frac{(c_2-1)L}{c_2} = L.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnosel'skiĭ's theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}_1 x + \mathcal{B}_1 x$. By Lemma 2.14 this fixed point is a solution of (1.1).

Next, we prove that $x \in \mathbb{D}_1$. We just need to prove that for all $t \in [0, T]$, $x(t) > L$. Otherwise, there exists $t^* \in [0, T]$ satisfying $x(t^*) = L$. From (2.7), we have

$$\begin{aligned}
L &= \frac{x(t^*+\tau)}{c(t^*+\tau)} \\
&\quad + \frac{1}{c(t^*+\tau)} \int_{t^*+\tau}^{t^*+\tau+T} G(t^*+\tau, s)\left\{a(s)c^\sigma(s)x^\sigma(s-\tau) - f(s,x(s-\tau))\right\} \triangle s \\
&\ge \frac{L}{c_2} + \int_{t^*+\tau}^{t^*+\tau+T} G(t^*+\tau, s)\, a(s) \left\{c^\sigma(s)x^\sigma(s-\tau) - \frac{f(s,x(s-\tau))}{a(s)}\right\} \triangle s.
\end{aligned}$$

From $\int_{t^*+\tau}^{t^*+\tau+T} G(t^*+\tau, s)\, a(s) \triangle s = 1$, it follows that

$$\int_{t^*+\tau}^{t^*+\tau+T} G(t^*+\tau, s)\, a(s)\left[F(s,x,y) - (c_2-1)L\right] \triangle s \le 0.$$

Noting that $F(s,x,y) \ge (c_2-1)L$ and $F(t_0,x) > (c_2-1)L$, $t_0 \in [0,T]$, we obtain

$$\int_{t^*+\tau}^{t^*+\tau+T} G(t^*+\tau, s)\, \gamma(s)\left[F(s,x,y) - (c_2-1)L\right] \triangle s > 0.$$

This is a contradiction. So, $x \in \mathbb{D}_1$. The proof is complete. □

Remark 3.4. *When $\mathbb{T} = \mathbb{R}$, Theorem 3.3 reduces to Theorem 1 of [11].*

In the case $-\infty < c(t) < -1$, we substitute conditions (3.3)–(3.5) with the following conditions respectively. We assume that there exist a negative constants c_3 and c_4 such that

$$c_3 \leq c(t) \leq c_4, \text{ for all } t \in [0, T], \tag{3.6}$$

$$c_4 < -1, \tag{3.7}$$

and for all $t \in [0, T]$, $x, y \in \mathbb{D}$

$$K - c_3 L \leq -F(t, x, y) \leq L - c_4 K. \tag{3.8}$$

Theorem 3.5. *Suppose (2.2)–(2.4) and (3.6)–(3.8) hold and there exists a $t_0 \in [0, T]$ such that $-F(t_0, x, y) > K - c_3 L$ for any $x, y \in \mathbb{D}$. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}_1$.*

The proof follows along the lines of Theorem 3.3, and hence we omit it.

Remark 3.6. *When $\mathbb{T} = \mathbb{R}$, Theorem 3.5 reduces to Theorem 2 of [11].*

4. Existence of positive periodic solutions in the case $|c(n)| < 1$

We express equation (2.5) as

$$\varphi(t) = (\mathcal{B}_2\varphi)(t) + (\mathcal{A}_2\varphi)(t) := (H_2\varphi)(t),$$

where $\mathcal{A}_2, \mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_2\varphi)(t) = \int_t^{t+T} G(t, s) [f(s, \varphi(s - \tau)) - a(s) c^\sigma(s) \varphi^\sigma(s - \tau)] \triangle s, \tag{4.1}$$

and

$$(\mathcal{B}_2\varphi)(t) = c(t) x(t - \tau). \tag{4.2}$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $0 \leq c(t) < 1$ and $-1 < c(t) \leq 0$ for all $t \in \mathbb{T}$.

Denote

$$H(t, x, y) = \frac{f(t, x)}{a(t)} - c^\sigma(t) y.$$

In the case $0 \leq c(t) < 1$, we assume that there exists positive constant c_1 such that

$$0 \leq c(t) \leq c_1, \text{ for all } t \in [0, T], \tag{4.3}$$

$$c_1 < 1, \tag{4.4}$$

and for all $t \in [0, T]$, $x, y \in \mathbb{D}$

$$L \leq H(t, x, y) \leq (1 - c_1) K. \tag{4.5}$$

In the case $-1 < c(t) \leq 0$, we assume that there exists negative constant c_2 such that

$$c_2 \leq c(t) \leq 0, \text{ for all } t \in [0, T], \tag{4.6}$$

$$c_2 > -1, \tag{4.7}$$

and for all $t \in [0, T]$, $x, y \in \mathbb{D}$

$$L - c_2 K \leq H(t, x, y) \leq K. \tag{4.8}$$

Lemma 4.1. *Suppose that (2.2)–(2.4), (4.3) and (4.4) hold. If $\mathcal{B}_2$ is given by (4.2), then $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathcal{B}_2$ be defined by (4.2). Obviously, $\mathcal{B}_2\varphi$ is continuous and it is easy to show that $(\mathcal{B}_2\varphi)(t+T) = (\mathcal{B}_2\varphi)(t)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}
&|(\mathcal{B}_2\varphi)(t) - (\mathcal{B}_2\psi)(t)| \\
&\leq |c(t)\varphi(t-\tau) - c(t)\psi(t-\tau)| \\
&\leq |c(t)| |\varphi(t-\tau) - \psi(t-\tau)| \\
&\leq c_1 \|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathcal{B}_2\varphi - \mathcal{B}_2\psi\| \leq c_1 \|\varphi - \psi\|$. Thus $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction by (4.4). $\square$

Lemma 4.2. *For $\mathcal{A}_2$ defined in (4.1), suppose that the conditions (2.2)–(2.4) and (4.3)–(4.5) hold. Then $\mathcal{A}_2 : \mathbb{D} \to \mathbb{B}$ is compact.*

Proof. Let $\mathcal{A}_2 : \mathbb{D} \to \mathbb{B}$ be defined by (4.1). Obviously, $\mathcal{A}_2\varphi$ is continuous and it is easy to show that $(\mathcal{A}_2\varphi)(t+T) = (\mathcal{A}_2\varphi)(t)$.

We show that $\mathcal{A}_2(\mathbb{D})$ is uniformly bounded. For $t \in [0, T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned}
|(\mathcal{A}_2\varphi)(t)| &= \left| \int_t^{t+T} G(t,s) \left[f(s, \varphi(s-\tau)) - a(s) c^\sigma(s) \varphi^\sigma(s-\tau) \right] \triangle s \right| \\
&\leq \int_t^{t+T} G(t,s) a(s) \left[\frac{f(s, \varphi(s-\tau))}{a(s)} - c^\sigma(s) \varphi^\sigma(s-\tau) \right] \triangle s \\
&\leq (1 - c_1) K \int_t^{t+T} G(t,s) a(s) \triangle s \\
&\leq (1 - c_1) K.
\end{aligned}$$

by (2.9) and (4.5). Thus from the estimation of $|(\mathcal{A}_2\varphi)(t)|$ we arrive

$$\|\mathcal{A}_2\varphi\| \leq (1 - c_1) K.$$

This shows that $\mathcal{A}_2(\mathbb{D})$ is uniformly bounded.

It remains to show that $\mathcal{A}_2(\mathbb{D})$ is equicontinuous. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $(\mathcal{A}_2\varphi_n)^\triangle(t)$ and show that it is uniformly bounded. By making use of (2.2) and (2.4) we obtain by taking the derivative in (4.1) that

$$(\mathcal{A}_2\varphi_n)^{\triangle}(t) = f(t,\varphi(t-\tau)) - a(t)\,c^{\sigma}(t)\,\varphi^{\sigma}(t-\tau) - a(t)\,(\mathcal{A}_2\varphi_n)^{\sigma}(t).$$

Consequently, by invoking (2.9), (4.5) and Lemma 2.13, we obtain

$$\begin{aligned}\left|(\mathcal{A}_2\varphi_n)^{\triangle}(t)\right| &\leq (1-c_1)\,K\,\|a\| + (1-c_1)\,K\,\|a\| \\ &= 2\,(1-c_1)\,K\,\|a\| \leq D,\end{aligned}$$

for some positive constant D. Hence the sequence $(\mathcal{A}_2\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}_2\varphi_{n_k})$ of $(\mathcal{A}_2\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}_2$ is continuous and $\mathcal{A}_2(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. □

Similar to the results in Section 3, we have

Theorem 4.3. *Suppose (2.2)–(2.4) and (4.3)–(4.5) hold and there exists a $t_0 \in [0,T]$ such that $H(t_0,x,y) > L$ for any $x,y \in \mathbb{D}$. Then equation (1.1) has a positive T-periodic solution in the subset $\mathbb{D}_1$.*

Theorem 4.4. *Suppose (2.2)–(2.4) and (4.6)–(4.8) hold and there exists a $t_0 \in [0,T]$ such that $H(t_0,x,y) > L - c_2K$ for any $x,y \in \mathbb{D}$. Then equation (1.1) has a positive T-periodic solution in the subset $\mathbb{D}_1$.*

Remark 4.5. *When $\mathbb{T} = \mathbb{R}$, Theorem 4.3 and Theorem 4.4 reduce to Theorem* 3 *and Theorem* 4 *of [11], respectively.*

Example 4.6. *Let $\mathbb{T} = \mathbb{Z}$. Consider the neutral difference equation*

$$\triangle(x(t) - 0.1x(t-5)) = -0.2x(t+1) + \frac{0.1}{x^2(t-5)+4} + 0.25. \tag{4.9}$$

Note that (4.9) of the form (1.1) with $T = 4$, $c(t) = 0.1$, $a(t) = 0.2$, $\tau = 5$ and $f(t,x) = \frac{0.1}{x^2+4} + 0.25$. It is easy to verify that the conditions of Theorem 4.3 are satisfied with $L = 1$, $K = 1.5$ and $t_0 = 0$. Thus (4.9) has at least one positive 4-periodic solution.

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References

1. M. Adıvar and Y. N. Raffoul, Existence of periodic solutions in totally nonlinear delay dynamic equations, Electron. J. Qual. Theory Differ. Equ. 2009, No. 1, 1–20.

2. A. Ardjouni and A. Djoudi, Existence of positive periodic solutions for two kinds of nonlinear neutral differential equations with variable delay, Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis 20 (2013) 357–366.

3. A. Ardjouni and A. Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale, Commun Nonlinear Sci Numer Simulat 17 (2012) 3061–3069.

4. A. Ardjouni and A. Djoudi, Existence of positive periodic solutions for a nonlinear neutral differential equation with variable delay, Appl. Math. E-Notes, 12 (2012), 94–101.

5. A. Ardjouni and A. Djoudi, Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale, Rend. Sem. Mat. Univ. Politec. Torino Vol. 68, 4 (2010), 349–359.

6. F. M. Atici, G. Sh. Guseinov and B. Kaymakcalan, Stability criteria for dynamic equations on time scales with periodic coefficients, Proceedings of the International Confernce on Dynamic Systems and Applications, III, 3 (1999), 43–48.

7. L. Bi, M. Bohner and M. Fan, Periodic solutions of functional dynamic equations with infinite delay, Nonlinear Anal. 68 (2008), 1226–1245.

8. M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser, Boston, 2001.

9. M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

10. T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover Publications, New York, 2006.

11. T. Candan, Existence of positive periodic solutions of first order neutral differential equations with variable coefficients, Appl. Math. Lett. 52 (2016) 142-148.

12. F. D. Chen, Positive periodic solutions of neutral Lotka-Volterra system with feedback control, Appl. Math. Comput. 162 (2005), No. 3, 1279–1302.

13. F. D. Chen and J. L. Shi, Periodicity in a nonlinear predator-prey system with state dependent delays, Acta Math. Appl. Sin. Engl. Ser. 21 (2005), No. 1, 49–60.

14. Y. M. Dib, M.R. Maroun and Y.N. Raffoul, Periodicity and stability in neutral nonlinear differential equations with functional delay, Electron. J. Differ. Equ., Vol. 2005 (2005), No. 142, pp. 1–11.

15. M. Fan and K. Wang, P. J. Y. Wong and R. P. Agarwal, Periodicity and stability in periodic n-species Lotka-Volterra competition system with feedback controls and deviating arguments, Acta Math. Sin. Engl. Ser. 19 (2003), No. 4, 801–822.

16. E. R. Kaufmann and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl. 319 (2006), No. 1, 315–325.

17. E. R. Kaufmann and Y. N. Raffoul, Periodicity and stability in neutral nonlinear dynamic equations with functional delay on a time scale, Electron. J. Differ. Equ., Vol. 2007 (2007), No. 27, 1–12.

18. E. R. Kaufmann, A nonlinear neutral periodic differential equation, Electron. J. Differ. Equ., Vol. 2010 (2010), No. 88, 1–8.

19. Y. Luo, W. Wang and J. Shen, Existence of positive periodic solutions for two kinds of neutral functional differential equations, Appl. Math. Lett. 21 (2008) 581–587.

20. Y. N. Raffoul, Positive periodic solutions in neutral nonlinear differential equations, Electron. J. Qual. Theory Differ. Equ. 2007, No. 16, 1–10.

21. D. S. Smart, Fixed point theorems; Cambridge Tracts in Mathematics, No. 66, Cambridge University Press, London-New York, 1974.

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