On Upper and Lower faintly ℐ-continuous Multifunctions

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ABSTRACT: The aim of this paper is to introduce and study upper and lower faintly ℐ-continuous multifunctions as a generalization of upper and lower ℐ-continuous multifunctions, respectively.

Key Words: Ideal topological spaces, ℐ-open sets, ℐ-closed sets, faintly ℐ-continuous multifunctions.

Contents

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [2,10,12,14,15]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [7] and Vaidyanathaswamy, [18]. An ideal ℐ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal ℐ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star \colon \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [18] of A with respect to τ and ℐ, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the ∗-topology, finer than τ is defined by $^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If ℐ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [5] introduced the notion of ℐ-open sets in topological spaces. In 1992, Abd El-Monsef et al. [1] further investigated ℐ-open sets and ℐ-continuous functions. Several characterizations and properties of ℐ-open sets were provided in [1,8]. Recently, Akdag [2] introduced and studied the concept of ℐ-continuous multifunctions in

topological spaces. In this paper, we introduce and study upper and lower faintly $\mathcal{I}$-continuous multifunctions on ideal topological space.

2. Preliminaries

For a subset A of a topological space (X, τ), (A) and $\mathbb{Z}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-open [5] if $S \subset \mathbb{Z}(S^*)$. The complement of an $\mathcal{I}$-closed set is said to be an $\mathcal{I}$-open set. The $\mathcal{I}$-closure [8] and the $\mathcal{I}$-interior [6], that can be defined in the same way as (A) and $\mathbb{Z}(A)$, respectively, will be denoted by $\mathcal{I}(A)$ and $\mathcal{I}\mathbb{Z}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\mathcal{I}O(X, x)$ (resp. $\mathcal{I}C(X, x)$). A subset N of a topological space (X, τ) is said to be $\mathcal{I}$-neighborhood of a point $x \in X$, if there exists an $\mathcal{I}$-open set V such that $x \in V \subset N$. A point $x \in X$ is called a θ-cluster point of A [17] if $(V) \cap A \neq \varnothing$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $_\theta(A)$. If $A = {}_\theta(A)$, then A is said to be θ-closed [17]. The complement of θ-closed set is said to be θ-open [17]. The union of all θ-open sets contained in a subset A is called the θ-interior of A and is denoted by $\mathbb{Z}_\theta(A)$. It follows from [17] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ on X. A subset S of a topological space (X, τ) is said to be preopen [9] if $S \subset \mathbb{Z}((S))$. By a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be lower $\mathcal{I}$-continuous [2] (resp. upper $\mathcal{I}$-continuous) multifunction if $F^-(V) \in \mathcal{I}O(X, \tau)$ (resp. $F^+(V) \in \mathcal{I}O(X, \tau)$) for every $V \in \sigma$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be lower faintly precontinuous [11] (resp. upper faintly precontinuous) multifunction if $F^-(V)$ is preopen (resp. $F^+(V)$ is preopen) in X for every $V \in \sigma_\theta$.

3. Faintly $\mathcal{I}$-continuous multifunctions

Definition 3.1. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be :*

1. *upper faintly $\mathcal{I}$-continuous at $x \in X$ if for each θ-open set V of Y containing $F(x)$, there exists $U \in \mathcal{I}O(X)$ containing x such that $F(U) \subset V$;*

2. *lower faintly $\mathcal{I}$-continuous at $x \in X$ if for each θ-open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \mathcal{I}O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;*

3. *upper (lower) faintly $\mathcal{I}$-continuous if it has this property at each point of X.*

Remark 3.2. *It is clear that every upper Ɉ-continuous multifunction is upper faintly Ɉ-continuous. But the converse is not true in general, as the following example shows.*

Example 3.3. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$, $\sigma = \{\emptyset, \{a\}, X\}$ and $\Im = \{\emptyset, \{a\}\}$. The multifunction $F : (X, \tau, \Im) \to (X, \sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper faintly Ɉ-continuous but is not upper Ɉ-continuous.*

Proposition 3.4. *Every faintly Ɉ-continuous multifunction is upper faintly pre-continuous.*

Proof. The proof follows from the fact that every Ɉ-open set is preopen. □

The following example shows that the converse of Proposition 3.4 in not true in general.

Example 3.5. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\Im = \{\emptyset, \{a\}\}$. The multifunction $F : (X, \tau) \to (X, \sigma)$ defined by $F(a) = \{a\}$, $F(b) = \{b, c\}$ and $F(c) = \{a, c\}$ is upper faintly precontinuous but is not upper faintly Ɉ-continuous.*

Remark 3.6. *The converse of Proposition 3.4 holds if $\Im = \{\emptyset\}$.*

Definition 3.7. *A sequence (x_α) is said to Ɉ-converge to a point x if for every Ɉ-open set V containing x, there exists an index α_0 such that for $\alpha \geq \alpha_0$, $\alpha_n \in V$. This is denoted by $x_\alpha \xrightarrow{\Im} x$.*

Theorem 3.8. *For a multifunction $F : (X, \tau, \Im) \to (Y, \sigma)$, the following statements are equivalent :*

1. *F is upper faintly Ɉ-continuous;*

2. *For each $x \in X$ and for each θ-open set V such that $x \in F^+(V)$, there exists an Ɉ-open set U containing x such that $U \subset F^+(V)$;*

3. *For each $x \in X$ and for each θ-closed set V such that $x \in F^+(Y \backslash V)$, there exists an Ɉ-closed set H such that $x \in X \backslash H$ and $F^-(V) \subset H$;*

4. *$F^+(V)$ is an Ɉ-open set for any θ-open set V of Y;*

5. *$F^-(V)$ is an Ɉ-closed set for any θ-closed set V of Y;*

6. *$F^-(Y \backslash V)$ is an Ɉ-closed set for any θ-open set V of Y;*

7. *$F^+(Y \backslash V)$ is an Ɉ-open set for any θ-closed set V of Y;*

8. *For each $x \in X$ and for each net (x_α) which Ɉ-converges to $x \in X$ and for each θ-open set V of Y such that $x \in F^+(V)$, the net (x_α) is eventually in $F^+(V)$.*

Proof. (1)⇔(2): Clear.

(2)⇔(3): Let $x \in X$ and V be a θ-open set of Y such that $x \in F^+(Y \backslash V)$. By (2), there exists an $\mathfrak{I}$-open set U containing x such that $U \subset F^+(Y \backslash V)$. Then $F^-(V) \subset X \backslash U$. Take $H = X \backslash U$. We have $x \in X \backslash H$ and H is $\mathfrak{I}$-open. The converse is similar.

(1)⇔(4): Let $x \in F^+(V)$ and V be a θ-open set of Y. By (1), there exists an $\mathfrak{I}$-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup\limits_{x \in F^+(V)} U_x$. Since any union of $\mathfrak{I}$-open sets is $\mathfrak{I}$-open, $F^+(V)$ is $\mathfrak{I}$-open. The converse can be shown similarly.

(4)⇔(5)⇔(6)⇔(7)⇔(8) : Clear.

(1)⇒(8): Let (x_α) be a net which $\mathfrak{I}$-converges to x in X and let V be any θ-open set of Y such that $x \in F^+(V)$. Since F is an upper faintly $\mathfrak{I}$-continuous multifunction, it follows that there exists an $\mathfrak{I}$-open set U of X containing x such that $U \subset F^+(V)$. Since (x_α) $\mathfrak{I}$-converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subset F^+(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^+(V)$.

(8) ⇒ (1): Suppose that (1) is not true. There exists a point x and a θ-open set V with $x \in F^+(V)$ such that $U \nsubseteq F^+(V)$ for each $\mathfrak{I}$-open set U of X containing x. Let $x_U \in U$ and $x_U \notin F^+(V)$ for each $\mathfrak{I}$-open set U of X containing x. Then for each $\mathfrak{I}$-neighborhood net (x_U), $x_U \xrightarrow{\mathfrak{I}} x$, but (x_U) is not eventually in $F^+(V)$. This is a contradiction. Thus, F is an upper faintly $\mathfrak{I}$-continuous multifunction. □

Theorem 3.9. *For a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$, the following statements are equivalent :*

1. *F is lower faintly $\mathfrak{I}$-continuous;*

2. *For each $x \in X$ and for each θ-open set V such that $x \in F^-(V)$, there exists an $\mathfrak{I}$-open set U containing x such that $U \subset F^-(V)$;*

3. *For each $x \in X$ and for each θ-closed set V such that $x \in F^-(Y \backslash V)$, there exists an $\mathfrak{I}$-closed set H such that $x \in X \backslash H$ and $F^+(V) \subset H$;*

4. *$F^-(V)$ is an $\mathfrak{I}$-open set for any θ-open set V of Y;*

5. *$F^+(V)$ is an $\mathfrak{I}$-closed set for any θ-closed set V of Y;*

6. *$F^+(Y \backslash V)$ is an $\mathfrak{I}$-closed set for any θ-open set V of Y;*

7. *$F^-(Y \backslash V)$ is an $\mathfrak{I}$-open set for any θ-closed set V of Y;*

8. *For each $x \in X$ and for each net (x_α) which $\mathfrak{I}$-converges to $x \in X$ and for each θ-open set V of Y such that $x \in F^-(V)$ the net (x_α) is eventually in $F^-(V)$.*

Proof. The proof is similar to that of Theorem 3.8. □

Lemma 3.10. *[1] Let A and B be subsets of a topological space (X, τ). If $A \in \mathfrak{I}O(X)$ and $B \in \tau$, then $A \cap B \in \mathfrak{I}O(B)$.*

Theorem 3.11. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction and $U \in \tau$. If F is a lower (upper) faintly $\mathfrak{I}$-continuous multifunction, then multifunction $F_{|_U} : U \to Y$ is a lower (upper) faintly $\mathfrak{I}$-continuous multifunction.*

Proof. Let V be any θ-open set of Y, $x \in U$ and $x \in F_{|_U}^{-}(V)$. Since F is a lower faintly $\mathfrak{I}$-continuous multifunction, there exists an $\mathfrak{I}$-open set G containing x such that $G \subset F^{-}(V)$. From here by Lemma 3.10, we obtain that $x \in G \cap U \in \mathfrak{I}O(A)$ and $G \cap U \subset F_{|_U}^{-}(V)$. This shows that the restriction multifunction $F_{|_U}$ is a lower faintly $\mathfrak{I}$-continuous.
The proof of the upper faintly $\mathfrak{I}$-continuity of $F_{|_U}$ can be done by the similar manner. $\square$

Lemma 3.12. *[13] The following hold for a multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$:*

  (i) $G_F^{+}(A \times B) = A \cap F^{+}(B)$ and

  (i) $G_F^{-}(A \times B) = A \cap F^{-}(B)$

for each subsets $A \subset X$ and $B \subset Y$.

Theorem 3.13. *Let $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ be a multifunction. If the graph multifunction of F is an upper faintly $\mathfrak{I}$-continuous, then F is upper faintly $\mathfrak{I}$-continuous.*

Proof. Let $x \in X$ and V be any θ-open subset of Y such that $x \in F^{+}(V)$. We obtain that $x \in G_F^{+}(X \times V)$ and that $X \times V$ is a θ-open set. Since the graph multifunction G_F is upper faintly $\mathfrak{I}$-continuous, it follows that there exists an $\mathfrak{I}$-open set U of X containing x such that $U \subset G_F^{+}(X \times V)$. Since $U \subset G_F^{+}(X \times V)$ $= X \cap F^{+}(V) = F^{+}(V)$. We obtain that $U \subset F^{+}(V)$. Thus, F is upper faintly $\mathfrak{I}$-continuous. $\square$

Theorem 3.14. *A multifunction $F : (X, \tau, \mathfrak{I}) \to (Y, \sigma)$ is lower faintly $\mathfrak{I}$-continuous if $G_F : (X, \tau, \mathfrak{I}) \to (X \times Y, \tau \times \sigma)$ is lower faintly $\mathfrak{I}$-continuous.*

Proof. Suppose that G_F is lower faintly $\mathfrak{I}$-continuous. Let $x \in X$ and V be any θ-open set of Y such that $x \in F^{-}(V)$. Then $X \times V$ is θ-open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \varnothing$. Since G_F is lower faintly $\mathfrak{I}$-continuous, there exists an $\mathfrak{I}$-open U containing x such that $U \subset G_F^{-}(X \times V)$; hence $U \subset F^{-}(V)$. This shows that F is lower faintly $\mathfrak{I}$-continuous. $\square$

Theorem 3.15. *Suppose that $(X, \tau, \mathfrak{I})$ is an ideal topological space and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : (X, \tau, \mathfrak{I}) \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from $(X, \tau, \mathfrak{I})$ to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the*

projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction, then $P_\alpha \circ F$ is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction for each $\alpha \in J$.

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a θ-open set in $(X_{\alpha0}, \tau_{\alpha0})$. Then $(P_{\alpha0} \circ F)^+(V_{\alpha0}) = F^+(P_{\alpha0}^+(V_{\alpha0})) = F^+(V_{\alpha0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (resp. $(P_{\alpha0} \circ F)^-(V_{\alpha0}) = F^-(P_{\alpha0}^-(V_{\alpha0})) = F^-(V_{\alpha0} \times \prod_{\alpha \neq \alpha0} X_\alpha)$). Since F is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction and since $V_{\alpha0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a θ-open set, it follows that $F^+(V_{\alpha0} \times \prod_{\alpha \neq 0} X_\alpha)$ (resp. $F^-(V_{\alpha0} \times \prod_{\alpha \neq \alpha0} X_\alpha)$) is an $\mathfrak{I}$-open set in (X, τ). This shows that $P_\alpha \circ F$ is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction. Hence, we obtain that $P_{\alpha0} \circ F$ is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction for each $\alpha \in J$. □

Theorem 3.16. *Suppose that for each $\alpha \in J$, $(X, \tau, \mathfrak{I})$ is an ideal topological space and $(Y_\alpha, \sigma_\alpha)$ is a topological space. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction, then each F_α is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction for each $\alpha \in J$.*

Proof. Let V_α be a θ-open set of Y_α. Then $V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a θ-open set. Since F is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction, it follows that $F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^+(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$ (resp. $F^-(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^-(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$) is an $\mathfrak{I}$-open set. Consequently, we obtain that $F_\alpha^+(V_\alpha)$ (resp. $F_\alpha^-(V_\alpha)$) is an $\mathfrak{I}$-open set. Thus, we show that F_α is an upper (lower) faintly $\mathfrak{I}$-continuous multifunction. □

Theorem 3.17. *Let F and G be upper faintly $\mathfrak{I}$-continuous and punctually closed multifunctions from a topological space (X, τ) to a θ-normal space (Y, σ). Then the set $K = \{x : F(x) \cap G(x) \neq \varnothing\}$ is $\mathfrak{I}$-closed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \varnothing$. Since F and G are punctually closed multifunctions and Y is a θ-normal space, it follows that there exist disjoint θ-open sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper faintly $\mathfrak{I}$-continuous multifunctions, then the sets $F^+(U)$ and $G^+(V)$ are $\mathfrak{I}$-open sets containing x. Let $H = F^+(U) \cup G^+(V)$. Then H is an $\mathfrak{I}$-open set containing x and $H \cap K = \varnothing$; hence K is $\mathfrak{I}$-closed in X. □

Definition 3.18. *A topological space (X, τ) is said to be $\mathfrak{I}$-T_2 [8] (resp. θ-T_2 [16]) if for each pair of distinct points x and y in X, there exists disjoint $\mathfrak{I}$-open (resp. θ-open) sets U and V in X such that $x \in U$ and $y \in V$.*

Theorem 3.19. *Let $F : (X, \tau, \Im) \to (Y, \sigma)$ be an upper faintly ℑ-continuous multifunction and punctually closed from a topological space X to a θ-normal space Y and let $F(x) \cap F(y) = \varnothing$ for each pair of distinct points x and y of X. Then X is an $\Im$-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \varnothing$. Since Y is θ-normal, it follows that there exist disjoint θ-open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint ℑ-open sets containing x and y, respectively and hence (X, τ) is $\Im$-T_2.   □

Definition 3.20. *Let $F : (X, \tau, \Im) \to (Y, \sigma)$ be a multifunction. The multigraph $G(F)$ is said to be $\Im$-θ-closed if for each $(x, y) \notin G(F)$, there exist ℑ-open set U and θ-open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \varnothing$.*

Theorem 3.21. *If a multifunction $F : (X, \tau, \Im) \to (Y, \sigma)$ is an upper faintly ℑ-continuous such that $F(x)$ is θ-compact relative to Y for each $x \in X$ and Y is a θ-T_2 space, then the multigraph $G(F)$ of F is $\Im$-θ-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \backslash G(F)$. That is $y \notin F(x)$. Since Y is θ-T_2 for each $z \in F(x)$, there exist disjoint θ-open sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(y)$. Then $\{U(z) : z \in F(x)\}$ is a θ-open cover of $F(x)$ and since $F(x)$ is θ-compact subset relative to Y, there exists a finite number of points, say, z_1, z_2, ..., z_n in $F(x)$ such that $F(x) \subset \cup\{U(z_i) : i = 1, 2, ..., n\}$. Put $U = \cup\{U(z_i): i = 1, 2, ..., n\}$ and $V = \cap \{V(y_i): i = 1, 2, ..., n\}$. Then U and V are θ-open sets in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \varnothing$. Since F is upper faintly ℑ-continuous multifunction, there exists an ℑ-open set W of X containing x such that $F(W) \subset U$. We have $(x, y) \in W \times V \subset (X \times Y) \setminus G(F)$. We obtain that $(W \times V) \cap G(F) = \varnothing$ and hence $G(F)$ is $\Im$-θ-closed in $X \times Y$.   □

Theorem 3.22. *Let $F : (X, \tau, \Im) \to (Y, \sigma)$ be a multifunction having $\Im$-θ-closed multigraph $G(F)$. If B is a θ-compact subset relative to Y, then $F^-(B)$ is $\Im$-closed in X.*

Proof. Let $x \in X \backslash F^-(B)$. For each $y \in B$, $(x, y) \notin G(F)$ and there exist an ℑ-open set $U(y) \subset X$ and a θ-open set $V(y) \subset Y$, containing x and y, respectively, such that $F(U(y)) \cap V(y) = \varnothing$. That is, $U(y) \cap F^-(V(y)) = \varnothing$. Then $\{V(y): y \in B\}$ is a θ-open cover of B and since B is θ-compact relative to Y, there exists a finite subset B_0 of B such that $B \subset \cup\{V(y) : y \in B_0\}$. Put $U = \cap\{U(y) : y \in B_0\}$. Then U is ℑ-open in X, $x \in U$ and $U \cap F^-(B) = \varnothing$; that is, $x \in U \subset X \backslash F^-(B)$. This shows that $F^-(B)$ is ℑ-closed in X.   □

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