Bol. Soc. Paran. Mat. (3s.) **v. 36** 4 (2018): 121–130.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i4.31125

On Quasi Compact Spaces and Some Functions

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ABSTRACT: In this paper, we introduce and investigate a new class of sets called $\mathcal{A}$-open sets which are weaker than cozero sets. Moreover, we obtain characterizations and preserving theorems of quasi compact spaces.

Key Words: Cozero set, Zero set, Quazi compact space, Z-closed space.

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1. Introduction

One of the fundamental ideas in all of mathematics is the notion of continuity. So much so that there has been a movement in recent years to categorize mathematics into two main parts, namely discrete mathematics and continuous mathematics. In topology there have been many variants of continuity considered in the literature. Recently papers [1,2,3,4,8,10] have introduced some new classes of functions via cozero sets. In this paper, we introduce and investigate a new class of sets called $\mathcal{A}$-open sets which are weaker than cozero sets. Moreover, we obtain characterizations and preserving theorems of quasi compact spaces.

A subset B of a topological space (X,τ) is called a cozero set if there is a continuous real-valued function g on X such that $B = \{x \in X : g(x) \neq 0\}$ [6]. The complement of a cozero set is called a zero set. It is well known [6] that the countable union of cozero sets is a cozero set and the intersection of two cozero sets is a cozero set, so the collection of all cozero subsets of (X,τ) is a base for a topology τ_z on X, called the complete regularization of τ. It is clear that $\tau_z \subseteq \tau$ in general. Furthermore, the space (X,τ) is completely regular if and only if $\tau_z = \tau$. In general for any topological space τ, we note that (X,τ_z) is completely regular. Thus (X,τ_z) is regular, and hence it is semi-regular. Therefore $(\tau_z)_s = \tau_z$. Now the inclusion $\tau_z \subseteq \tau$ implies that $(\tau_z)_s \subseteq \tau_s$. That is, we have $\tau_z \subseteq \tau_s$, for any topological space (X,τ).

2010 *Mathematics Subject Classification:* 54C05, 54C08, 54C10.
Submitted February 26, 2016. Published December 30, 2016

Definition 1.1. *[7] A set G in a topological space X is said to be z-open if for each $x \in G$ there exists a cozero set H such that $x \in H \subseteq G$, or equivalently if G is expressible as the union of cozero sets. The complement of a z-open set will be referred to be as a z-closed set. The family of all z-open sets of X is denoted by τ_z and is called the complete regularization of τ. It is clear that $\tau_z \subseteq \tau$ in general.*

2. Quasi Compact Spaces

Definition 2.1. *[5](1) A topological space X is said to be quasi compact or Z-compact if every cover of X by cozero sets admits a finite sub cover. And it is clear (X, τ) is quasi compact if and only if (X, τ_z) is compact.*
(2) A subset A of a space X is said to be quasi compact relative to X if every cover of A by cozero sets of X has a finite subcover.

Theorem 2.2. *Let B be a quasi compact relative to X and A be a cozero set of X contained in B. Then $B - A$ is quasi compact relative to X.*

Proof: Let $\mathcal{U}$ be a cover of $B - A$ by cozero sets of X. Then $\mathcal{U} \cup \{A\}$ is a cover of B consisting of cozero sets. Since B is quasi compact relative to X, there exists a finite subcover $\{U_i : 1 \leq i \leq n\}$ such that $B \subseteq A \cup \{U_i : 1 \leq i \leq n\}$ which implies $B - A \subseteq \{U_i : 1 \leq i \leq n\}$. This implies that $B - A$ is quasi compact relative to X. $\square$

Theorem 2.3. *A zero set contained in a set which is quasi compact relative to X is quasi compact relative to X.*

Proof: Let V be a quasi compact relative to X and B be a zero set such that $B \subseteq V$. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be a cover of B consisting of cozero sets of X. Then $\mathcal{U} \cup \{X - B\}$ is a cover of V consisting of cozero sets of X because $X - B$ a cozero set. Since V is quasi compact relative to X, there exists a finite subcollection $\{U_i : 1 \leq i \leq n\}$ of $\mathcal{U}$ such that $V \subseteq \{X - B\} \cup \{U_i : 1 \leq i \leq n\}$. It follows that $B \subseteq \{U_i : 1 \leq i \leq n\}$. Hence B is quasi compact relative to X. $\square$

Theorem 2.4. *The intersection of a set which is quasi compact relative to X and a zero set is quasi compact relative to X.*

Proof: Let us suppose that A is quasi compact relative to X and B is a zero set in a topological space (X, τ). We will show that $A \cap B$ is quasi compact relative to X. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by cozero sets in X. Then $\mathcal{U} \cup \{X - B\}$ is a cover of A by cozero sets of X. Since A is quasi compact relative to X, there exists a finite subfamily $\{U_{\alpha_i} : 1 \leq i \leq n\}$ of $\mathcal{U}$ such that $A \subseteq \cup_{i=1}^n \{U_{\alpha_i}\} \cup \{X - B\}$. Thus we obtain $A \cap B \subseteq \cup_{i=1}^n \{U_{\alpha_i}\}$. This implies that $A \cap B$ is quasi compact relative to X. $\square$

Theorem 2.5. *A finite union of sets which are quasi compact relative to X is quasi compact relative to X.*

Proof: Let $B = \cup_{i=1}^{n}\{B_i : 1 \leq i \leq n\}$, where B_i is quasi compact relative to X. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be any cover of B by cozero sets. Then $\mathcal{U}$ covers each B_i. Since B_i is quasi compact relative to X, there exists a finite subcollection Δ_i of Δ such that $B_i \subseteq \cup_{\alpha\in\Delta_i}\{U_\alpha\}$. It follows that $B \subseteq \cup\{U_\alpha : \alpha \in \cup_{i=1}^{n}\Delta_i\}$. Hence B is quasi compact relative to X. $\square$

Definition 2.6. *A function $f : X \to Y$ is said to be cozero irresolute if $f^{-1}(V)$ is a cozero set in X for each cozero set V in Y.*

Theorem 2.7. *If f is a cozero irresolute surjection from a quasi compact space X onto Y, then Y is quasi compact.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be any cover of Y by cozero sets in Y. Since f is cozero irresolute, each $f^{-1}(U_\alpha)$ is a cozero set in X and $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ forms a cover of X consisting of cozero sets. Since X is quasi compact, there exists a finite subcollection of $\mathcal{V}$, say $\mathcal{V}^* = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ which covers X. It follows that the finite subcollection $\{U_{\alpha_i} : 1 \leq i \leq n\}$ covers Y. Hence Y is quasi compact. $\square$

Definition 2.8. *[11] A space X is functionally Hausdorff if for each $x, y \in X$, $x \neq y$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) \neq f(y)$. And it is clear that each functionally Hausdorff space is Hausdorff.*

Proposition 2.9. *[9] For a topological space X, the following statements are equivalent.*

1. *X is functionally Hausdorff.*
2. *Every pair of distinct points in X are contained in disjoint cozero sets.*

Theorem 2.10. *If $X - K$ is countable and K is a quasi compact subset of a functionally Hausdorff space X, then K is a zero set.*

Proof: Let $x \in X - K$, for any $y \in K$, using the functionally Hausdorff property, we can two cozero sets U_y and V_y with $x \in U_y$, $y \in V_y$ and $U_x \cap V_y = \phi$. Since we obviously have $K \subseteq \cup_{y\in K}V_y$, by quasi compact, there exist points $y_1, \cdots, y_n \in K$, such that $K \subseteq V_{y_1} \cup \cdots \cup V_{y_n}$. Then $x \in \cap_{i=1}^{n}U_{y_i} = D_x$ which is a cozero set containing x. Now we can write $X - K = \cup_{x\in X-K}D_x$ which is a cozero set since the countable union of cozero sets is a cozero set and hence K a zero set. $\square$

3. On $\mathcal{A}$-open Sets

In this section we introduce the following notion:

Definition 3.1. *A subset A of a space X is said to be $\mathcal{A}$-open if for every $x \in A$, there exists a cozero set $U_x \subseteq X$ containing x such that $U_x - A$ is finite. The complement of an $\mathcal{A}$-open subset is said to be $\mathcal{A}$-closed.*

The family of all $\mathcal{A}$-open subsets of a space (X,τ) is denoted by $\mathcal{A}O(X)$. For each $x \in X$, $\mathcal{A}O(X,x)$ denotes the family $\{U \in \mathcal{A}O(X) : x \in U\}$.

Lemma 3.2. *For a subset of a topological space, every cozero set is $\mathcal{A}$-open.*

Proof: Let A be a cozero set. For each $x \in A$, there exists a cozero set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$. Therefore, A is $\mathcal{A}$-open. □

Example 3.1. *Let $X = \mathbb{R}$ be the set of all real numbers. Let τ_{co} be the cofinite topology on X. Since cofinite topology on X is connected, then every continuous function $f : (\mathbb{R},\tau_{co}) \to (\mathbb{R},\tau_u)$ is constant, hence the set of all cozero sets in (X,τ_{co}) is $\{X,\phi\}$. Thus $X - \{1,2,3\}$ is $\mathcal{A}$-open but it is not a cozero set.*

Theorem 3.3. *Let (X,τ) be a topological space. Then $(X,\mathcal{A}O(X))$ is a topological space.*

Proof: (1): We have $\phi, X \in \mathcal{A}O(X)$.
(2): Let $U, V \in \mathcal{A}O(X)$ and $x \in U \cap V$. Then there exist cozero sets $G, H \in X$ containing x such that $G \setminus U$ and $H \setminus V$ are finite. And $(G \cap H) \setminus (U \cap V) = (G\cap H)\cap[(X\setminus U)\cup(X\setminus V)] \subseteq [G\cap(X\setminus U)]\cup[H\cap(X\setminus V)]$. Thus $(G\cap H)\setminus(U\cap V)$ is finite and $G \cap H$ is a cozero set. Hence $U \cap V \in \mathcal{A}O(X)$.
(3): Let $\{U_i : i \in I\}$ be a family of $\mathcal{A}$-open subsets of X and $x \in \cup_{i\in I}U_i$. Then $x \in U_j$ for some $j \in I$. This implies that there exists a cozero set V of X containing x such that $V \setminus U_j$ is finite. Since $V \setminus \cup_{i\in I}U_i \subseteq V \setminus U_j$, then $V \setminus \cup_{i\in I}U_i$ is finite. Thus $\cup_{i\in I}U_i \in \mathcal{A}O(X)$. □

Lemma 3.4. *Let (X,τ) be a topological space. Then the intersection of a cozero set and an $\mathcal{A}$-open set is $\mathcal{A}$-open.*

Proof: Let U be a cozero set and A be $\mathcal{A}$-open. Then, by Lemma 3.2, U is $\mathcal{A}$-open and by Theorem 3.3 the intersection is $\mathcal{A}$-open. □

Lemma 3.5. *A subset A of a space X is $\mathcal{A}$-open if and only if for every $x \in A$, there exist a cozero set U containing x and a finite subset C such that $U - C \subseteq A$.*

Proof: Let A be $\mathcal{A}$-open and $x \in A$, then there exists a cozero set U_x containing x such that $U_x - A$ is finite. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exist a cozero set U_x containing x and a finite subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is a finite set. □

Theorem 3.6. *Let X be a space and $C \subseteq X$. If C is $\mathcal{A}$-closed, then $C \subseteq K \cup B$ for some zero set K and a finite subset B.*

Proof: If C is $\mathcal{A}$-closed, then $X - C$ is $\mathcal{A}$-open and hence for every $x \in X - C$, there exist a cozero set U containing x and a finite set B such that $U - B \subseteq X - C$. Thus $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$. Let $K = X - U$. Then K is a zero set such that $C \subseteq K \cup B$. □

Definition 3.7. *A function $f : X \to Y$ is said to be pre-cozero if $f(V)$ is a cozero set in Y for each cozero set V in X.*

Proposition 3.8. *If $f : X \to Y$ is pre-cozero, then the image of an $\mathcal{A}$-open set of X is $\mathcal{A}$-open in Y.*

Proof: Let $f : X \to Y$ be pre-cozero and W an $\mathcal{A}$-open subset of X. Let $y \in f(W)$, there exists $x \in W$ such that $f(x) = y$. Since W is $\mathcal{A}$-open, there exists a cozero set U such that $x \in U$ and $U - W = C$ is finite. Since f is pre-cozero, $f(U)$ is a cozero set in Y such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$ is finite. Therefore, $f(W)$ is $\mathcal{A}$-open in Y. □

Proposition 3.9. *If $f : X \to Y$ is cozero irresolute injective and A is $\mathcal{A}$-open in Y, then $f^{-1}(A)$ is $\mathcal{A}$-open in X.*

Proof: Assume that A is an $\mathcal{A}$-open subset of Y. Let $x \in f^{-1}(A)$. Then $f(x) \in A$ and there exists a cozero set V containing $f(x)$ such that $V - A$ is finite. Since f is cozero irresolute, $f^{-1}(V)$ is a cozero set containing x. Thus $f^{-1}(V) - f^{-1}(A) = f^{-1}(V - A) =$ finite. It follows that $f^{-1}(A)$ is $\mathcal{A}$-open in X. □

4. On $\mathcal{A}$-open Sets and Quasi Compact Spaces

In this section we obtain characterizations of quasi compact spaces via $\mathcal{A}$-open set.

Theorem 4.1. *For any space X, the following properties are equivalent:*

1. *X is quasi compact;*
2. *Every $\mathcal{A}$-open cover of X has a finite subcover.*

Proof: (1) $\Rightarrow$ (2): Let $\{U_\alpha : \alpha \in \Lambda\}$ be any $\mathcal{A}$-open cover of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is $\mathcal{A}$-open, there exists a cozero set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)}$ and $V_{\alpha(x)} \backslash U_{\alpha(x)}$ is finite. The family $\{V_{\alpha(x)} | x \in X\}$ is a cozero cover of X and X is quasi compact. There exists a finite subset, says $\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)$ such that $X = \cup\{V_{\alpha(x_i)} | i \in F = \{1, 2, ..., n\}\}$. Now, we have

$$\begin{aligned} X =& \cup_{i\in F} \{(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cup U_{\alpha(x_i)}\} \\ =& [\cup_{i\in F}(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)})] \cup [\cup_{i\in F} U_{\alpha(x_i)}]. \end{aligned}$$

For each $\alpha(x_i)$, $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda_{\alpha(x_i)}$ of Λ such that $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)} \subseteq \cup\{U_\alpha | \alpha \in \Lambda_{\alpha(x_i)}\}$. Therefore, we have $X \subseteq [\cup_{i\in F}(\cup\{U_\alpha | \alpha \in \Lambda_{\alpha(x_i)}\})] \cup [\cup_{i\in F} U_{\alpha(x_i)}]$.
(2) $\Rightarrow$ (1): Since every cozero set is $\mathcal{A}$-open, the proof is obvious. □

In any topological space, since every cozero set is open, the following corollary is obvious.

Corollary 4.2. *If a space X is compact, then it is quasi compact.*

Theorem 4.3. *For any space X, the following properties are equivalent:*

1. *X is quasi compact;*
2. *Every proper $\mathcal{A}$-closed set is quasi compact with respect to X.*

Proof: (1) $\Rightarrow$ (2): Let A be a proper $\mathcal{A}$-closed subset of X. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by cozero sets of X. Now for each $x \in X - A$, there is a cozero set V_x such that $V_x \cap A$ is finite. Since $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X - A\}$ is a cozero cover of X and X is quasi compact, there exists a finite subcover $\{U_{\alpha_i} : i \in F_1 = \{1, 2, ..., n\}\} \cup \{V_{x_i} : i \in F_2 = \{1, 2, ..., m\}\}$. Since $\cup_{i \in F_2}(V_{x_i} \cap A)$ is finite, so for each $x_j \in \cup_{i \in F_2}(V_{x_i} \cap A)$, there is $U_{\alpha(x_j)} \in \{U_\alpha : \alpha \in \Lambda\}$ such that $x_j \in U_{\alpha(x_j)}$ and $j \in F_3$ where F_3 is finite . Hence $\{U_{\alpha_i} : i \in F_1\} \cup \{U_{\alpha(x_j)} : j \in F_3\}$ is a finite subcover of $\{U_\alpha : \alpha \in \Lambda\}$ and it covers A. Therefore, A is quasi compact relative to X.
(2) $\Rightarrow$ (1): Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cozero cover of X. We choose and fix one $\alpha_0 \in \Lambda$. Then $X - V_{\alpha_0} \subset \cup\{V_\alpha : \alpha \in \Lambda - \{\alpha_0\}\}$ is a cozero cover of a $\mathcal{A}$-closed set $X - V_{\alpha_0}$. There exists a finite subset Λ_0 of $\Lambda - \{\alpha_0\}$ such that $X - V_{\alpha_0} \subset \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore, $X = \cup\{V_\alpha : \alpha \in \Lambda_0 \cup \{\alpha_0\}\}$. This shows that X is quasi compact. □

Corollary 4.4. *If a space X is quasi compact and A is zero set, then A is quasi compact relative to X.*

Definition 4.5. *A function $f : X \to Y$ is said to be $\mathcal{A}$-continuous if $f^{-1}(V)$ is $\mathcal{A}$-open in X for each open set V in Y.*

Theorem 4.6. *A function $f : X \to Y$ is $\mathcal{A}$-continuous if and only if for each point $x \in X$ and each open set V in Y with $f(x) \in V$, there is an $\mathcal{A}$-open set U in X such that $x \in U$ and $f(U) \subseteq V$.*

Proof: *Sufficiency.* Let V be open in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exists an $U_x \in \mathcal{A}O(X)$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$. Then by Theorem 3.3 $f^{-1}(V)$ is $\mathcal{A}$-open.
Necessity . Let $f(x) \in V$. Then $x \in f^{-1}(V) \in \mathcal{A}O(X)$ since $f : X \to Y$ is $\mathcal{A}$-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$. □

Theorem 4.7. *Let f be an $\mathcal{A}$-continuous function from a space X onto a space Y. If X is quasi compact, then Y is compact.*

Proof: Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an $\mathcal{A}$-open cover of X. Since X is quasi compact, by Theorem 4.1, there exists a finite subset Λ_0 of Λ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Lambda_0\}$; hence $Y = \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore Y is compact. □

Definition 4.8. *A function $f : X \to Y$ is said to be $\mathcal{A}_z$-continuous if $f^{-1}(V)$ is $\mathcal{A}$-open in X for each cozero set V in Y.*

It is clear that a function $f : X \to Y$ is $\mathcal{A}_z$-continuous if and only if for each point $x \in X$ and each cozero set V in Y with $f(x) \in V$, there is an $\mathcal{A}$-open set U in X such that $x \in U$ and $f(U) \subseteq V$.

Definition 4.9. *A function $f : X \to Y$ is said to be weakly $\mathcal{A}_z$-continuous if for each $x \in X$ and each cozero set V of Y containing $f(x)$, there exists $U \in \mathcal{A}O(X, x)$ such that $f(U) \subseteq Cl(V)$.*

Now we state the following theorem whose proof follows from Theorem 4.1.

Theorem 4.10. *Let f be an $\mathcal{A}_z$-continuous function from a space X onto a space Y. If X is quasi compact, then Y is quasi compact.*

Definition 4.11. *A topological space X is said to be Z-closed if for every cozero set cover $\{U_\alpha : \alpha \in \Lambda\}$ of X there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.*

Theorem 4.12. *Let $f : X \to Y$ be a weakly $\mathcal{A}_z$-continuous surjection. If X is quasi compact, then Y is Z-closed.*

Proof: Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cozero cover of Y. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is weakly $\mathcal{A}_z$-continuous, there exists an $\mathcal{A}$-open set $U_{\alpha(x)}$ of X containing x such that $f(U_{\alpha(x)}) \subseteq Cl(V_{\alpha(x)})$. Now $\{U_{\alpha(x)} : x \in X\}$ is an $\mathcal{A}$-open cover of the quasi compact space X. So by Theorem 4.1 there exist a finite numbers of points, says, $x_1, x_2, ..., x_n$ in X such that $X = \cup\{U_{\alpha(x_i)} : 1 \leq i \leq n\}$. Thus $Y = f(\cup\{U_{\alpha(x_i)} : 1 \leq i \leq n\}) = \cup\{f(U_{\alpha(x_i)}) : 1 \leq i \leq n\}) \subseteq \cup\{Cl(V_{\alpha(x_i)}) : 1 \leq i \leq n\}$. This shows that Y is Z-closed.

□

Definition 4.13. *A function $f : X \to Y$ is said to be $\mathcal{A}_z$-closed (resp. $\mathcal{A}$-closed) if $f(A)$ is $\mathcal{A}$-closed in Y for each zero set (resp. closed) set A of X.*

Theorem 4.14. *If $f : X \to Y$ is an $\mathcal{A}_z$-closed surjection such that $f^{-1}(y)$ is quasi compact relative to X for each $y \in Y$ and Y is quasi compact, then X is quasi compact.*

Proof: Let $\{U_\alpha : \alpha \in \Lambda\}$ be any cozero cover of X. For each $y \in Y$, $f^{-1}(y)$ is quasi compact relative to X and there exists a finite subset $\Lambda(y)$ of Λ such that $f^{-1}(y) \subset \cup\{U_\alpha : \alpha \in \Lambda(y)\}$. Now we put $U(y) = \cup\{U_\alpha : \alpha \in \Lambda(y)\}$ since a countable union of cozero sets is a cozero set so $U(y)$ is a cozero set and $V(y) = Y - f(X - U(y))$. Then, since f is $\mathcal{A}_z$-closed, $V(y)$ is an $\mathcal{A}$-open set in Y containing y such that $f^{-1}(V(y)) \subset U(y)$. Since $\{V(y) : y \in Y\}$ is an $\mathcal{A}$-open cover of Y, by Theorem 4.1 there exist a finite numbers of points, says, $y_1, y_2, ..., y_n$ in Y such that $Y = \cup\{V(y_i) : 1 \leq i \leq n\}$. Therefore, $X = f^{-1}(Y) = \cup\{f^{-1}(V(y_i)) : 1 \leq i \leq n\} \subseteq \cup\{U(y_i) : 1 \leq i \leq n\} = \cup\{U_\alpha : \alpha \in \Lambda(y_i), 1 \leq i \leq n\}$. This shows that X is quasi compact. □

Theorem 4.15. *If $f : X \to Y$ is an $\mathcal{A}$-closed surjection such that $f^{-1}(y)$ is compact in X for each $y \in Y$ and Y is quasi compact, then X is compact.*

Proof: The proof is analogous to that of Theorem 4.14. □

5. Z-closed

The intersection of all $\mathcal{A}$-closed sets of X containing A is called the $\mathcal{A}$-closure of A and is denoted by $Cl_a(A)$. And the union of all $\mathcal{A}$-open sets of X contained in A is called the $\mathcal{A}$-interior and is denoted by $Int_a(A)$.

Lemma 5.1. *Let A be a subset of a space X. Then*

1. *A is $\mathcal{A}$-closed in X if and only if $A = Cl_a(A)$.*
2. *$Cl_a(X \setminus A) = X \setminus Int_a(A)$.*
3. *$Cl_a(A)$ is $\mathcal{A}$-closed in X.*
4. *$x \in Cl_a(A)$ if and only $A \cap G \neq \phi$ for each $\mathcal{A}$-open set G containing x.*

Theorem 5.2. *Let every cozero set in a space X be infinite. If A is a cozero set, then $Cl_a(A) = Cl_z(A)$.*

Proof: Clearly $Cl_a(A) \subseteq Cl_z(A)$. Let $x \in Cl_z(A)$ and B be an $\mathcal{A}$-open subset containing x. Then by Lemma 3.5, there exist a cozero subset V containing x and a finite set C such that $V \setminus C \subseteq B$. Thus $(V \setminus C) \cap A \subseteq B \cap A$ and so $(V \cap A) \setminus C \subseteq B \cap A$. Since $x \in V$ and $x \in Cl_z(A)$, $V \cap A \neq \phi$. Since $V \cap A$ is cozero, by the hypothesis $V \cap A$ is infinite and so is $(V \cap A) \setminus C$. Then $B \cap A$ is not finite. Therefore, $B \cap A \neq \phi$ and by Lemma5.1 $x \in Cl_a(A)$. Hence, $Cl_a(A) = Cl_z(A)$. □

A subset S of X is said to be Z-closed relative to X if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of S by cozero sets in X, there exists a finite subset Λ_0 of Λ such that $S \subseteq \{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.

Theorem 5.3. *For an open set G of a topological space X, the following properties are equivalent:*

1. *G is Z-closed relative to X;*
2. *For each cover $\{U_\alpha : \alpha \in \Lambda\}$ of G by $\mathcal{A}$-open sets of X, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $G \subseteq \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.*

Proof: (1)⇒ (2): Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of G and $U_\alpha \in \mathcal{A}O(X)$. For each $x \in G$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is $\mathcal{A}$-open, there exists a cozero set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)}$ and $V_{\alpha(x)} \setminus U_{\alpha(x)}$ is finite. The family $\{V_{\alpha(x)} : x \in G\}$ is a cozero cover of G and G is Z-closed relative to X. There exists a finite subset, says, $x_1, x_2, \cdots, x_n$ such that $G \subseteq \cup\{Cl(V_{\alpha(x_i)}) : i \in F\}$, where $F = \{1, 2, ..., n\}$. Now, we have

$$G \subseteq \cup_{i\in F}\,\{Cl((V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cup U_{\alpha(x_i)})\}$$
$$=[\cup_{i\in F}(Cl(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}))] \cup [\cup_{i\in F} Cl(U_{\alpha(x_i)})].$$

For each $\alpha(x_i)$, $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda_{\alpha(x_i)}$ of Λ such that $(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cap G \subseteq \cup\{U_\alpha : \alpha \in \Lambda_{\alpha(x_i)}\}$ and $Cl(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cap G \subseteq Cl((V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cap G) \subseteq \cup\{Cl(U_\alpha) : \alpha \in \Lambda_{\alpha(x_i)}\}$. Therefore, we have $G \subseteq [\cup_{i\in F}(\cup\{Cl(U_\alpha) : \alpha \in \Lambda_{\alpha(x_i)}\})] \cup [\cup_{i\in F} Cl(U_{\alpha(x_i)})]$. Hence G is Z-closed relative to X.
(2)$\Rightarrow$ (1): Since every cozero set is $\mathcal{A}$-open, the proof is obvious. □

Corollary 5.4. *For any topological space X, the following properties are equivalent:*

1. *X is Z-closed;*
2. *For each $\mathcal{A}$-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.*

Theorem 5.5. *If a topological space (X, τ) is Z-closed, then for each $\mathcal{A}$-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = Cl_z(\cup\{U_\alpha : \alpha \in \Lambda_0\})$.*

Proof: Let $\{U_\alpha : \alpha \in \Lambda\}$ be any $\mathcal{A}$-open cover of X. If X is Z-closed, by Corollary 5.4 there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$. Then since $\tau_z \subseteq \tau$, $X \subseteq \cup\{Cl_z(U_\alpha) : \alpha \in \Lambda_0\} = Cl_z(\cup\{U_\alpha : \alpha \in \Lambda_0\})$. □

The following lemma is well known and will be stated without proof.

Lemma 5.6. *A topological space is a T_1-space if and only if every finite set is closed.*

Proposition 5.7. *Let (X, τ) be a T_1-space. Then the following properties hold:*

1. *For each $A \in \mathcal{A}(X, x)$, there exists $V_x \in \tau$ such that $x \in V_x \subseteq A$,*
2. *$\mathcal{A}(X, \tau) \subseteq \tau$.*

Proof: (1) Let A be any $\mathcal{A}$-open set and $x \in A$, then there exists a cozero set U_x containing x such that $U_x - A$ is finite. Let $C = U_x - A = U_x \cap (X - A)$. Then $x \in U_x - C \subseteq A$. Since $\tau_z \subseteq \tau$, by Lemma 5.6 $V_x = U_x - C$ is an open set and $x \in V_x \subseteq A$.
(2) This is an immediate consequence of (1). □

Theorem 5.8. *If (X, τ) is a Z-closed T_1-space, then for each $\mathcal{A}$-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = Cl_a(\cup\{U_\alpha : \alpha \in \Lambda_0\})$.*

Proof: By Proposition 5.7 (2), $\mathcal{A}(X, \tau) \subseteq \tau$ and $Cl(A) \subseteq Cl_a(A)$ for any subset A of X. Therefore, the proof follows form Corollary 5.4. □

Acknowledgement

The authors wishes to thank the referees for useful comments and suggestions.

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